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Some Applications of p-normed Algebras

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As the metric generalizations of Banach algebras W. Żelazko considered, in his paper [1], some types of topological algebras. One of them is not always convex in the sense of a linear space, that is the p-normed algebra, $0<p\leqq 1$. The purpose of this note is to give some applications of this theory of p-normed algebras, not so trivial in case $0<p<1$.

§1. We shall give the definition and the fundamental properties of the p-normed algebra due to W. Żelazko.

DEFINITION. *A p-normed linear space A is a linear space over the complex number field C with p-norm $\|x\|$, $0<p\leqq 1$, i.e., a functional on A satisfying*

1) $\|x\|\geqq 0$ *and* $\|x\|=0$ *iff* $x=0$
2) $\|x+y\|\leqq\|x\|+\|y\|$
3) $\|\lambda x\|=|\lambda|^p\|x\|$ *where λ is complex and x, y elements of A.*

A p-normed algebra is a complete p-normed linear space in which multiplication is defined satisfying

$$\|xy\|\leqq\|x\|\,\|y\|,\ \|e\|=1 \text{ where } e \text{ is the identity of } A.$$

If a p-normed algebra A is a field we call A a p-normed field.

THEOREM 1. *A p-normed field A is isomorphic and homeomorphic with the complex number field.*

PROPOSITION 1. *Every ideal of A is contained in a maximal ideal. Every maximal ideal is closed and codimension 1 and there is a 1–1 correspondence between multiplicative linear functionals and maximal ideals given by*

$$M=\{x\in A\,;\ \varphi_M(x)=0\}\,.$$

$\varphi_M(x)=\lambda$, if $x=m+\lambda e$, $m\in M$ and this decomposition follows from the fact that $A=M\oplus\{\lambda e\}$. Consequently each multiplicative linear functional is continuous.

PROPOSITION 2. *If $\mathfrak{M}_A$ is the compact space of all maximal ideals of A (in $\mathfrak{M}_A$ the weak topology is introduced) then there is a continuous homomorphism of A into the algebra $C(\mathfrak{M}_A)$, the space of all continuous* *functions on $\mathfrak{M}_A$, given by*

$$x \rightarrow \varphi_M(x) \equiv x(M) .$$

Moreover the inequality

$$\sup_M |x(M)|^p \leqq \|x\| \text{ holds for each } x \text{ of } A.$$

THEOREM 2. *An element x of A is invertible if and only if $x(M) \neq 0$, $M \in \mathfrak{M}_A$ or equivalently if and only if $\varphi(x) \neq 0$ for each multiplicative linear functional φ.*

§ 2. APPLICATIONS. As in the theory of Banach algebras the following facts, not found in [1], though simple, seem to be some of significance.

1°) *Suppose $f(z)$ is a holomorphic function in the unit disc U such that*

$$\text{if } f(z) = \sum_{n=0}^{\infty} a_n z^n, \ \sum_{n=0}^{\infty} |a_n|^p < \infty \text{ for a fixed } p, \ 0 < p < 1 \tag{1}$$

and $|f(z)| > 0$ for each z in $\bar{U}$.

Then $1/f(z) = \sum_{n=0}^{\infty} c_n z^n$, $\sum_{n=0}^{\infty} |c_n|^p < \infty$.

PROOF. Let $A_p(U)$ be the space of all holomorphic functions f in the unit disc U expressed as $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\sum_{n=0}^{\infty} |a_n|^p < \infty$. It is clear that $\lambda f(z)$ belongs to $A_p(U)$ for $\lambda \in C$ and $f \in A_p(U)$.

Since sums of holomorphic functions are holomorphic, and holds the inequality

$$|a+b|^p \leqq |a|^p + |b|^p \text{ for } p, \ 0 < p \leqq 1 , \tag{2}$$

it is easy to see that with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$ their sum $f(z) + g(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^n$ also belongs to $A_p(U)$.

Thus $A_p(U)$ becomes a p-normed linear space under the p-norm $\|f\| = \sum_{n=0}^{\infty} |a_n|^p$. Moreover $A_p(U)$ is complete under this p-norm. Further $A_p(U)$ becomes a p-normed algebra, under pointwise multiplication.

