Natural Science Report, Ochanomizu University, Vol. 21, No. 2, 1970
お茶の水女子大学 自然科学報告 第21巻 第2号

The Level Density of a Fermion System in a Parabolic Potential II

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(Received September 10, 1970)

The method of our previous paper[1], by which we calculated the level density of a Fermion system in a parabolic potential, is applied to the nucleus under the assumption that protons and neutrons are moving substantially in the central potential field. In this case we get an interesting result that the level density drops sharply at the magic numbers, in addition to the dependence of the level density on the angular momentum.

§ 1. The quasi-shell model

The nuclear shell model requires that the Hamiltonian H of the nucleus which is composed of N_n neutrons and N_p protons, is defined as follows

$$H = H_n + H_p, \quad H_n = \sum_{i=1}^{N_n} H_i, \quad H_p = \sum_{i=N_n+1}^{A} H_i, \quad A = N_p + N_n$$
$$H_i = -\frac{h}{2M_i}\nabla^2 - V_0[1-(\Gamma_i/R)^2] - f(r_i)\boldsymbol{l}_i \cdot \boldsymbol{s}_i . \tag{1}$$

If we think of comparatively heavy nuclei, we had better choose the individual angular momentum $\boldsymbol{j} = \boldsymbol{l} + \boldsymbol{s}$ as a good quantum number because of the strong $j-j$ coupling.

We get the energy ε_τ as follows

when $\quad j = l + 1/2$

$$\varepsilon_\tau = 2\varepsilon(n-1) + \left(j + \frac{1}{2}\right)(\varepsilon + a) + a \tag{2—a}$$

when $\quad j = l - 1/2$

$$\varepsilon'_\tau = 2\varepsilon(n-1) + \left(j + \frac{1}{2}\right)(\varepsilon - a) + a - \varepsilon \tag{2—b}$$

where

$$\varepsilon \cong h\omega \cong 40/A^{1/3}\,\mathrm{MeV}, \qquad a = \frac{1}{2}f(r) \cong \frac{6.61}{A^{2/3}}\,\mathrm{MeV},$$

the Coulomb effect for protons being neglected.

The energies ε_τ and ε'_τ are given by the radial quantum number n ranging over 1, 2, $\cdots$ and the angular momentum j ranging over 1/2, 3/2, $\cdots$.

§ 2. The quantum number and partition function

As stated above, we assume that a nucleus is composed of protons and neutrons of spin 1/2 moving independently in a central potential.

We can get the grand partition function for such a system which is marked by the proton number N_p, the neutron number N_n, the total energy E, the total angular momentum J and its z- component M,

$$Z = \text{trace} \exp\left[\alpha_n N_n + \alpha_p N_p - \beta E + \gamma M\right]. \tag{3}$$

We can also introduce the level density by

$$Z = \iiiint \rho(N_n, N_p, E, M) \cdot \exp\left[\alpha_n N_n + \alpha_p N_p - \beta E + \gamma M\right] dN_n dN_p dE dM \tag{4}$$

where

$$\rho(N_n, N_p, E, M) = \sum_{i,n,z} \delta_{N_n n} \delta_{N_p z} \delta_{MM_i(n+z)}\,.$$

The proton number N_n, the neutron number N_p, the total energy E, and the z-component M of the total angular momentum J are obtained by summing over eigenstates τ and we have

$$\begin{aligned} Z &= \prod_\tau (1+\exp[\alpha_n - \beta\varepsilon_\tau^{(n)} + m_\tau^{(n)}\gamma])(1+\exp[\alpha_p - \beta\varepsilon_\tau^{(p)} + m_\tau^{(p)}\gamma]) \\ &= \prod_{i=1}^{4} \prod_{njm} (1+Z_{(nj)}^{(i)}\zeta^m)(1+Z_{(nj)}^{(i)}\zeta^{-m}) \end{aligned} \tag{5}$$

where from (3) we set

$$\begin{aligned} Z_{nj}^{(1)} &= \exp(\alpha_n - \beta\varepsilon_{nj}^{(n)})\,, & Z_{nj}^{(2)} &= \exp(\alpha_n - \beta\varepsilon_{nj}'^{(n)}) \\ Z_{nj}^{(3)} &= \exp(\alpha_p - \beta\varepsilon_{nj}^{(p)})\,, & Z_{nj}^{(4)} &= \exp(\alpha_p - \beta\varepsilon_{nj}'^{(p)})\,. \end{aligned} \tag{6}$$

§ 3. An approximate evaluation of log Z

The calculations proceed roughly parallel to those of the previous paper.

