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On a Generalization of the Spaces Introduced by M. De Wilde and Closed Graph Theorem

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M. De Wilde defined in [1] certain types of linear topological spaces (in the sequel we suppose every linear topological space is Hausdorff) for which the closed graph theorem concerning linear mappings from Banach spaces is valid, i.e. the spaces with the *réseau* of type (P), (K), and $(\mathcal{E})$. In this paper we prove that all the types (P), (K), and $(\mathcal{E})$ coincide, consider a generalization of the spaces defined by M. De Wilde, and prove another type of closed graph theorem generalizing and simplifying the result obtained in [1], succeeding the investigation in the papers [2], [3], and [4].

§1. Equivalence of the *réseau* of type (P), (K), and $(\mathcal{E})$.

A system $\mathfrak{R}$ of subsets $E_{n_1, n_2, \cdots, n_k}$ $(k, n_1, n_2, \cdots, n_k=1, 2, \cdots)$ of a linear topological space E satisfying $E=\bigcup_{n_1=1}^{\infty} E_{n_1}$ and

$$E_{n_1, n_2, \cdots, n_k}=\bigcup_{n_{k+1}=1}^{\infty} E_{n_1, n_2, \cdots, n_k, n_{k+1}}$$

for every finite sequence $\{n_1, n_2, \cdots, n_k\}$ of natural numbers is called a *réseau*[1] of E.

A *réseau* $\mathfrak{R}$ is said to be of *type* (P), if for any sequence $\{n_i\}$ of natural numbers there exists a sequence $\{\lambda_i\}$ of positive numbers such that $\{\sum_{i=1}^{n} \mu_i x_i \mid n=1, 2, \cdots\}$ converges in E for all μ_i in $[0, \lambda_i]$ and all x_i in $E_{n_1, n_2, \cdots, n_i}$.

If for any sequence $\{n_i\}$ of natural numbers there exists a sequence $\{\lambda_i\}$ of positive numbers such that for all μ_i in $[0, \lambda_i]$ and all x_i in $E_{n_1, n_2, \cdots, n_i}$ the closure of $\{\sum_{i=1}^{n} \mu_i x_i \mid n=1, 2, \cdots\}$ is compact (resp. for any

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1) Though M. De Wilde defined in [1] for a locally convex space E, here we do not assume local convexity since it is not necessary.

sequence of $\{\sum_{i=1}^{n}\mu_i x_i \mid n=1, 2, \cdots\}$ there exists a subsequence converging in E), then $\mathcal{R}$ is said to be *a réseau of type* (K) (resp. *type* $(\mathcal{E})$).

PROPOSITION 1. *The réseau of type* (P), (K), *and* $(\mathcal{E})$ *coincide.*

PROOF. It is clear that type (P) is type (K) and type $(\mathcal{E})$. Now, we show that type (K) is type (P). Let

$$\mathcal{R}=\{E_{n_1, n_2, \cdots, n_k} \mid k, n_1, n_2, \cdots, n_k=1, 2, \cdots\}$$

