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Schroedinger Equations Soluble in Terms of Hypergeometric Functions

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As a sequel to the foregoing paper,[1] types of the Schroedinger equation soluble in terms of hypergeometric functions are determined, with accompanying potentials and eigenvalues.

§1. The Hypergeometric equations

Our aim in this paper is to determine types of Schroedinger equation

$$\phi''(x)+(E-V(x))\phi(x)=0 \tag{1}$$

that can be reduced to the hypergeometric equation

$$t(1-t)w''(t)+\{c-(a+b+1)t\}w'(t)-abw(t)=0 \tag{2}$$

through the change of variables

$$w(t)=s(x)\phi(x) \quad \text{and} \quad t=t(x) \tag{3}$$

under the condition that the potential $V(x)$ depends on only the variable x while the eigenvalue E depends on only a parameter a, or, in other words, the term $\phi''(x)/\phi(x)$ splits into the sum of a function of x and a function of the parameter a, b and c being either dependent on a or numerical constants. Substitution of (3) in (2) leads to

$$\frac{\phi''}{\phi}+\left[2\frac{s'}{s}-\frac{t''}{t'}+\frac{c-(a+b+1)t}{t(1-t)}t'\right]\frac{\phi'}{\phi}$$
$$+\frac{s''}{s}-\frac{t''}{t'}\frac{s'}{s}+\frac{c-(a+b+1)t}{t(1-t)}\frac{s'}{s}t'-\frac{abt'^2}{t(1-t)}=0\,. \tag{4}$$

Comparison of (4) with (1) leads to the vanishing of the coefficient of ϕ'/ϕ, or

$$2\frac{s'}{s}=\frac{t''}{t'}-\frac{c-(a+b+1)t}{t(1-t)}t'$$
$$=\frac{t''}{t'}-\left(\frac{c}{t}+\frac{c-a-b-1}{1-t}\right)t'$$

hence, to the following expression of s in terms of t

$$s=\text{const.}\ (1-t)^{(c-a-b-1)/2}t^{-c/2}\sqrt{t'} \tag{5}$$

where the abbreviations $t'=dt/dx$, $s'=ds/dx$, $\psi'=d\psi/dx$, etc. are tacitly understood.

Elimination of s from (4) by means of (5) gives

$$U\equiv-\psi''/\psi=\left(\frac{s'}{s}\right)'-\left(\frac{s'}{s}\right)^2-\frac{abt'^2}{t(1-t)}$$

$$=\left(\frac{t''}{2t'}\right)'-\left(\frac{t''}{2t'}\right)^2+\frac{t'^2}{4t^2(1-t)^2}$$

$$\{[1-(a-b)^2]t^2+2[c(a+b-1)-2ab]t+2c-c^2\}\ . \tag{6}$$

§ 2. The determination of U

For the present U may be regarded as the sum of four terms, the first term being a function of x, each of three other terms being a product of a function of x and a function of a. In short U may be represented as

$$U=T(x)+A(a)X(x)+B(a)Y(x)+C(a)Z(x)\ . \tag{7}$$

The condition set forth in § 1 that U splits into the sum of a function of x and a function of a requires that

$$A'(a)X'(x)+B'(a)Y'(x)+C'(a)Z'(x)=0 \tag{8}$$

for any values of a and x.

Three cases may be distinguished.

Case 1. $X'(x)$, $Y'(x)$ and $Z'(x)$ are linearly independent. It follows then that

$$A'(a)=B'(a)=C'(a)=0\ ,$$

hence that parameters are numerical constants. This case must be excluded.

Case 2. Only one of X', Y' and Z' is independent, for example, X'. It follows then that

$$Y'=\rho X',\ Z'=\sigma X',\ \rho,\ \sigma \text{ being numerical constants.}$$

The above assumption gives two independent differential equations to be satisfied by a single function $t(x)$, so that the function $t(x)$ is reduced to a numerical constant. This case must be excluded also.

Case 3. Two of X', Y' and Z', for example, Y' and Z', are linearly independent. This case leads to a relation between X', Y' and Z', for example,

$$X'+\rho Y'+\sigma Z'=0\ . \tag{9}$$

Elimination of X' from (8) by means of (9) leads to the relation

$$(B'-\rho A')Y'+(C'-\sigma A')Z'=0\,.$$

Since Y' and Z' are assumed to be linearly independent, it follows that

$$\left.\begin{aligned} B'&=\rho A' \quad \text{or} \quad B=\rho A+\rho_1 \\ C'&=\sigma A' \quad \text{or} \quad C=\sigma A+\sigma_1 \end{aligned}\right\} \tag{10}$$

ρ_1, σ_1 being numerical constants.