Taking $\log Z$ instead of Z

$$\log Z = \sum_{i=1}^{4} \sum_{njm} \log\left[(1+Z_{nj}^{(i)})^2 + 4Z_{nj}^{(i)} \sinh^2(m\gamma/2)\right] \tag{7}$$

we expand $\log Z$ in power series of γ^2 and get, up to the first order in γ^2,

$$\log Z = \sum_{i=1}^{4} (p_i + q_i r^2) = p + q r^2 \tag{8—a}$$

$$p_i = 2 \sum_{n,j} \left(j + \frac{1}{2} \right) \log (1 + Z_{nj}^{(i)}) \tag{8—b}$$

$$q_i = \sum_{nj} 4 Z_{nj}^{(i)} / (1 + Z_{nj}^{(i)})^2 \cdot \{ \sum_{m=1/2}^{j} m^2 \} / 4 . \tag{8—c}$$

To evaluate p_i we use the formula of Mellin transform

$$\log (1+x) = \frac{1}{2\pi i} \int \frac{\pi}{s \sin \pi s} x^s ds , \qquad 0 < \Re s \equiv \sigma < 1 \tag{9}$$

the path of integration extending from $\sigma - i\infty$ to $\sigma + i\infty$.

We get by use of (6)

$$p_i = \frac{4}{2\pi i} \int \frac{\pi}{s \sin \pi s} \exp [\beta \delta_i s] \cdot \frac{ds}{(1 - e^{-2\beta \varepsilon s})(1 - e^{-\beta \xi_i s})^2}$$

$$= \beta f(\delta_i) + \frac{1}{\beta} h(\delta_i) + 0(\beta^{-3}) \tag{10}$$

where

$$\xi_1 = \xi_3 = \varepsilon + a , \qquad \xi_2 = \xi_4 = \varepsilon - a ,$$

$$\delta_1 = \alpha_n / \beta - (\varepsilon + 2a), \quad \delta_2 = \alpha_p / \beta - (\varepsilon + 2a), \quad \delta_3 = \alpha_n / \beta, \quad \delta_4 = \alpha_p / \beta .$$

We can evaluate p_i by use of Cauchy's formula about complex integration and approximate it as follows

$$\begin{aligned} f(\delta_i) = {} & \frac{\delta_i^4}{12 \varepsilon \xi_i} + \frac{\delta_i^3}{3 \xi_i^2} + \frac{5 \delta_i^2}{12 \varepsilon} + \frac{\delta_i^2}{\xi_i} + \frac{\varepsilon \delta_i^2}{3 \xi_i} \\ & - \frac{\xi_i}{\pi^2} \frac{1}{\sin (2\pi\varepsilon/\xi_i)} \left\{ \left(1 + \frac{\delta_i}{\xi_i} \right) \sin \left[(\delta_i + \varepsilon) \frac{2\pi}{\xi_i} \right] \right. \\ & \left. + \frac{1}{\pi} \cos \left[(\delta_i + \varepsilon) \frac{2\pi}{\xi_i} \right] + \frac{\varepsilon}{\xi_i} \frac{\cos (2\pi \delta_i / \xi_i)}{\sin (2\pi\varepsilon/\xi_i)} \right\} \end{aligned} \tag{11}$$

$$\begin{aligned} h(\delta_i) = {} & \frac{\pi^2}{6} \left[\frac{\delta_i^2}{\varepsilon \xi_i^2} + \frac{2\delta_i}{\varepsilon \xi_i} + \frac{2\delta_i}{\xi_i^2} + \frac{5}{6\varepsilon} + \frac{2}{\xi_i} + \frac{2\varepsilon}{3\xi_i^2} \right. \\ & + \frac{4}{\xi_i} \frac{1}{\sin (2\pi\varepsilon/\xi_i)} \left\{ \left(1 + \frac{\delta_i}{\xi_i} \right) \sin \left[(\delta_i + \varepsilon) \frac{2\pi}{\xi_i} \right] \right. \\ & \left. \left. + \frac{\varepsilon}{\xi_i} \frac{\cos (2\pi \delta_i / \xi_i)}{\sin (2\pi\varepsilon/\xi_i)} \right] \right\} . \end{aligned} \tag{12}$$