be a *réseau* of type (K). Therefore, for any sequence $\{n_i\}$ of natural numbers, there exist positive numbers λ_i $(i=1, 2, \cdots)$ such that $\{\sum_{i=1}^{n}\mu_i x_i \mid n=1, 2, \cdots\}^-$ is compact for all μ_i in $[0, \lambda_i]$ and all x_i in $E_{n_1, \cdots, n_k}$. We put $E_i=\bigcup_{\mu\in[0,\lambda_i]}\mu E_{n_1, n_2, \cdots, n_i}$ for each i, then $E_{i+1}\subset E_i$ since we may suppose $\lambda_i \downarrow_i$. We will show that if $\{\sum_{i=1}^{n} x_i \mid n=1, 2, \cdots\}^-$ is compact for all x_i in E_i $(i=1, 2, \cdots)$, then $\{\sum_{i=1}^{n} z_i \mid n=1, 2, \cdots\}$ converges in E for all z_i in $\frac{1}{i}E_i$ $(i=1, 2, \cdots)$. Suppose $\sum_{i=1}^{\infty} z_i$ does not converge for some $z_i\in\frac{1}{i}E_i$ $(i=1, 2, \cdots)$, then the sequence $z_1+z_2+\cdots+z_n$ $(n=1, 2, \cdots)$ is not Cauchy sequence in E since $z_1+z_2+\cdots+z_n$ belongs to the compact set $\{\sum_{i=1}^{n} z_i \mid n=1, 2, \cdots\}^-$ for each n. Therefore, for some circled neighbourhood U of 0 in E and for any natural number m_0 there exist natural numbers n and m such that $\frac{1}{m}x_m+\frac{1}{m+1}x_{m+1}+\cdots+\frac{1}{n}x_n\notin U$ or $\frac{m}{m}x_m+\frac{m}{m+1}x_{m+1}+\cdots+\frac{m}{n}x_n\notin mU$. Therefore we can find a sequence $x'_i\in E_i$ $(i=1, 2, \cdots)$ such that the set of elements $x'_m+x'_{m+1}+\cdots+x'_n$ $(m<n)$ is not bounded. This is a contradiction, since every $x'_m+x'_{m+1}+\cdots+x'_n$ is contained in the compact set $K-K$ (where $K=\{\sum_{i=1}^{n} x'_n \mid n=1, 2, \cdots\}^-$) and hence bounded.

In the same way, we can show that type $(\mathcal{E})$ is type (P).

We make it a rule to call a *W-sequence* countable subsets B_n $(B_n\supset B_{n+1}, n=1, 2, \cdots)$ of E satisfying the following condition (I).

(I) *There exist positive numbers* λ_n $(n=1, 2, \cdots)$ *such that* $\{\sum_{n=1}^{k}\mu_n x_n \mid k=1, 2, \cdots\}$ *converges in* E *for all* μ_n *in* $[0, \lambda_n]$ *and all* x_n *in* B_n.

In a linear topological space E, we may consider a system $\mathcal{S}=\{C_{n,m} \mid n, m=1, 2, \cdots\}$ such that $E=\bigcup_{m=1}^{\infty}C_{n,m}$ for each n instead of a *réseau*

$\mathcal{R}=\{E_{n_1, n_2, \cdots, n_k} | k, n_1, n_2, \cdots, n_k=1, 2, \cdots\}$ of E, because from every *réseau* $\mathcal{R}$ we can get a countable covering of E such that

$$\{E_{n_1, n_2, \cdots, n_k, n_{k+1}} \cup E^C_{n_1, \cdots, n_k} | n_{k+1}=1, 2, \cdots\}$$

for any finite sequence $\{n_1, n_2, \cdots, n_k\}$, conversely for any system $\mathcal{S}$ we can choose a *réseau* $\mathcal{R}$ such that $E_{n_1, n_2, \cdots, n_k}=C_{1, n_1} \cap C_{2, n_2} \cap \cdots \cap C_{k, n_k}$ for each finite sequence $\{n_1, n_2, \cdots, n_k\}$.

If we state a *réseau* of type (P) by a system $\mathcal{S}$ of E, then it turns out the following.

For every sequence $\{m_n\}$ of natural numbers, $B_n=C_{1, m_1} \cap C_{2, m_2} \cap \cdots \cap C_{n, m_n}$ $(n=1, 2, \cdots)$ is a W-sequence.

Therefore, in the sequel we make use of a system $\mathcal{S}$ instead of a *réseau* $\mathcal{R}$ of E.

§ 2. A generalization of the spaces introduced by M. De Wilde.

We make it a rule to call a *$*$-W-sequence* countable subsets B_n $(B_n \supset B_{n+1}, n=1, 2, \cdots)$ of E such that a countable subfamily of $\{B_n\}$ is a W-sequence.