For if $f, g \in A_p(U)$, then $f(z)g(z) = \sum_{n=0}^{\infty} \left(\sum_{i+j=0}^{n} a_i b_j \right) z^n$ and hence

$$\|fg\| = \sum_{n=0}^{\infty} \left| \sum_{i+j=0}^{n} a_i b_j \right|^p \leqq \sum_{n=0}^{\infty} \sum_{i+j=0}^{n} |a_i b_j|^p = \|f\| \ \|g\| .$$

Also, the constant function 1 is the identity of $A_p(U)$, and $\|1\| = 1$.

Now put $f_0(z) = z$, then $f_0 \in A_p(U)$ and $\|f_0\| = 1$. If φ is any multiplicative linear functional on $A_p(U)$ and $\varphi(f_0) = \alpha$, then, by virtue of proposition 2, we have $|\alpha|^p \leqq 1$, and consequently $|\alpha| \leqq 1$. If f is given by (1) then $f = \sum a_n f_0^n$.

Since this series converges in $A_p(U)$ and φ is continuous on $A_p(U)$, we conclude $\varphi(f)=f(\alpha)(f\in A_p(U))$. Our hypothesis that f vanishes at no point of $\bar{U}$ asserts that f is not in the kernel of any multiplicative linear functional and which shows from theorem 2, that f is invertible in $A_p(U)$. But this is what we have to show.

2°) *Suppose $f_1, \cdots, f_n$ are members of the above-mentioned algebra $A_p(U)$, such that $|f_1(z)|+\cdots+|f_n(z)|>0$ for every $z\in \bar{U}$. Then there exist $g_1, \cdots, g_n\in A_p(U)$ such that $\sum_{i=1}^{n} f_i(z)\, g_i(z)=1$ $(z\in \bar{U})$.*

PROOF. The set J of all functions $\sum_{i=1}^{n} f_i g_i$, where the g_i are arbitrary members of $A_p(U)$, is an ideal of $A_p(U)$. We have to prove that J contains the identity 1 of $A_p(U)$, i.e., there is no maximal ideal containing J. By theorem 2, we have only to prove that there is no multiplicative linear functional φ on $A_p(U)$ into the complex number field such that $\varphi(f_i)=0$ for every i $(1\leqq i\leqq n)$.

Put $f_0(z)=z$ and $\varphi(f_0)=\alpha$, as before. By the same reason as 1°) our hypothesis implies that $|f_i(\alpha_i)|>0$ for at least one index i, $1\leqq i\leqq n$ $(\alpha\in U)$, follows $\varphi(f_i)\neq 0$. We have proved that to each $\varphi\in \mathfrak{M}_{A_p}$ there corresponds at least one of the given function f_i such that $\varphi(f_i)\neq 0$, and which, as remarked, is to be proved.

REMARK 1. Similarly as in case of Banach algebras, we have also determined all maximal ideals of $A_p(U)$, in the course of the preceding proof, since each is the kernel of some $\varphi\in \mathfrak{M}_{A_p}$: If $\alpha\in \bar{U}$ and if M_α is the set of all $f\in A_p(U)$ such that $f(\alpha)=0$, then M_α is a maximal ideal of $A_p(U)$ and all maximal ideals of $A_p(U)$ are obtained in this way.

REMARK 2. Let $A(U)$ be the space of all continuous functions in the closure of the unit disc $\bar{U}$ whose restrictions to the open unit disc U are holomorphic. If $f(z)=\sum_{n=0}^{\infty} a_n z^n$, $f\in A_p(U)$, then, by reason of the inequality (2), follows $(\sum_{n=0}^{\infty}|a_n|)^p\leqq \sum_{n=0}^{\infty}|a_n|^p$.

Thus $f(z)$ is continuous in the closure of the open unit disc U. Hence $A_p(U)$ is a subspace of $A(U)$ in sence of a linear space.