The second term in (8—c) may be approximated as in the previous paper,

$$\begin{aligned} q(\delta_i) &= \sum_{nj} \frac{4 Z_{nj}^{(i)}}{(1 + Z_{nj}^{(i)})^2} \cdot \frac{1}{12} \left\{ \left(j + \frac{1}{2} \right)^3 - \frac{1}{4} \left(j + \frac{1}{2} \right) \right\} \\ &= \frac{1}{48 \varepsilon^4} (\delta_i + \xi_i)(\delta_i + \xi_i + 1) \{ (\delta_i + \xi_i)^2 + \varepsilon (\delta_i + \xi_i) - \varepsilon^2 / 2 \} . \end{aligned} \tag{13}$$

§ 4. An approximate evaluation of the level density

One may get the level density ρ by the inverse Fourier transform of Z as

$$\rho(N_n, N_p, E, M) = \frac{1}{(2\pi i)^3}\iiint d\alpha_n d\alpha_p d\beta Z_M \exp(-\alpha_n N_n - \alpha_p N_p + \beta E) \quad (14)$$

$$Z_M = \frac{1}{2\pi i}\int_{-\pi i}^{\pi i} Z \exp(-\gamma M) d\gamma .$$

The function Φ is defined by

$$\Phi = \log Z_M - \alpha_n N_n - \alpha_p N_p + \beta E . \quad (15\text{—}a)$$

One can deduce three relations for a saddle point

$$\frac{\partial \Phi}{\partial \alpha_n} = 0 , \quad \frac{\partial \Phi}{\partial \alpha_p} = 0 , \quad \frac{\partial \Phi}{\partial \beta} = 0 . \quad (15\text{—}b)$$

On this condition, ρ may be evaluated approximately by use of the saddle point method as

$$\rho(N_n, N_p, E, M) = \exp \Phi^* / (2\pi)^{3/2} D^{1/2} \quad (16)$$

where Φ^* is the value which satisfies the relations of (15—b) simultaneously, D standing for the determinant

$$D = \begin{vmatrix} \dfrac{\partial^2 \Phi}{\partial \beta^2} & \dfrac{\partial^2 \Phi}{\partial \beta \partial \alpha_n} & \dfrac{\partial^2 \Phi}{\partial \beta \partial \alpha_p} \\ \dfrac{\partial^2 \Phi}{\partial \beta \partial \alpha_n} & \dfrac{\partial^2 \Phi}{\partial \alpha_n^2} & 0 \\ \dfrac{\partial^2 \Phi}{\partial \beta \partial \alpha_p} & 0 & \dfrac{\partial^2 \Phi}{\partial \alpha_p^2} \end{vmatrix} .$$

An approximate value of Z_M is given by the substitution of p in $\log Z$ as

$$Z_M = \frac{1}{2\pi}\sqrt{\frac{\pi}{q}} \exp\left(p - \frac{M^2}{4q}\right) \quad (17)$$

and the Φ will be, up to the order $1/\beta$,

$$\Phi = \beta f(\delta) + g(\delta) + \frac{1}{\beta} h(\delta) + \beta(E - N_n \delta_n - N_p \delta_p) \quad (18)$$

where

$$\delta_n = \alpha_n / \beta , \qquad \delta_p = \alpha_p / \beta$$

$$g(\delta) = -\frac{M^2}{4q(\delta)} - \frac{1}{2}\log q(\delta) - \log 2\sqrt{\pi} . \quad (19)$$

If the values of δ and E at $\beta = \infty$ are denoted by δ_0 and E_0, the parameters at the saddle point will be given by

$$\delta_n = \delta_{no} - \frac{g'(\delta_{no})}{f''(\delta_{no})} \cdot \frac{1}{\beta}, \qquad \delta_p = \delta_{po} - \frac{g'(\delta_{po})}{f''(\delta_{po})} \cdot \frac{1}{\beta}$$
$$\beta = \left\{ \frac{1}{E-E_0} \sum_{k=n,p} \frac{2h(\delta_{ko})f''(\delta_{ko}) - g'^2(\delta_{ko})}{2f''(\delta_{ko})} \right\}^{1/2} \tag{20}$$

since $N_n = f'(\delta_{n0})$, $N_p = f'(\delta_{p0})$, $E_0 = N_n\delta_{n0} + N_p\delta_{p0} - f'(\delta_{n0}) - f'(\delta_{p0})$, δ_{n0} and δ_{p0} are independently determined by the neutron number N_n and the proton number N_p.