Then, we consider a linear topological space E with a system $\mathcal{S}=\{C_{n, m} | n, m=1, 2, \cdots\}$ satisfying the following condition (II).

(II) *For every sequence $\{m_n\}$ of natural numbers, $B_n=C_{1, m_1} \cap C_{2, m_2} \cap \cdots \cap C_{n, m_n}$ $(n=1, 2, \cdots)$ is a $*$-W-sequence.*

The class of our spaces, as in the case of the spaces considered by M. De Wilde, is closed by the following operations:

(1) *the image by a continuous linear mapping*
(2) *the sequentially closed subspace*
(3) *the product space of countable spaces*
(4) *the inductive limit of countable spaces*

PROOF OF (1). Let E be a linear topological space with a system $\mathcal{S}=\{C_{n, m} | n, m=1, 2, \cdots\}$ satisfying the condition (II). Let F be a linear topological space such that $F=\varphi(E)$ by a continuous linear mapping φ. Then F is a linear topological space with a system $\{\varphi(C_{n, m}) | n, m=1, 2, \cdots\}$ satisfying the condition (II).

PROOF OF (2). Let $\mathcal{S}=\{C_{n, m} | n, m=1, 2, \cdots\}$ be a system of a linear topological space E satisfying the condition (II). Then, for each sequentially closed subspace F, $\{C_{n, m} \cap F | n, m=1, 2, \cdots\}$ is a system of F satisfying the condition (II).

PROOF OF (3). For each n, let E_n be a linear topological space with a system $\mathcal{S}^{(n)}=\{C_{l,m}^{(n)}\,|\,l, m=1, 2, \cdots\}$ satisfying the condition (II) and p_n the projection from the product space $E=\prod_{n=1}^{\infty} E_n$ of E_n $(n=1, 2, \cdots)$ to E_n. Then the system of coverings $\{p_n^{-1}(C_{l,m}^{(n)})\,|\,m=1, 2, \cdots\}$ of E $(n, l,=1, 2, \cdots)$ satisfying the condition (II).

PROOF OF (4). For each n, let E_n be a linear topological space with a system $\mathcal{S}^{(n)}=\{C_{l,m}^{(n)}\,|\,l, m=1, 2, \cdots\}$ satisfying the condition (II). Then, the inductive limit E of E_n $(n=1, 2, \cdots)$ is a linear topological space with the system of coverings $\{C_{l,m}^{(n)}\cup(E\setminus E_n)\,|\,m=1, 2, \cdots\}$ of E $(n, l,=1, 2, \cdots)$ satisfying the condition (II).

To prove the closed graph theorem for our space, we make use of the following

PROPOSITION 2. *For every sequence* $\{B_n\}$ $(B_n\supset B_{n+1}, n=1, 2, \cdots)$ *of subsets of* E*, the following two conditions* (1) *and* (2) *are equivalent.*

(1) $\{B_n\}$ *is a* $*$*-W-sequence.*

(2) *For* $\{B_n\}$*, there exists a complete metric group* G *and a continuous homomorphism* φ *from* G *to* E *such that for each neighbourhood* U *of* 0 *in* G, $\varphi(U)$ *absorbs some* B_n.

PROOF. "(1)$\Rightarrow$(2)" By the condition (1), there exist a sequence $\{n_i\}$ of natural numbers and a sequence $\{\lambda_i\}$ of positive numbers such that $\{\sum_{i=1}^{n}\mu_i x_i\,|\,n=1, 2, \cdots\}$ converges in E for all μ_i in $[0, \lambda_i]$ and all x_i in B_{n_i}. We put $U_i=\bigcup_{|\mu|\in[0,\lambda_i]}\mu B_{n_i}$ $(i=1, 2, \cdots)$. Then we have $U_{i+1}\subset U_i$ $(i=1, 2, \cdots)$, since we may assume $\lambda_i\downarrow_i$.