This gives

$$U=T+A(X+\rho Y+\sigma Z)+\rho_1 Y+\sigma_1 Z\,.$$

Hence $X+\rho Y+\sigma Z$ must be a numerical constant, otherwise A will be a constant. Return to the original expression (6) of U leads to

$$\begin{aligned} \frac{t'^2}{4t^2(1-t)^2}(t^2+\rho t+\sigma)&=\text{constant} \\ &=k \end{aligned} \tag{11}$$

and changes relations (10) into the following two relations

$$2[c(a+b-1)-2ab]=\rho[1-(a-b)^2]+\rho_1 \tag{12}$$

$$2c-c^2=\sigma[1-(a-b)^2]+\sigma_1 \tag{13}$$

which may be rewritten as

$$(1+\rho+\sigma)(a-b)^2-(a+b-c)^2=\rho+\rho_1+\sigma+\sigma_1\equiv\mu \tag{14}$$

$$\sigma(a-b)^2-(c-1)^2=\sigma+\sigma_1-1\equiv\lambda \tag{15}$$

The U takes finally the following form

$$\left.\begin{aligned} U&=E-V(x) \\ E&=-k(a-b)^2 \\ V(x)&=\frac{kt^2(1-t)^2}{t^2+\rho t+\sigma}\left\{\frac{1}{t^2(1-t)^2}+\frac{2}{t^2+\rho t+\sigma}\right. \\ &\qquad\left.+\frac{1-2t}{t(1-t)}\frac{2t+\rho}{t^2+\rho t+\sigma}-\frac{5}{4}\frac{(2t+\rho)^2}{(t^2+\rho t+\sigma)^2}\right\} \\ &\qquad -k\frac{t^2+(\mu-\lambda-1)t+\lambda+1}{t^2+\rho t+\sigma} \end{aligned}\right\} \tag{16}$$

The potential $V(x)$ must be a real function of x, so that all constants k, ρ, σ, ρ_1, σ_1, λ, μ must be real. The eigenvalue E must be real, so $(a-b)^2$ must be also real.

The wave function $\psi(x)$ takes the following form

$$\psi(x)=t^{(c-1)/2}(1-t)^{(a+b-c)/2}(t^2+\rho t+\sigma)^{1/4}w(t) \tag{17}$$

§ 3. Potentials

The correspondence between t and x is to be determined from the equation (11), so that the constant k must be chosen so as to make t'^2 positive for a relevant interval of t.

When the polynomial $t^2+\rho t+\sigma$ is positive for $-\infty<t<\infty$, the constant k must be chosen positive. Then, any of three intervals of t, $-\infty<t<0$, $0<t<1$, $1<t<\infty$, corresponds to the whole range of x, $-\infty<x<\infty$.

If the equation to determine $t(x)$ is set as

$$\frac{(t^2+\rho t+\sigma)^{1/2}t'}{2t(1-t)}=\kappa>0, \qquad \kappa^2=k \tag{18}$$

in the three intervals of t, the integration of (18) gives in the neighbourhoods of $t=0$, $t=1$ and $t=\infty$

$$2\kappa x=\sqrt{\sigma}\cdot\log t+\text{const},$$

$$2\kappa x=-\sqrt{\tau}\cdot\log(1-t)+\text{const},$$

and

$$2\kappa x=-\log(t-1)+\text{const},$$

or

$$t=\text{const}\cdot e^{2\kappa x/\sqrt{\sigma}}$$

$$1-t=\text{const}\cdot e^{-2\kappa x/\sqrt{\tau}}$$

$$1/(t-1)=\text{const}\cdot e^{2\kappa x}$$

where $\tau=1+\rho+\sigma$.

Since the potential $V(x)$ is a rational function of $t(x)$, it is of short range force, excluding a Coulomb potential. This is an important property of the potentials derived in this paper.

When the polynomial $t^2+\rho t+\sigma$ has two real zeros α and β, the correspondence between t and x is a little complicated in the interval where the polynomial vanishes. The interval of t limited by a zero and one of 0, 1 and ∞ can be made to correspond to a half interval of x, $0<x<\infty$ or $-\infty<x<0$. Each of other intervals of t corresponds to the whole interval of x as before.