3°) Let A be the space of all formal power series $\sum_{-\infty}^{\infty} a_n X^n$, where $\{a_n\}_{-\infty}^{\infty}$ is a sequence of complex numbers satisfying $\sum_{-\infty}^{\infty}|a_n|^p\alpha_n<\infty$ for a fixed sequence of positive numbers $\{\alpha_n\}_{-\infty}^{\infty}$ and p, $0<p<1$. If we define a p-norm of $x=\sum_{-\infty}^{\infty} a_n X^n$ in A by $||x||=\sum_{-\infty}^{\infty}|a_n|^p\alpha_n$, then A is a complete p-normed linear space under usual operations on power series. We shall prove the following facts, obtained by I. Gelfand [2], also hold in A.

With each pair $x=\sum_{-\infty}^{\infty} a_n X^n$ and $y=\sum_{-\infty}^{\infty} b_n X^n$ in A, their formal product

$x*y$ is, by definition, $x*y=\sum_{n=-\infty}^{\infty}(\sum_{k=-\infty}^{\infty}a_k b_{n-k})X^n$.

Then, the necessary and sufficient condition for the formal product $x*y$ of each pair of x and y in A to be again in A is that there exists a positive constant c, for the sequence $\{\alpha_n\}_{-\infty}^{\infty}$, such that $\alpha_{m+n}\leqq c\alpha_m\alpha_n$.

PROOF OF NECESSITY. First, to show that there exists a positive constant c such that $||x*y||\leqq c||x||\ ||y||$ for every x, y of A, we shall prepare the following lemma, known in case of Banach spaces as Gelfand's lemma, which can be proved in almost the same way as in Gelfand's lemma.

LEMMA. *Let $L(x)$ be a subadditive p-homogeneous functional defined on a complete p-normed linear space X, i.e., a functional L on X satisfying*

1) $0\leqq L(x)<\infty$
2) $L(x+y)\leqq L(x)+L(y)$
3) $L(\lambda x)=|\lambda|^p L(x)$ *for any complex λ and a fixed p with* $0<p\leqq 1$.

Then it is necessary and sufficient for $L(x)$ to be bounded is that $L(x)$ is lower semicontinuous on X.

Hence we shall make use of the term "a p-convex functional" for "a subadditive p-homogeneous functional".

Now fix $y=\sum_{-\infty}^{\infty}b_n X^n$. Since $|\sum_{k=-N}^{N}a_k b_{n-k}|^p$ is, for each $N\geqq 0$, continuous p-homogeneous functional in x, $\sup_N|\sum_{k=-N}^{N}a_k b_{n-k}|^p=|\sum_{k=-\infty}^{\infty}a_k b_{n-k}|^p$ is a lower semicontinuous functional on A. If we denote $K_y(x)=||x*y||$, then

$$K_y(x)=\sum_{n=-\infty}^{\infty}\alpha_n|\sum_{k=-\infty}^{\infty}a_k b_{n-k}|^p=\sup_N\sum_{n=-N}^{N}\alpha_n|\sum_{k=-\infty}^{\infty}a_k b_{n-k}|^p$$

which shows K_y is a p-convex and lower semicontinuous functional on A. Then, by virtue of the above-mentioned lemma, K_y is bounded on $||x||\leqq 1$ and therefore $K(y)=\sup_{||x||\leqq 1}||x*y||$ exists for every y of A. Moreover, since K is also a p-convex continuous functional, again by the lemma, we see that K is bounded, i.e., there exists a positive constant c such that $K(y)\leqq c||y||$. Consequently we obtain $||x*y||\leqq c||x||\ ||y||$. Now as x and y are arbitrary, taking $x=X^m$ and $y=X^n$, we have $\alpha_{m+n}\leqq c\alpha_m\alpha_n$.