We set $U_A=\{\sum_{j=1}^{k}x_{n_j}\,|\,n_j\in A, x_{n_j}\in U_{n_j}, k=1, 2, \cdots\}$ for every infinite subset A of the set $\boldsymbol{N}$ of all natural numbers.

Let $A_1=\{n_1, n_2, \cdots\}$ be a subset of $\boldsymbol{N}$ satisfying $n_{i+1}-n_i>1$ $(i=1, 2, \cdots)$ and $n_{i+1}-n_i\to\infty$ $(i\to\infty)$. Then we choose a subset $A_2=\{m_1, m_2, \cdots\}$ of $\boldsymbol{N}$ such that $m_1>n_2$, $m_{i+1}-m_i>1$ $(i=1, 2, \cdots)$, $m_{i+1}-m_i\to\infty$ $(i\to\infty)$, and that there exist at least two elements of A_1 between m_i and m_{i+1} for each $i=1, 2, \cdots$. Then we have $U_{A_1}\supset U_{A_2}+U_{A_2}$ for A_1 and A_2. Continuing this process, we can find a sequence $A_i=\{n_1^{(i)}, n_2^{(i)}, \cdots, n_k^{(i)}, \cdots\}$ $(i=1, 2, \cdots)$ of subsets of $\boldsymbol{N}$ such that $U_{A_i}\supset U_{A_{i+1}}+U_{A_{i+1}}$ for each i. Moreover, by virtue of the condition $n_{k+1}^{(i)}-n_k^{(i)}\to\infty$ $(k\to\infty)$ for each i, we can choose $\{A_i\}$ satisfying $A_i\cap A_j=\phi$ $(i\neq j)$. (In this case, necessarily $\lim_{i\to\infty}\min A_i=\infty$.)

Then we define the topology $\mathcal{T}'$ of the additive group E by taking $\{U_{A_i}\}$ as the fundamental neighbourhood system of 0 in E. Let i be the identity mapping from $\{E, \mathfrak{T}'\}$ to E and G the completion of $\{E, \mathfrak{T}'\}$. Now we will show that this identity mapping i can be extended to a continuous homomorphism φ from G to E. To prove that every $\mathfrak{T}'$-Cauchy sequence $\{y_n\}$ of E converges in E, it is sufficient to prove that for all x_i in U_{A_i}. Here each x_i is the sum of finite number of elements in U_j $(j\in A_i)$. Let y_n $(n=1, 2, \cdots)$ be a sequence of elements of E such that $y_n\in U_n$ and every $y_n\neq 0$ appears as a summand of x_i for some i. Then the sequence $\{\sum_{i=1}^{n} y_i \mid n=1, 2, \cdots\}$ converges to some element x in E. For every neighbourhood U of 0 in E, there exists a neighbourhood V of 0 in E such that $V+V\subset U$. We can see easily that there exists a natural number n such that $\sum_{j=1}^{\infty} y_{n_j}\in V$ for every subsequence $n_j>n$, and for this n there exists a natural number m such that $1, 2, \cdots n\notin A_{m+1}$. Then we have $x-\sum_{i=1}^{m} x_i\in V+V\subset U$. Thus $\{\sum_{i=1}^{n} x_i \mid n=1, 2, \cdots\}$ converges to x. Since two different $\mathfrak{T}'$-Cauchy sequences $\{y_n\}$ and $\{z_n\}$ converging to any element $\bar{y}$ in G, converge to the same element y in E, we define a mapping φ from G to E by $\varphi(\bar{y})=y$. It is clear that φ is a homomorphism and so, we will show that φ is continuous. Let $\{\bar{y}_n\}$ be a sequence of G converging to 0. For each n, there exists a $\mathfrak{T}'$-Cauchy sequence $\{y_m^{(n)}\}$ of E converging to some element y_n in E. From the double sequence $\{y_m^{(n)} \mid n, m=1, 2, \cdots\}$, we can choose some $\mathfrak{T}'$-Cauchy sequence $\{y_{m_n}^{(n)}\}$ converging to 0 in E. By using this sequence, we can prove that $\{y_n\}$ converges to 0 in E.