In the neighbourhood of $t=\alpha$, the integration of (18) yields

$$\kappa x\sim\frac{\beta-\alpha}{3\alpha(1-\alpha)}(\alpha-t)^{3/2}. \tag{19}$$

The dominant term of the $V(x)$ as expressed in (16) in the neighbourhood of $t=\alpha$

$$-\frac{4\kappa^2t^2(1-t)^2}{t^2+\rho t+\sigma}\frac{5}{4}\frac{(2t+\rho)^2}{(t^2+\rho t+\sigma)^2}$$

turns out to be

$$\sim -\frac{5}{36}\frac{1}{x^2}$$

by virtue of the relation (19). In other words, the potential of this paper can have two poles of second order at most.

§ 4. Wave functions

As is well known, the hypergeometric equation has two independent solutions in the neighbourhood of each of three singular points $t=0$, 1 and ∞ having leading terms

$$\begin{array}{llll} 1 & \text{and} & t^{1-c} & \text{at } t=0 \\ 1 & \text{and} & (1-t)^{c-a-b} & \text{at } t=1 \\ t^{-a} & \text{and} & t^{-b} & \text{at } t=\infty\,. \end{array}$$

Therefore the wave function $\psi(x)$ expressed as by (17) has two leading terms

$$\begin{array}{llll} t^{(c-1)/2} & \text{and} & t^{(1-c)/2} & \text{at } t=0 \\ (1-t)^{(a+b-c)/2} & \text{and} & (1-t)^{(c-a-b)/2} & \text{at } t=1 \\ t^{(a-b)/2} & \text{and} & t^{(b-a)/2} & \text{at } t=\infty\,. \end{array}$$

The physical requirement that the wave function remains finite in the relevant interval of x entails that the real part of at least one exponent at both limits of the corresponding interval of t must be zero or negative. Among the exponents there are two relations expressed as (14) and (15). Further, it must be remembered that any solution at one of the singularities can be expressed as a linear combination of two independent solutions at any of other singularities. These things combine to determine three constants a, b and c.

For example, the wave function at $x=0$ may be expressed as

$$\left.\begin{aligned} &\psi=t^{(c-1)/2}(1-t)^{(a+b-c)/2}(t^2+\rho t+\sigma)^{1/4}\{Aw_1(t)+Bw_2(t)\} \\ &w_1(t)=F(a,\, b\,;\, c\,;\, t) \\ &w_2(t)=t^{1-c}F(a-c+1,\, b-c+1\,;\, 2-c\,;\, t) \end{aligned}\right\} \tag{20}$$

where $F(a, b\,; c\,; t)$ denotes the hypergeometric function, A and B being arbitrary constants.

When $\sigma>0$, $\tau>0$ and $4\sigma-\rho^2>0$, the interval of t, $0<t<1$ may correspond to the interval of x, $-\infty<x<\infty$.

Since the exponents are related through the equations (14) and (15), and since $(a-b)^2$ must be real, the exponents must be real or purely imaginary. Since τ and σ are assumed to be positive, $c-1$ is real for $(a-b)^2>\lambda/\sigma$ and purely imaginary for $(a-b)^2<\lambda/\sigma$, while

$a+b-c$ is real for $(a-b)^2 > \mu/\tau$ and purely imaginary for $(a-b)^2 < \mu/\tau$.

For $(a-b)^2 < \mathrm{Min}\,(\mu/\tau, \lambda/\sigma)$, both $c-1$ and $a+b-c$ are purely imaginary, so that the finiteness of the wave function is automatically satisfied, $(a-b)^2$ constituting continuous spectra, except for special cases $c-1=0$ or $a+b-c=0$.

For $(a-b)^2 > \mathrm{Max}\,(\mu/\tau, \lambda/\sigma)$, both $c-1$ and $a+b-c$ are real, so that only one solution is allowable at $t=0$ and $t=1$. Therefore one allowable solution at $t=0$ must be a constant multiple of one allowable solution at $t=1$. This imposes a condition on three constants a, b and c.

