On Projective Killing Tensor

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§ 0. **Introduction.** Let M^n be an n dimensional Riemannian space with positive definite metric g_{ab} with respect to local coordinate system $\{x^a\}$[1)] and denote the operator of the covariant derivation by ∇_c.

A vector field v^c is called an infinitesimal isometry or a Killing vector if it satisfies

$$\mathcal{L}(v)g_{ab} = \nabla_a v_b + \nabla_b v_a = 0\,, \tag{0.1}$$

where $\mathcal{L}(v)$ means the operator of Lie derivation with respect to v^c, and $v_b = g_{bc}v^c$. In 1948, S. Bochner [1][2)] introduced the notion of Killing tensor of degree p (>1) as a generalization of Killing vector. It is a skew symmetric tensor $v_{a_1\cdots a_p}$ satisfying

$$\nabla_b v_{a_1\cdots a_p} + \nabla_{a_1} v_{ba_2\cdots a_p} = 0\,. \tag{0.2}$$

Its first non-parallel example of degree $p>1$ was found in 1955 by T. Fukami and S. Ishihara [2] for $p=2$ on the 6 dimensional sphere S^6. As a generalization of S^6 the present author [7] introduced and studied in 1959 an almost Hermitian space whose fundamental form is a Killing tensor of degree 2.[3)] The Killing tensor on S^6 was the only example of non-parallel Killing tensor of degree p (>1) known until 1967 when Y. Ogawa [5] found them of degree odd in a Sasakian space in a chance studying C-harmonic forms. These works called our attentions to Killing tensor, and we had recent works which contain among others a new characterization of space of constant curvature, i. e.,

THEOREM 0.1. *A necessary and sufficient condition for a Riemannian space to be a space of constant curvature is that for any point* m *and any skew symmetric constants* $C_{a_1\cdots a_p}$ *and* $C_{a_1\cdots a_{p+1}}$ *there exists locally a Killing tensor* $v_{a_1\cdots a_p}$ *satisfying* $v_{a_1\cdots a_p}(m) = C_{a_1\cdots a_p}$ *and* $(\nabla_{a_1} v_{a_2\cdots a_{p+1}})(m) = C_{a_1\cdots a_{p+1}}$, ([9], [10]).

Killing tensor in the Euclidean space and on the sphere of con-

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1) Indices $a, b, c, \cdots$ run over 1 to n.

2) The number in brackets refers to the paper in Bibliography.

3) We shall identify a skew symmetric tensor $v_{a_1\cdots a_p}$ with the differential form $v=(1/p!)\ v_{a_1\cdots a_p}dx^{a_1}\wedge\cdots\wedge dx^{a_p}$.

stant curvature have been determined completely.

The Lie derivative of Christoffel symbols $\left\{ {a \atop cb} \right\}$ with respect to any vector field v^c satisfies the identity

$$\mathcal{L}(v)\left\{ {a \atop cb} \right\} = \frac{1}{2} g^{ae}\{\nabla_c \mathcal{L}(v) g_{be} + \nabla_b \mathcal{L}(v) g_{ce} - \nabla_e \mathcal{L}(v) g_{cb}\}$$

$$= \nabla_c \nabla_b v^a + R_{ecb}{}^a v^e ,$$

and a vector field v^c satisfying

$$\mathcal{L}(v)\left\{ {a \atop cb} \right\} = 0 \tag{0.3}$$

is called an infinitesimal affine transformation or an affine Killing vector, where $R_{ecb}{}^a$ is the Riemannian curvature tensor.

We remark that (0.3) is equivalent to

$$\nabla_c \mathcal{L}(v) g_{be} + \nabla_b \mathcal{L}(v) g_{ce} - \nabla_e \mathcal{L}(v) g_{cb} = 0 . \tag{0.4}$$

A Killing vector is clearly an affine Killing vector, and the converse is true for the compact case. An affine Killing tensor is defined by an analogous equation to (0.4) taking account of the resemblance of (0.1) and (0.2).[4)]

Corresponding to the case of Killing vector we know

THEOREM 0.2. *A Killing tensor is an affine Killing tensor, and the converse is true for a compact Riemannian space,* ([9], [10]).

It was in 1952 that K. Yano [13] introduced first conformal Killing tensor as a generalization of conformal Killing vector. A conformal Killing vector or an infinitesimal conformal transformation is a vector field v^c which satisfies

$$\mathcal{L}(v) g_{ab} = \nabla_a v_b + \nabla_b v_a = 2\rho g_{ab} ,$$

where ρ is a scalar function. He named a skew symmetric tensor $v_{a_1 \cdots a_p}$ a conformal Killing tensor if it satisfies

$$\nabla_b v_{a_1 \cdots a_p} + \nabla_{a_1} v_{b a_2 \cdots a_p} = 2 g_{b a_1} \rho_{a_2 \cdots a_p} , \tag{0.5}$$

where $\rho_{a_2 \cdots a_p}$ is a skew symmetric tensor. Unfortunately this definition does not introduce new notion at all, because it is proved that $\rho_{a_2 \cdots a_p}$ in (0.5) vanishes identically and (0.5) reduces to (0.2). This can be seen by comparing two equations which are obtained from (0.5) by transvection with g^{ba_1} and $g^{a_1 a_2}$.