PROOF OF SUFFICIENCY. If there exists a positive constant c such that $\alpha_{m+n}\leqq c\alpha_m\alpha_n$ then $x*y$ is contained in A and $||x*y||\leqq c||x||\ ||y||$, since

$$x*y=\sum_{n=-\infty}^{\infty}(\sum_{k=-\infty}^{\infty}a_k b_{n-k})X^n$$

and

$$||x*y||=\sum_{n=-\infty}^{\infty}|\sum_{k=-\infty}^{\infty}a_k b_{n-k}|^p\alpha_n=\sum_{m+n=-\infty}^{\infty}|\sum_{k=-\infty}^{\infty}a_k b_{m+n-k}|^p\alpha_{m+n}$$

$$\leqq c \sum_{m+n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |a_k b_{m+n-k}|^p \alpha_m \alpha_n$$

$$\leqq c\left(\sum_{n=-\infty}^{\infty} |a_n|^p \alpha_n\right)\left(\sum_{m=-\infty}^{\infty} |b_m|^p \alpha_m\right) = c\|x\|\,\|y\|\,.$$

Moreover, putting for $x=\sum_{-\infty}^{\infty} a_n X^n$, $\|x\|' = \sup_m \frac{\sum_{n=-\infty}^{\infty} \alpha_{m+n}|a_n|^p}{\alpha_m}$, we obtain another equivalent p-norm to the original p-norm defined already, satisfying the multiplicative inequality $\|x*y\|' \leqq \|x\|'\|y\|'$ and $\|1\|'=1$ for the identity 1 of A. Thus A becomes a p-normed algebra with the identity under the p-norm $\|x\|'$.

4°) *For an element* $x=\sum_{-\infty}^{\infty} a_n X^n$ *of the above-mentioned p-normed algebra A, it is necessary and sufficient to be invertible in A is that the function* $\Phi(r,t)=\sum_{n=-\infty}^{\infty} a_n r^n e^{int}$ *vanishes at no point of* $r_1 \leqq r \leqq r_2$ *and* $0 \leqq t \leqq 2\pi$, *where* $r_1 = (\lim_{n\to-\infty} \alpha_n^{\frac{1}{n}})^{\frac{1}{p}}$, $r_2 = (\lim_{n\to+\infty} \alpha_n^{\frac{1}{n}})^{\frac{1}{p}}$.

PROOF. Put $x_0 = X$, then $x_0 \in A$. If φ is any multiplicative linear functional on A and $\varphi(x_0)=re^{it}$ $(0\leqq t\leqq 2\pi)$, then $\varphi(x_0{}^n)=r^n e^{int}$. As the series $\sum_{n=-\infty}^{\infty} a_n x_0{}^n$ converges in A and φ is continuous, we have $\varphi(\sum_{n=-\infty}^{\infty} a_n x_0{}^n) = \sum_{n=-\infty}^{\infty} a_n r^n e^{int}$, and consequently $\varphi(x)=\Phi(r,t)$. Hence, by theorem 2, we have only to find r_1 and r_2. Now since, by virtue of proposition 2, $\|x_0{}^n\|' = \sup_m \frac{\alpha_{m+n}}{\alpha_m}$ we have $|r^n e^{int}|^p \leqq \sup_m \frac{\alpha_{m+n}}{\alpha_m}$. And consequently $r^p \leqq \left(\sup_m \frac{\alpha_{m+n}}{\alpha_m}\right)^{\frac{1}{n}}$, from which we obtained $r^p \leqq \lim_{n\to-\infty}\left(\sup_m \frac{\alpha_{m+n}}{\alpha_m}\right) = r_2{}^p$ and $r^p \geqq \lim_{n\to-\infty}\left(\sup_m \frac{\alpha_{m+n}}{\alpha_m}\right)^{\frac{1}{n}} = r_1{}^p$.

On the other hand as the inequality $\frac{\alpha_n}{\alpha_0} \leqq \sup_m \frac{\alpha_{m+n}}{\alpha_m} \leqq c\alpha_n$ holds for $n=0, \pm1, \pm2, \cdots$, we have $r_1 = (\lim_{n\to-\infty} \alpha_n^{\frac{1}{n}})^{\frac{1}{p}}$ and $r_2 = (\lim_{n\to+\infty} \alpha_n^{\frac{1}{n}})^{\frac{1}{p}}$.

We have proved $\varphi(x) \neq 0$ for each $\varphi \in \mathfrak{M}_A$ which, by theorem 2, is to be proved.

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