"(2)$\Rightarrow$(1)" By the assumption (2), there exist a complete metric group G and a continuous homomorphism φ from G to E such that $\varphi(U)$ absorbs some B_n for all neighbourhood U of 0 in G. Since the topology of G is complete metric group, we can choose a fundamental neighbourhood system $\{U_i\}$ of 0 in G such that $\sum_{i=1}^{\infty} g_i$ converges for all g_i in U_i $(i=1, 2, \cdots)$. For each U_i, $\varphi(U_i)$ absorbs some B_{n_i}, so, for each n_i, there exists a $\lambda_i>0$ such that $\mu_i B_{n_i}\subset\varphi(U_i)$ for all μ_i in $[0, \lambda_i]$. Therefore, for every y_i in $\mu_i B_{n_i}$ there exists an element g_i in U_i satisfying $y_i=\varphi(g_i)$. Since $\sum_{i=1}^{\infty} g_i$ converges in G and φ is continuous, we can see that $\sum_{i=1}^{\infty} y_i$ converges in E. That is, $\{\sum_{i=1}^{n} \mu_i x_i \mid n=1, 2, \cdots\}$ converges in E for all μ_i in $[0, \lambda_i]$ and all x_i in B_{n_i}.

Thus the proof is completed.

THEOREM. *Every linear mapping φ with sequentially closed graph from an $\mathfrak{F}$-space F into a linear topological space E with a system $\mathcal{S}$ satisfying the condition* (II) *is continuous.*

PROOF. Since the graph $G(\varphi)$ of φ is a sequentially closed subspace of the product space $E \times F$ of two linear topological spaces E and F with a system $\mathcal{S}$ satisfying the condition (II) respectively, it has also a system satisfying the condition (II). Therefore, it is sufficient to prove that every continuous linear mapping φ from a linear topological space E with a system $\mathcal{S}$ satisfying the condition (II) onto an $\mathfrak{F}$-space F is open. There exists a $\mathfrak{s}$-W-sequence $\{B_k\}$ of E such that $\varphi(B_k)$ is of second category in F for each k. By virtue of the Proposition, there exist a complete metric group G and continuous homomorphism f from G to E such that for every neighbourhood U of 0 in G, $f(U)$ absorbs some B_k so that $\varphi \cdot f(U)$ is of second category in F. For any neighbourhood V of 0 in E, since there exists a neighbourhood U of 0 in G such that $f(U) \subset V$, $\varphi(V)$ is of second category in F, and hence the closure of $\varphi(V)$ has an interior point. Now the proof is completed by virtue of the following fact.

Let φ be a continuous homomorphism from a complete metric group X to a metric group Y. If for every neighbourhood U of 0 in X the closure of $\varphi(U)$ has an interior point, then $\varphi(U)$ is an neighbourhood of 0 in Y.

Especially, if a system $\{C_{n,m} \mid n, m=1, 2, \cdots\}$ satisfy the property $C_{n,m} \uparrow_m$ for each n, then our space coincide with a linear topological space which has countable S-filters Φ_n $(n=1, 2, \cdots)$ (i.e. each Φ_n has a countable basis $\{S_k\}$ such that $\bigcap_k S_k = \phi$) satisfying the following condition (III).

(III) *For any filter Ψ in E which is disjoint from every Φ_n $(n=1, 2, \cdots)$, there exist a complete metric group G and a continuous homomorphism f from G into E such that for any neighbourhood U of 0 in E, $f(U)$ absorbs some element B in Ψ.*

We discussed in [4] about a linear topological space which has countable S-filters Φ_n $(n=1, 2, \cdots)$ satisfying the condition (III).

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