The author [11] and T. Kashiwada [3] changed its definition and they call a skew symmetric tensor $v_{a_1 \cdots a_p}$ a conformal Killing tensor if there exists a skew symmetric tensor $\rho_{a_1 \cdots a_{p-1}}$ called the associated ten-

4) S. Tachibana, [9]. S. Tachibana and T. Kashiwada, [10]. Also see § 3 in this paper.

sor such that

$$\nabla_b v_{a_1\cdots a_p}+\nabla_{a_1}v_{ba_2\cdots a_p}$$
$$=2g_{ba_1}\rho_{a_2\cdots a_p}-\sum_{i=2}^{p}(-1)^i(g_{ba_i}\rho_{a_1\cdots\hat{a}_i\cdots a_p}+g_{a_1a_i}\rho_{ba_2\cdots\hat{a}_i\cdots a_p}),$$

where $\hat{a}_i$ means that a_i is deleted. This definition seems natural because of the following theorems.

THEOREM 0.3. *For any point m of an n dimensional Riemannian space and any skew symmetric constants $C_{a_1\cdots a_p}$, if there exists locally a conformal Killing tensor $v_{a_1\cdots a_p}$ of degree p $(1<p<n-1)$ satisfying $v_{a_1\cdots a_p}(m)=C_{a_1\cdots a_p}$, then the space is conformally flat,* ([3], [11]).

THEOREM 0.4. *In a space of constant curvature with $R\neq 0$, a conformal Killing tensor $v_{a_1\cdots a_p}$ of degree p $(<n)$ is uniquely decomposed in the form*

$$v_{a_1\cdots a_p}=w_{a_1\cdots a_p}+q_{a_1\cdots a_p}, \tag{0.6}$$

where $w_{a_1\cdots a_p}$ is a Killing tensor and $q_{a_1\cdots a_p}$ is a closed conformal Killing tensor. In this case, $q_{a_1\cdots a_p}$ is of the form

$$q_{a_1\cdots a_p}=-\frac{n(n-1)}{R}\nabla_{a_1}\rho_{a_2\cdots a_p}, \tag{0.7}$$

where $\rho_{a_2\cdots a_p}$ is the associated tensor of $v_{a_1\cdots a_p}$ and R denotes the scalar curvature. Conversely, if $w_{a_1\cdots a_p}$ and $\rho_{a_2\cdots a_p}$ are Killing tensors then $v_{a_1\cdots a_p}$ given by (0.6) *and* (0.7) *is a conformal Killing tensor,* ([3], [11]).[5]

The following theorem gives some meaning to our conformal Killing tensor too.

THEOREM 0.5. *Let M^n be a compact Riemannian space which is locally isometric to the direct product of p $(\leqq n/2)$ dimensional Riemannian space and an $n-p$ dimenisonal one. Then M^n can not admit a conformal Killing tensor of degree r satisfying $3(r-1)<2p$ which is not a Killing tensor* ([6]).[6]

A vector field v^c is called an infinitesimal projective transformation or a projective Killing vector if there exists a scalar function θ called the associated function satisfying

$$\mathcal{L}(v)\left\{{a \atop cb}\right\}=\nabla_c\nabla_b v^a+R_{ecb}{}^a v^e=\theta_c\delta_b^a+\theta_b\delta_c^a, \tag{0.8}$$

where $\theta_a=\nabla_a\theta$.

The purpose of this paper is to introduce projective Killing tensor

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5) This is a generalization of Yano-Nagano's theorem for conformal Killing vector in Einstein spaces, [15]. We remark that in a space of constant curvature the covariant derivative $\nabla_{a_1}v_{a_2\cdots a_p}$ of a Killing tensor $v_{a_2\cdots a_p}$ is a conformal Killing tensor whose associated tensor is $(-R/n(n-1))v_{a_2\cdots a_p}$, [3], [11].

6) This is a generalization of Tachibana's theorem for conformal Killing vector [8]. As to conformal Killing vector in the complete case, see Y. Tashiro [12].

as a generalization of projective Killing vector. Though we do not know whether it is worthy to study such tensor, it would seem partial without it.

§ 1 will be devoted to notations and preliminaries. In § 2 we shall treat of a tensor field on the sphere S^n in the Euclidean space which will be taken as a model of projective Killing tensor. We shall give in § 3 an analogous formula for any skew symmetric tensor v to $\mathcal{L}(v)\left\{ {a \atop cb} \right\}$ and define affine Killing tensor. The definition of projective Killing tensor will be given in § 4 and we shall discuss such tensor in a space of constant curvature in § 5 to get the main result corresponding to Theorem 0.4. We shall give also the corresponding theorem for projective Killing tensor to the following

THEOREM 0.6. *In a compact n dimensional Riemannian space of negative constant curvature, there exists no (conformal) Killing tensor of degree p* ($\leqq n/2$) *other than the zero tensor*, ([3]).

§ 1. **Preliminaries.** Consider a Riemannian space M^n keeping notations in § 0. Let $v_{a_1\cdots a_p}$ be a skew symmetric tensor identified with the differential p-form $v=(1/p!)v_{a_1\cdots a_p}dx^{a_1}\wedge\cdots\wedge dx^{a_p}$. The exterior derivative is

$$dv=\frac{1}{(p+1)!}(dv)_{a_1\cdots a_{p+1}}dx^{a_1}\wedge\cdots\wedge dx^{a_{p+1}},$$

where

$$(dv)_{ca_1\cdots a_p}=\nabla_c v_{a_1\cdots a_p}-\sum_{j=1}^{p}\nabla_{a_j}v_{a_1\cdots c\cdots a_p}.$$

The index c in the last term appears at the j-th position of index, i. e. $v_{a_1\cdots c\cdots a_p}=v_{a_1\cdots a_{j-1}ca_{j+1}\cdots a_p}$ and we shall express this, if necessary, by $v_{a_1\cdots c(j)\cdots a_p}$.