For example, when $c-1>0$ and $a+b-c>0$, it follows that $A\neq 0$, $B=0$. A formula concerning the hypergeometric function,

$$F(a, b\,;\, c\,;\, t) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a, b\,;\, a+b-c+1\,;\, 1-t)$$

$$+\frac{\Gamma(a+b-c)\Gamma(c)}{\Gamma(a)\Gamma(b)}(1-t)^{c-a-b} F(c-a, c-b\,;\, c-a-b+1\,;\, 1-t)\,, \tag{21}$$

makes obvious that the second term of the right member must be removed for the finiteness of the wave function, thus entailing that

$$\frac{1}{\Gamma(a)\Gamma(b)} = 0\,. \tag{22}$$

Other relevant conditions to determine a, b and c are

$$c-1 = \sqrt{\sigma(a-b)^2-\lambda} > 0 \tag{23}$$

$$a+b-c = \sqrt{\tau(a-b)^2-\mu} > 0\,. \tag{24}$$

Hence

$$\left.\begin{aligned} &1-2a = -p-\sqrt{\sigma p^2-\lambda}-\sqrt{\tau p^2-\mu} \\ &1-2b = p-\sqrt{\sigma p^2-\lambda}-\sqrt{\tau p^2-\mu} \\ &p = a-b\,. \end{aligned}\right\} \tag{25}$$

The equation (22) is satisfied by $a=0, -1, -2, \cdots$ and $b=0, -1, -2, \cdots$. These values of a or b may happen to give real values of p. However the assumption $\sigma>0$, $\tau>0$ and $4\sigma-\rho^2>0$ involves that $\sqrt{\sigma}+\sqrt{\tau}>1$. Hence the equations (25) cannot be satisfied by infinitely many values of a or b. Consequently $(a-b)^2$ constitutes a finite number of discrete levels. This tallies with the property mentioned earlier that the potential in this paper is of short range force. It is to be added that p satisfies an equation of fourth degree for given a or b.

For $\mathrm{Min}\,(\mu/\tau, \lambda/\sigma) < (a-b)^2 < \mathrm{Min}\,(\mu/\tau, \lambda/\sigma)$, one of $c-1$ and $a+b-c$ is purely imaginary, ensuring the finiteness of the wave function at one of the singularities $t=0$ and $t=1$. Consequently $(a-b)^2$ constitutes continuous spectra.

§ 5. A special case $\sigma=\tau=0$

A special case $\sigma=\tau=0$ gives the potential[2)]

$$V=k\left\{\frac{\lambda+{}^1/_4}{t}+\frac{\mu+{}^1/_4}{1-t}\right\}$$

$$=k\left\{\frac{{}^1/_4-(c-1)^2}{t}+\frac{{}^1/_4-(a+b-c)^2}{1-t}\right\} \tag{26}$$

while the function $t(x)$ is to be determined by

$$\frac{t'^2}{4t(1-t)}=k\,. \tag{27}$$

In the interval of t, $0<t<1$ the constant k may be replaced by κ^2. Equation (27) leads then to $t=\sin^2 \kappa x$ and

$$V=\kappa^2\left\{\frac{{}^1/_4-(c-1)^2}{\sin^2 x}+\frac{{}^1/_4-(a+b-c)^2}{\cos^2 x}\right\}. \tag{28}$$

Only when both $c-1$ and $a+b-c$ are real, there may appear finite levels.

In the interval of t, $1<t<\infty$, replacement of k by $-\kappa^2$ in (27) leads to $t=\mathrm{Cos}^2 \kappa x$ and

$$V=\kappa^2\left\{-\frac{{}^1/_4-(c-1)^2}{\mathrm{Cos}^2 \kappa x}+\frac{{}^1/_4-(a+b-c)^2}{\mathrm{Sin}^2 \kappa x}\right\}. \tag{29}$$

Only when both $a+b-c$ and $a-b$ are real, there may appear finite levels.

In the interval of t, $-\infty<t<0$, replacement of k by $-\kappa^2$ in (27) leads to $t=-\mathrm{Sin}^2 \kappa x$ and

$$V=\kappa^2\left\{\frac{{}^1/_4-(c-1)^2}{\mathrm{Sin}^2 \kappa x}+\frac{{}^1/_4-(a+b-c)^2}{\mathrm{Cos}^2 \kappa x}\right\}. \tag{30}$$

Only when both $c-1$ and $a-b$ are real, there may appear finite discrete levels.

References

1) N. Kubota, M. Yamaguchi and G. Iwata: Natural Science Report, Ochanomizu University, 20 (1969), 31.
2) E. Teller and G. Poeschel: Z. Phys. 83 (1933), 143.