One may get the approximate value β

$$\beta = \sqrt{\{h_0/(E-E_0)\}}, \qquad h_0 = h(\delta_{n0}) + h(\delta_{p0})$$

because of the relations

$$2h(\delta_{n0})f''(\delta_{n0}) \gg g'(\delta_{n0})^2$$
$$2h(\delta_{p0})f''(\delta_{p0}) \gg g'(\delta_{p0})^2$$

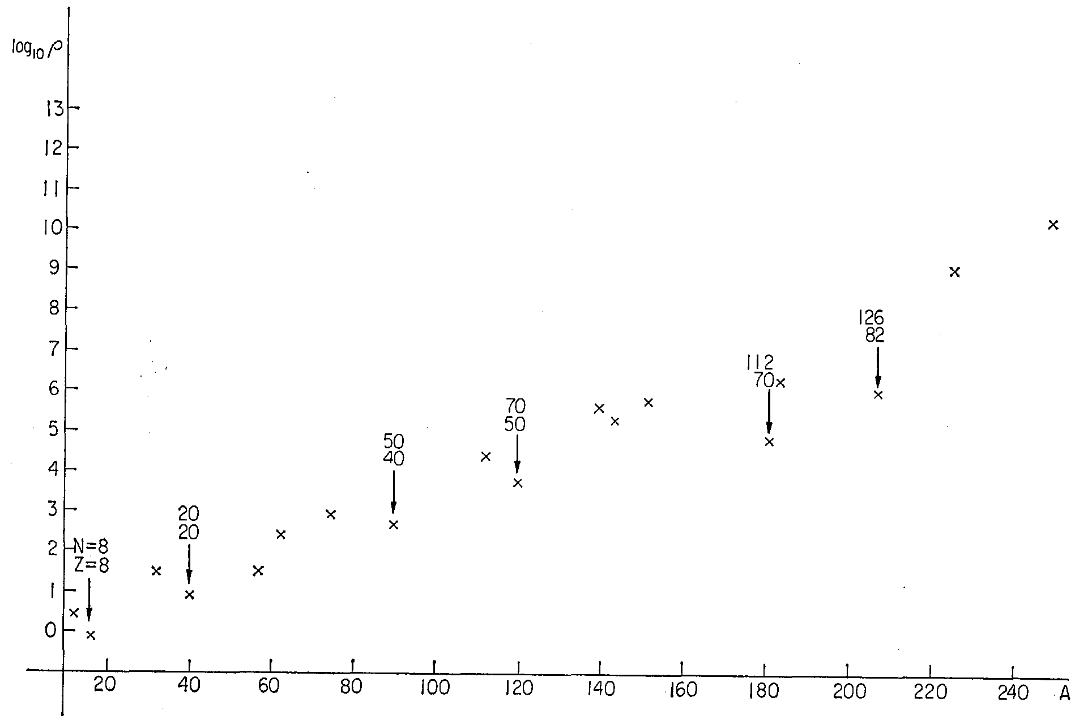
and the Φ will be approximated by

$$\Phi = g(\delta_0) + \sqrt{\{h_0(E-E_0)\}}. \tag{21}$$

We get the level density $\rho(N_n, N_p, E, M)$ by use of the above results

$$\rho(N_n, N_p, E, M) = \exp\left[\sqrt{\{h_0(E-E_0)\}} - M^2/4q\right]/\sqrt{2}(2\pi)^2(qD)^{1/2} \tag{22}$$

and the level density with the total angular momentum J will be given by



$$\rho(N_n, N_p, E; J) = \rho(N_n, N_p, E, M)_{M=J} - \rho(N_n, N_p, E, M)_{M=J+1}$$
$$= (2J+1) \exp [\surd\{h_0'(E-E_0)\} \qquad (23)$$
$$-J(J+1)/4q] \cdot /16\surd 2\, \pi^2 q^{3/2}.$$

An essential improvement on the previous calculation is the inclusion of the residues on the imaginary axis. This inclusion results in the sharp drops of the level density at the magic numbers.

In the graph, the level density with $J=0$ is plotted versus mass number A, where we take excited energy $E-E_0=8$ MeV.

Reference

[1] N. Matsushita and G. Iwata: Natural Science Report 20 (1969), 31.