A differential p-form v is called to be closed if $dv=0$, and to be derived if there is a $(p-1)$-form u such that $v=du$. The exterior co-derivative δv is

$$\delta v=\frac{1}{(p-1)!}(\delta v)_{a_1\cdots a_{p-1}}dx^{a_1}\wedge\cdots\wedge dx^{a_{p-1}}$$

with

$$(\delta v)_{a_1\cdots a_{p-1}}=-\nabla^b v_{ba_1\cdots a_{p-1}},$$

where $\nabla^b=g^{bc}\nabla_c$. v is called to be coclosed when it satisfies $\delta v=0$.

The Ricci identity for any tensor, say $T_{ba}{}^f$, is

$$\nabla_e\nabla_c T_{ba}{}^f-\nabla_c\nabla_e T_{ba}{}^f=-R_{ecb}{}^r T_{ra}{}^f-R_{eca}{}^r T_{br}{}^f+R_{ecr}{}^f T_{ba}{}^r.$$

The following identity is proved easily for any skew symmetric tensor $v_{a_1\cdots a_p}$:

$$\nabla^c\nabla^b v_{cba_3\cdots a_p}=0 \tag{1.1}$$

by taking account of Ricci identity.

The Ricci tensor and the scalar curvature are defined by

$$R_{bc}=R_{abc}{}^{a}, \qquad R=g^{bc}R_{bc},$$

and put

$$k=\frac{R}{n(n-1)}.$$

A Riemannian space M^n is called an Einstein space or a space of constant curvature, if

(1.2) $$R_{bc}=(n-1)kg_{bc}, \qquad \text{or}$$

(1.3) $$R_{abcd}=k(g_{bc}g_{ad}-g_{ac}g_{bd})$$

is satisfied respectively. As well known, k is constant for these spaces when $n>2$.

Let M^n be a Riemannian space immersed in a Riemannian space M^{n+1}. With respect to local coordinates $\{y^a\}$ in M^n and $\{x^\lambda\}$[7] in M^{n+1}, M^n is represented locally by $x^\lambda=x^\lambda(y)$. If we put

$$B_a{}^\lambda=\frac{\partial x^\lambda}{\partial y^a},$$

the Riemannian metric g_{bc} of M^n is related with the one $G_{\mu\nu}$ of M^{n+1} by $g_{bc}=G_{\mu\nu}B_b{}^\mu B_c{}^\nu$. The second fundamental tensor is defined by

$$H_{ba}{}^\lambda=\nabla_b B_a{}^\lambda=\frac{\partial B_a{}^\lambda}{\partial y^b}-\begin{Bmatrix}c\\ ba\end{Bmatrix}B_c{}^\lambda-B_b{}^\mu\begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix}B_a{}^\nu,$$

where $\begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix}$ are the Christoffel's symbols formed from $G_{\mu\nu}$ and ∇_b means the Van der Waerden-Bortolotti differential operator.

Let N^λ be the unit normal vector field defined locally on M^n, then there exists a symmetric tensor H_{ba} on M^n such that

$$H_{ba}{}^\lambda=H_{ba}N^\lambda,$$

and we have

$$\nabla_b N^\lambda=\frac{\partial N^\lambda}{\partial y^b}+B_b{}^\mu\begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix}N^\nu=-H_b{}^cB_c{}^\lambda,$$

where $H_b{}^c=H_{ba}g^{ac}$.

§2. A tensor field on S_+^n. Let E^{n+1} be an $n+1$ dimensional Euclidean space with orthogonal coordinate $\{x^\lambda\}$. Consider the upper hemi-sphere S_+^n:

$$(x^1)^2+\cdots+(x^{n+1})^2=1, \qquad x^{n+1}>0$$

7) Indices $\lambda, \mu, \nu, \cdots$ run over 1 to $n+1$.

as a Riemannian space imbedded in E^{n+1}. We shall denote by N^λ the unit (inward) normal vector field defined globally on S_+^n and take $\{x^a\}$ as a local coordinate there. If we put

$$f = x^{n+1}$$

the following equations hold good:

$$N^\lambda = -x^\lambda, \qquad B_a^{\ \lambda} = \begin{cases} \delta_a^{\ b}, & \text{if } \lambda = b, \\ -\dfrac{x^a}{f}, & \text{if } \lambda = n+1, \end{cases}$$

$$g_{ab} = \delta_{ab} + \frac{x^a x^b}{f^2}, \qquad g^{ab} = \delta^{ab} - x^a x^b,$$

$$\left\{ {a \atop bc} \right\} = x^a g_{bc}, \qquad R_{abc}{}^e = g_{bc}\delta_a^{\ e} - g_{ac}\delta_b^{\ e},$$

$$\nabla_b B_a^{\ \lambda} = H_{ab} N^\lambda = g_{ab} N^\lambda, \qquad \nabla_b N^\lambda = -B_c^{\ \lambda}.$$

It is known ([12], [4]) that f is a special concircular scalar field, i. e., it satisfies

$$\nabla_b \nabla_c f = -f g_{bc}.$$

Now denote by E^n the n-plane defined by $x^{n+1} = 1$. If we consider the projection from the origin of E^{n+1}, it is a diffeomorphism between S_+^n and E^n and induces a projective transformation on S_+^n from an affine transformation on E^n. Especially a projective Killing vector field v^a on S_+^n is obtained by the projection from a constant vector field on E^n which is extended naturally to a constant one u^λ $(= u_\lambda)$ over E^{n+1}. They are related by

$$v_a = f u_\lambda B_a^{\ \lambda}.$$

Generalizing this geometrical fact, we shall adopt

$$v_{a_1 \cdots a_p} = f u_{\lambda_1 \cdots \lambda_p} B_{a_1}{}^{\lambda_1} \cdots B_{a_p}{}^{\lambda_p}$$

as the model of projective Killing tensor whose exact definition will be given in § 4, where $u_{\lambda_1 \cdots \lambda_p}$ is a constant skew symmetric tensor over E^{n+1}.

Define a skew symmetric tensor $\theta_{a_1 \cdots a_{p-1}}$ for this $v_{a_1 \cdots a_p}$ by

$$\theta_{a_1 \cdots a_{p-1}} = f u_{\mu \lambda_1 \cdots \lambda_{p-1}} N^\mu B_{a_1}{}^{\lambda_1} \cdots B_{a_{p-1}}{}^{\lambda_{p-1}},$$

then we have

$$\nabla_b \theta_{a_1 \cdots a_{p-1}} = (\nabla_b f) u_{\mu \lambda_1 \cdots \lambda_{p-1}} N^\mu B_{a_1}{}^{\lambda_1} \cdots B_{a_{p-1}}{}^{\lambda_{p-1}} - v_{b a_1 \cdots a_{p-1}}.$$

Simple computations show the validity of the following equations:

$$\nabla_b v_{a_1 \cdots a_{p-1}} = (\nabla_b f) u_{\lambda_1 \cdots \lambda_p} B_{a_1}{}^{\lambda_1} \cdots B_{a_p}{}^{\lambda_p} + \sum_{i=1}^{p} (-1)^{i-1} g_{b a_i} \theta_{a_1 \cdots \hat{a}_i \cdots a_p},$$

(2.1)
$$\nabla_c \nabla_b v_{a_1\cdots a_p} = -g_{bc} v_{a_1\cdots a_p} + \sum_{i=1}^{p} g_{ca_i} v_{a_1\cdots b\cdots a_p}$$
$$+ \sum_{i=1}^{p} (-1)^{i-1} (g_{ba_i} \nabla_c \theta_{a_1\cdots \hat{a}_i\cdots a_p} + g_{ca_i} \nabla_b \theta_{a_1\cdots \hat{a}_i \cdots a_p}) .$$

§ 3. **Affine Killing tensor.** Let $v_{a_1\cdots a_p}$ be a skew symmetric tensor field in a Riemannian space M^n. We shall put

(3.1)
$$A_{ba_1,a_2\cdots a_p} = \nabla_b v_{a_1\cdots a_p} + \nabla_{a_1} v_{ba_2\cdots a_p}$$

which corresponds to (0.1). Next, corresponding to (0.4), we shall consider

(3.2)
$$T_{cb,a_1\cdots a_p} = (1/2)(\nabla_c A_{ba_1,a_2\cdots a_p} + \nabla_b A_{ca_1,a_2\cdots a_p} - \nabla_{a_1} A_{cb,a_2\cdots a_p}) .$$

By virtue of Ricci identity we can get the following equation

(3.2)
$$T_{cb,a_1\cdots a_p} = \nabla_c \nabla_b v_{a_1\cdots a_p} + (1/2) \sum_{i=1}^{p} R_{cba_i}{}^{e} v_{a_1\cdots e\cdots a_p}$$
$$-(1/2)(R_{ba_1c}{}^{e} + R_{ca_1b}{}^{e}) v_{ea_2\cdots a_p}$$
$$-(1/2) \sum_{i=2}^{p} (R_{ba_1a_i}{}^{e} v_{ca_2\cdots e\cdots a_p} + R_{ca_1a_i}{}^{e} v_{ba_2\cdots e\cdots a_p}) .$$

We shall call a skew symmetric tensor $v_{a_1\cdots a_p}$ an affine Killing tensor if its $T_{cb,a_1\cdots a_p}$ vanishes identically. Though this definition differs from ones in [9] and [10], Theorem 0.2 is still true. The proof is as follows.

For a Killing tensor $v_{a_1\cdots a_p}$ its $A_{ba_1,a_2\cdots a_p}$ vanishes and hence $T_{cb,a_1\cdots a_p} = 0$ which shows that $v_{a_1\cdots a_p}$ is an affine Killing tensor. Conversely, let $v_{a_1\cdots a_p}$ be a skew symmetric tensor and consider $A_{ba_1,a_2\cdots a_p}$ given by (3.1). Taking account of Ricci identity we can get

$$\nabla^b A_{ba_1,a_2\cdots a_p} = \nabla^b \nabla_b v_{a_1\cdots a_p} + R_{a_1}{}^{e} v_{ea_2\cdots a_p}$$
$$- \sum_{i=2}^{p} R^{b}{}_{a_1a_i}{}^{e} v_{ba_2\cdots e\cdots a_p} + \nabla_{a_1} \nabla^b v_{ba_2\cdots a_p} ,$$
$$\nabla^b (A_{ba_1,a_2\cdots a_p} v^{a_1\cdots a_p}) = \nabla^b A_{ba_1,a_2\cdots a_p} v^{a_1\cdots a_p} + (1/2) A_{ba_1,a_2\cdots a_p} A^{ba_1,a_2\cdots a_p} .$$

Integrating the last equation we have

THEOREM 3.1. *In a compact orientable Riemannian space M^n, the integral formula*

$$\int_M \{(\nabla^b \nabla_b v_{a_1\cdots a_p} + R_{a_1}{}^{e} v_{ea_2\cdots a_p} - \sum_{i=2}^{p} R^{b}{}_{a_1a_i}{}^{e} v_{ba_2\cdots e\cdots a_p}) v^{a_1\cdots a_p}$$
$$+ (\nabla_{a_1} \nabla^b v_{ba_2\cdots a_p}) v^{a_1\cdots a_p} + (1/2) A_{bc,a_2\cdots a_p} A^{bc,a_2\cdots a_p}\} d\sigma = 0$$

holds good for any skew symmetric tensor field $v_{a_1\cdots a_p}$, where $d\sigma$ means the volume element of M^n and $A_{ba_1,a_2\cdots a_p}$ is given by (3.1).

Assume M^n to be compact orientable and let $v_{a_1\cdots a_p}$ be an affine Killing tensor. Then, taking account of (3.3), we have

(3.4) $$\nabla^b\nabla_b v_{a_1\cdots a_p}+R_{a_1}{}^e v_{ea_2\cdots a_p}-\sum_{i=2}^{p} R^b{}_{a_1a_i}{}^e v_{ba_2\cdots e\cdots a_p}=0\,,$$

(3.5) $$\nabla_c\nabla^b v_{ba_2\cdots a_p}=0\,.$$

Thus $A_{ba_1,a_2\cdots a_p}=0$ follows from Thcorem 3.1.

We remark that in a space of constant curvature the equation (3.3) becomes the following simple form:

(3.6) $$T_{cb,a_1\cdots a_p}=\nabla_c\nabla_b v_{a_1\cdots a_p}+k(g_{cb}v_{a_1\cdots a_p}-\sum_{i=1}^{p} g_{ca_i}v_{a_1\cdots b\cdots a_p})\,.$$

§ 4. **Projective Killing tensor.** We shall define projective Killing tensor in this section. A projective Killing vector is defined by (0.8) whose left hand side corresponds to $T_{cb,a_1\cdots a_p}$. Thus the problem is how to determine for $T_{cb,a_1\cdots a_p}$ the form corresponding to the right hand side of (0.8). Now let take $v_{a_1\cdots a_p}$ on S_+^n in § 2 and calculate (3.6) for it. Substituting (2.1) and $k=1$ in (3.6) we can get

$$T_{cb,a_1\cdots a_p}=\sum_{i=1}^{p}(-1)^{i-1}(g_{ca_i}\nabla_b\theta_{a_1\cdots\hat{a}_i\cdots a_p}+g_{ba_i}\nabla_c\theta_{a_1\cdots\hat{a}_i\cdots a_p})\,.$$

Thus it is natural that we would admit the following definition. A skew symmetric tensor $v_{a_1\cdots a_p}$ will be called a projective Killing tensor of degree p if there exists a skew symmetric tensor $\theta_{a_1\cdots a_{p-1}}$ such that

(4.1) $$\begin{aligned}&\nabla_c\nabla_b v_{a_1\cdots a_p}+(1/2)\sum_{i=1}^{p} R_{cba_i}{}^e v_{a_1\cdots e\cdots a_p}\\&\quad-(1/2)(R_{ba_1c}{}^e+R_{ca_1b}{}^e)v_{ea_2\cdots a_p}\\&\quad-(1/2)\sum_{i=2}^{p}(R_{ba_1a_i}{}^e v_{ca_2\cdots e\cdots a_p}+R_{ca_1a_i}{}^e v_{ba_2\cdots e\cdots a_p})\\&=\sum_{i=1}^{p}(-1)^{i-1}(g_{ca_i}\nabla_b\theta_{a_1\cdots\hat{a}_i\cdots a_p}+g_{ba_i}\nabla_c\theta_{a_1\cdots\hat{a}_i\cdots a_p})\,.\end{aligned}$$

$\theta_{a_1\cdots a_{p-1}}$ is called the associated tensor and $\theta=(1/(p-1)!)\theta_{a_2\cdots a_p}dx^{a_2}\wedge\cdots\wedge dx^{a_p}$ the associated form.

Especially, (4.1) becomes (0.8) when $p=1$. Thus our definition of projective Killing tensor is a generalization of projective Killing vector.

By transvection (4.1) with g^{cb} it follows

(4.2) $$\begin{aligned}&\nabla^r\nabla_r v_{a_1\cdots a_p}+R_{a_1}{}^e v_{ea_2\cdots a_p}-\sum_{i=2}^{p} R^b{}_{a_1a_i}{}^e v_{ba_2\cdots e\cdots a_p}\\&=2\sum_{i=1}^{p}(-1)^{i-1}\nabla_{a_i}\theta_{a_1\cdots\hat{a}_i\cdots a_p}=2(d\theta)_{a_1\cdots a_p}\,.\end{aligned}$$

On the other hand, transvecting (4.1) with g^{ba_1}, we have

$$(4.3)\qquad \nabla_c\nabla^r v_{ra_2\cdots a_p}=(n-p+2)\nabla_c\theta_{a_2\cdots a_p}+\sum_{i=2}^{p}(-1)^{i-1}g_{ca_i}\nabla^r\theta_{ra_2\cdots\hat{a}_i\cdots a_p}$$

i. e.

$$(4.4)\qquad -\nabla_c(\delta v)_{a_2\cdots a_p}=(n-p+2)\nabla_c\theta_{a_2\cdots a_p}+\sum_{i=2}^{p}(-1)^{i}g_{ca_i}(\delta\theta)_{a_2\cdots\hat{a}_i\cdots a_p}.$$

Some simple computations show the following equation to be valid.

$$(4.5)\qquad -(d\delta v)_{ca_2\cdots a_p}=(n-p+2)(d\theta)_{ca_2\cdots a_p}.$$

We proceed to give some applications of these equations.

Let $v_{a_1\cdots a_p}$ be a projective Killing tensor whose associated form θ is closed. Then we have (3.4) from (4.2), and have

$$(\nabla_c\nabla^r v_{ra_2\cdots a_p})v^{ca_2\cdots a_p}=0$$

from (4.3). Thus $A_{ba_1,a_2\cdots a_p}=0$ follows from Theorem 3.1, which means that $v_{a_1\cdots a_p}$ is a Killing tensor. Thus we have

THEOREM 4.1. *In a compact Riemannian space, a projective Killing tensor is a Killing tensor if the associated form is closed.*

Let v be a projective Killing tensor which is coclosed. Then the associated form is closed by (4.5). Hence

THEOREM 4.2. *In a compact Riemannian space, a coclosed projective Killing tensor is a Killing tensor.*

When $p=1$ these theorems reduce to well known theorems on projective Killing vectors.

As is well known, a projective Killing vector which is a conformal Killing vector at a time is affine. We shall generalize this to the case of tensor. Let $v_{a_1\cdots a_p}$ be a conformal Killing tensor and $\rho=(1/(p-1)!)\rho_{a_2\cdots a_p}dx^{a_2}\wedge\cdots\wedge dx^{a_p}$ its associated form. By definition, $v_{a_1\cdots a_p}$ is a skew symmetric tensor satisfying

$$\nabla_b v_{a_1\cdots a_p}+\nabla_{a_1}v_{ba_2\cdots a_p}$$
$$=2g_{ba_1}\rho_{a_2\cdots a_p}-\sum_{i=2}^{p}(-1)^i(g_{ba_i}\rho_{a_1\cdots\hat{a}_i\cdots a_p}+g_{a_1a_i}\rho_{ba_2\cdots\hat{a}_i\cdots a_p}).$$

It is known that the following equations are valid,[8]

$$(4.6)\qquad \nabla^r v_{ra_2\cdots a_p}=(n-p+1)\rho_{a_2\cdots a_p},\qquad -\delta v=(n-p+1)\rho,$$

$$(4.7)\qquad \nabla^r\nabla_r v_{a_1\cdots a_p}+R_{a_1}{}^e v_{ea_2\cdots a_p}-\sum_{i=2}^{p}R^b{}_{a_1a_i}{}^e v_{ba_2\cdots e\cdots a_p}$$
$$=-(n-p)\nabla_{a_1}\rho_{a_2\cdots a_p}+(d\rho)_{a_1\cdots a_p}.$$

Now suppose that $v_{a_1\cdots a_p}$ is a projective Killing tensor at a time. Then, comparing (4.7) with (4.2), we have

8) T. Kashiwada, [3], S. Sato, [6].

$$2(d\theta)_{a_1\cdots a_p} = -(n-2)\nabla_{a_1}\rho_{a_2\cdots a_p} + (d\rho)_{a_1\cdots a_p},$$

from which it follows that $\nabla_{a_1}\rho_{a_2\cdots a_p}$ is skew symmetric. Substituting $(d\rho)_{a_1\cdots a_p} = p\nabla_{a_1}\rho_{a_2\cdots a_p}$ into the last equation we get

$$2pd\theta = (2p-n)d\rho, \tag{4.8}$$

On the other hand, $-d\delta v = (n-p+2)d\theta$ follows from (4.5) and taking account of (4.6) we have

$$(n-p+2)d\theta = (n-p+1)d\rho. \tag{4.9}$$

Eliminating $d\rho$ from (4.8) and (4.9) we get $(n+2)(n-p)d\theta = 0$ and hence θ is closed for $p<n$. Consequently by virtue of Theorem 4.1 we obtain

THEOREM 4.3. *In a compact n dimensional Riemannian space, if a projective Killing tensor of degree p $(<n)$ is a conformal Killing tensor at a time, then it is a Killing tensor.*

§ 5. Projective Killing tensor in a space of constant curvature.

In this section we shall restrict our attention to a space M^n of constant curvature and assume $n>2$.

A projective Killing tensor $v_{a_1\cdots a_p}$ is a skew symmetric tensor satisfying

$$\nabla_c\nabla_b v_{a_1\cdots a_p} = k(-g_{cb}v_{a_1\cdots a_p} + \sum_{i=1}^{p} g_{ca_i}v_{a_1\cdots b\cdots a_p}) + \sum_{i=1}^{p}(-1)^{i-1}(g_{ca_i}\nabla_b\theta_{a_1\cdots\hat{a}_i\cdots a_p} + g_{ba_i}\nabla_c\theta_{a_1\cdots\hat{a}_i\cdots a_p}), \tag{5.1}$$

where $\theta_{a_1\cdots a_{p-1}}$ is skew symmetric.

Operating ∇^{a_1} to (5.1) and changing a_1 to e, we have

$$\begin{aligned}\nabla^e\nabla_c\nabla_b v_{ea_2\cdots a_p} = {} & k(-g_{cb}\nabla^e v_{ea_2\cdots a_p} + \nabla_c v_{ba_2\cdots a_p} + \sum_{i=2}^{p} g_{ca_i}\nabla^e v_{ea_2\cdots b\cdots a_p}) \\ & + \sum_{i=2}^{p}(-1)^{i-1}(g_{ca_i}\nabla^e\nabla_b\theta_{ea_2\cdots\hat{a}_i\cdots a_p} + g_{ba_i}\nabla^e\nabla_c\theta_{ea_2\cdots\hat{a}_i\cdots a_p}) \\ & + \nabla_c\nabla_b\theta_{a_2\cdots a_p} + \nabla_b\nabla_c\theta_{a_2\cdots a_p}.\end{aligned} \tag{5.2}$$

First we shall calculate the left hand side of (5.2) which is the sum of the following a_1 to a_4.

$$a_1 = \nabla_c\nabla_e\nabla_b v^e{}_{a_2\cdots a_p}, \qquad a_2 = -R_{ecb}{}^f\nabla_f v^e{}_{a_2\cdots a_p},$$

$$a_3 = R_{ecf}{}^e\nabla_b v^f{}_{a_2\cdots a_p}, \qquad a_4 = -\sum_{i=2}^{p} R_{eca_i}{}^f\nabla_b v^e{}_{a_2\cdots f\cdots a_p}.$$

Taking account of (1.2), (1.3) and (4.3) we have

$$a_1 = \nabla_c(\nabla_b\nabla_e v^e{}_{a_2\cdots a_p} + R_{ebf}{}^e v^f{}_{a_2\cdots a_p} - \sum_{i=2}^{p} R_{eba_i}{}^f v^e{}_{a_2\cdots f\cdots a_p})$$

$$= (n-p+2)\nabla_c\nabla_b\theta_{a_2\cdots a_p} + \sum_{i=2}^{p}(-1)^{i-1}g_{ba_i}\nabla_c\nabla^r\theta_{ra_2\cdots\hat{a}_i\cdots a_p}$$

$$+ (n-p)k\nabla_c v_{ba_2\cdots a_p},$$

$$a_2 = k(-g_{cb}\nabla^r v_{ra_2\cdots a_p} + \nabla_c v_{ba_2\cdots a_p}),$$

$$a_3 = (n-1)k\nabla_b v_{ca_2\cdots a_p},$$

$$a_4 = -(p-1)k\nabla_b v_{ca_2\cdots a_p}.$$

Substituting these values to the left hand side of (5.2) we have

$$(5.3)\qquad (n-p)k(\nabla_c v_{ba_2\cdots a_p} + \nabla_b v_{ca_2\cdots a_p}) = k\sum_{i=2}^{p} g_{ca_i}\nabla^r v_{ra_2\cdots b\cdots a_p}$$

$$-(n-p+1)\nabla_c\nabla_b\theta_{a_2\cdots a_p} + \nabla_b\nabla_c\theta_{a_2\cdots a_p}$$

$$+\sum_{i=2}^{p}(-1)^{i-1}g_{ba_i}(\nabla_r\nabla_c\theta^r{}_{a_2\cdots\hat{a}_i\cdots a_p} - \nabla_c\nabla_r\theta^r{}_{a_2\cdots\hat{a}_i\cdots a_p})$$

$$+\sum_{i=2}^{p}(-1)^{i-1}g_{ca_i}\nabla^r\nabla_b\theta_{ra_2\cdots\hat{a}_i\cdots a_p}.$$

On the other hand, it follows by Ricci identity that

$$\nabla_b\nabla_c\theta_{a_2\cdots a_p} = \nabla_c\nabla_b\theta_{a_2\cdots a_p} - k\sum_{i=2}^{p}(g_{ca_i}\theta_{a_2\cdots b\cdots a_p} - g_{ba_i}\theta_{a_2\cdots c\cdots a_p}),$$

$$\nabla^r\nabla_c\theta_{ra_2\cdots\hat{a}_i\cdots a_p} = \nabla_c\nabla^r\theta_{ra_2\cdots\hat{a}_i\cdots a_p} + (n-p+1)k\theta_{ca_2\cdots\hat{a}_i\cdots a_p}.$$

Substituting these two equations into (5.3) we can get

$$(5.4)\qquad (n-p)k(\nabla_c v_{ba_2\cdots a_p} + \nabla_b v_{ca_2\cdots a_p}) = -(n-p)\nabla_c\nabla_b\theta_{a_2\cdots a_p}$$

$$+\sum_{i=2}^{p}(-1)^{i-1}(n-p)kg_{ba_i}\theta_{ca_2\cdots\hat{a}_i\cdots a_p}$$

$$+\sum_{i=2}^{p}g_{ca_i}\{k\nabla^r v_{ra_2\cdots b\cdots a_p} + (-1)^{i-1}\nabla_b\nabla^r\theta_{ra_2\cdots\hat{a}_i\cdots a_p}$$

$$+(-1)^{i-1}(n-p+2)k\theta_{ba_2\cdots\hat{a}_i\cdots a_p}\}.$$

Transvecting (5.4) with g^{ba_2} we have

$$k\nabla^r v_{rca_3\cdots a_p} = \nabla_c\nabla^r\theta_{ra_3\cdots a_p} + (n-p+2)k\theta_{ca_3\cdots a_p},$$

and substituting this equation into (5.4) we have for $n>p$

$$(5.5)\qquad k(\nabla_c v_{ba_2\cdots a_p} + \nabla_b v_{ca_2\cdots a_p}) = -\nabla_c\nabla_b\theta_{a_2\cdots a_p} - k\sum_{i=2}^{p}g_{ba_i}\theta_{a_2\cdots c\cdots a_p}.$$

When $p=1$, (5.5) reduces to

$$k(\nabla_c v_b + \nabla_b v_c) = -\nabla_c\nabla_b\theta = (-1/2)(\nabla_c\nabla_b\theta + \nabla_b\nabla_c\theta),$$

where θ is a scalar function. If $k\neq 0$, the vector w^b defined by

$$w_b = v_b + (1/2k)\nabla_b\theta$$

is a Killing vector. Thus a projective Killing vector v^b is represented in the form

$$v_b = w_b - (1/2k)\nabla_b\theta ,$$

where w^b is a Killing vector and $\nabla_b\theta$ is a derived projective Killing vector. This fact has been got by K. Yano and T. Nagano [15] for an Einstein space. We shall generalize this to the case of tensor.

From (5.5) we have

$$\nabla_c\nabla_b\theta_{a_2\cdots a_p} = B_{cb,a_2\cdots a_p} - k\sum_{i=2}^{p} g_{ba_i}\theta_{a_2\cdots c\cdots a_p} \tag{5.6}$$

for a projective Killing tensor $v_{a_1\cdots a_p}$, where

$$B_{cb,a_2\cdots a_p} = -k(\nabla_c v_{ba_2\cdots a_p} + \nabla_b v_{ca_2\cdots a_p}) .$$

Now we put $u_{a_1\cdots a_p} = (d\theta)_{a_1\cdots a_p}$ which is by definition

$$u_{ba_2\cdots a_p} = \nabla_b\theta_{a_2\cdots a_p} - \sum_{j=2}^{p}\nabla_{a_j}\theta_{a_2\cdots b\cdots a_p} .$$

Thus it holds that

$$\nabla_c u_{ba_2\cdots a_p} = \nabla_c\nabla_b\theta_{a_2\cdots a_p} - \sum_{j=2}^{p}\nabla_c\nabla_{a_j}\theta_{a_2\cdots b\cdots a_p} . \tag{5.7}$$

On the other hand we have from (5.6)

$$\nabla_c\nabla_{a_j}\theta_{a_2\cdots b(j)\cdots a_p} = B_{ca_j,a_2\cdots b(j)\cdots a_p} - k\sum_{i\neq j} g_{a_j a_i}\theta_{a_2\cdots c(i)\cdots b(j)\cdots a_p} - kg_{a_j b}\theta_{a_2\cdots c(j)\cdots a_p}$$

for $2\leqq j\leqq p$. Substituting the last equations and (5.6) into (5.7), we get

$$\nabla_c u_{ba_2\cdots a_p} = B_{cb,a_2\cdots a_p} - \sum_{j=2}^{p} B_{ca_j,a_2\cdots b(j)\cdots a_p} - k\sum_{j=2}^{p}\sum_{i\neq j} g_{a_j a_i}\theta_{a_2\cdots c(i)\cdots b(j)\cdots a_p} ,$$

from which it follows

$$\begin{aligned}\nabla_c u_{ba_2\cdots a_p} + \nabla_b u_{ca_2\cdots a_p} &= 2B_{cb,a_2\cdots a_p} - \sum_{j=2}^{p}(B_{ca_j,a_2\cdots b\cdots a_p} + B_{ba_j,a_2\cdots c\cdots a_p}) \\ &= -(p+1)k\sum_{j=2}^{p}(\nabla_c v_{ba_2\cdots a_p} + \nabla_b v_{ca_2\cdots a_p}) .\end{aligned}$$

Thus we have

$$\nabla_c\{(p+1)kv_{ba_2\cdots a_p} + u_{ba_2\cdots a_p}\} + \nabla_b\{(p+1)kv_{ca_2\cdots a_p} + u_{ca_2\cdots a_p}\} = 0 .$$

If we assume $k\neq 0$ and put

$$w_{ba_2\cdots a_p} = v_{ba_2\cdots a_p} + \frac{1}{(p+1)k}u_{ba_2\cdots a_p} ,$$

it is a Killing tensor. Hence we have

THEOREM 5.1. *In a space of constant curvature with non-vanishing* $k=R/n(n-1)$, *any projective Killing tensor* $v_{a_1\cdots a_p}$, $(p<n)$, *is decomposed uniquely in the form*

$$v_{a_1\cdots a_p}=w_{a_1\cdots a_p}+q_{a_1\cdots a_p},$$

where $w_{a_1\cdots a_p}$ *is a Killing tensor and* $q_{a_1\cdots a_p}$ *is a closed projective Killing tensor.* $q_{a_1\cdots a_p}$ *is given by exactly*

$$q_{a_1\cdots a_p}=-\frac{1}{(p+1)k}(d\theta)_{a_1\cdots a_p}$$

in terms of the associated tensor $\theta_{a_2\cdots a_p}$.

The uniqueness follows from the following lemma.

LEMMA. *In a space of constant curvature* $(k\neq 0)$, *if a Killing tensor of degree* p $(<n)$ *is closed, then it is the zero tensor,* ([3], [11]).

COROLLARY 5.2. *In a space of constant curvature* $(k\neq 0)$, *if* $\theta_{a_2\cdots a_p}$ $(p<n)$ *is the associated tensor of a projective Killing tensor, then* $(d\theta)_{a_1\cdots a_p}$ *is a projective Killing tensor whose associated tensor is* $-(p+1)k\theta_{a_2\cdots a_p}$.

We shall give an application which corresponds to Theorem 0.6.

Under the assumption in Corollary 5.2, we have

$$d\delta d\theta=(p+1)(n-p+2)kd\theta$$

from (4.5), because of $\delta w=0$. For the compact case, making use of integral and taking account of what d and δ are dual each other, we can get

THEOREM 5.3. *In a compact space of negative constant curvature, any projective Killing tensor is a Killing tensor.*

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