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Operators on Almost Kählerian Spaces

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Introduction

Frölicher-Nijenhuis have shown in [1] some interesting results about the derivations of scalar valued or vector valued differential forms on a differentiable manifold. They defined an operator $\bar{\wedge}$ and seeked after some identities between several derivations. On the other hand, Yano-Ako [4] got many differential concomitants on a manifold, and especially a concomitant S for a pure tensor on an almost complex space. A covariant almost analytic tensor u is defined by $Su=0$ when u is pure. For a skew-symmetric covariant tensor which is not necessarily supposed to be pure, the operator T corresponding to S can be written in terms of the derivations of Frölicher-Nijenhuis. Then it is naturally considered that a p-form u satisfying $Tu=0$ has the properties which are closely related to that of covariant almost analytic tensors. We call such forms to be covariant pseudo analytic.

We firstly define some operators on an almost Kählerian space and search for identities between them in § 1 and § 2. The formulas mainly related to the Laplacian operator are got in § 3. We shall give the conditions of integrability of an almost Kählerian structure and it is shown that there exist adjoint relations of operators in a compact case. The last § 4 is devoted to studying the pseudo analytic forms in compact Kählerian spaces.

§ 1. Definitions and notations

Let M^n be an almost Hermitian space with the positive definite Riemannian metric $g_{\lambda\mu}$ [1)] and the almost complex structure $\varphi_\lambda{}^\mu$. ∇_λ means the operator of covariant derivation with respect to the Riemannian connection. A differential p-form u is expressed by its coefficients $u_{\lambda_1\cdots\lambda_p}$ which are the components of skew-symmetric covariant tensors of degree p. Then u is written as

$$u=\frac{1}{p!}u_{\lambda_1\cdots\lambda_p}dx^{\lambda_1}\wedge\cdots\wedge dx^{\lambda_p}$$

1) The indices λ, μ, $\cdots$ run over the range $1, 2, \cdots, n$.

where (x^λ) is a local coordinates system. We denote by $\mathfrak{F}^p$ the set of all p-forms on M^n. The exterior differential d is an operator $\mathfrak{F}^p \to \mathfrak{F}^{p+1}$, and the dual mapping $*: \mathfrak{F}^p \to \mathfrak{F}^{n-p}$ with respect to the Riemannian structure is defined by

$$(1.1) \qquad ({}^*u)_{\lambda_1 \cdots \lambda_{n-p}} = \frac{\sqrt{g}}{p!} g^{\mu_1\sigma_1} \cdots g^{\mu_p\sigma_p} u_{\mu_1\cdots\mu_p} \epsilon_{\sigma_1\cdots\sigma_p\lambda_1\cdots\lambda_{n-p}}$$

for a p-form u, where g is the positive determinent $|g_{\lambda\mu}|$. The codifferential operator $\delta: \mathfrak{F}^p \to \mathfrak{F}^{p-1}$ is given by

$$(1.2) \qquad \delta u = (-1)^p {}^*d{}^*u \qquad u \in \mathfrak{F}^p .$$

Then we have for a p-form $u = (u_{\lambda_1\cdots\lambda_p})$

$$(1.3) \qquad (\delta u)_{\lambda_2\cdots\lambda_p} = -\nabla^\rho u_{\rho\lambda_2\cdots\lambda_p} \qquad (p \geqq 1),$$
$$\delta u = 0 \qquad (p = 0).$$

An almost Hermitian structure $(\varphi_\lambda{}^\mu, g_{\lambda\mu})$ is called to be almost Kählerian (almost semi-Kählerian) if the 2-form $\varphi = (\varphi_{\lambda\mu})$, $\varphi_{\lambda\mu} = \varphi_\lambda{}^\rho g_{\rho\mu}$, is closed (coclosed). An almost Kählerian space is necessarily almost semi-Kählerian, and the form φ is harmonic. We define some differential operator[2] on an almost Kählerian space M^n. They are $\Gamma, \gamma, \mathfrak{D}$: $\mathfrak{F}^p \to \mathfrak{F}^{p+1}$ and C, c, ϑ: $\mathfrak{F}^p \to \mathfrak{F}^{p-1}$. Their representations for a p-form $u = (u_{\lambda_1\cdots\lambda_p})$ are given by the following formulas:

$$(1.4) \qquad (\Gamma u)_{\lambda_0\cdots\lambda_p} = \sum_{\alpha=0}^{p} (-1)^\alpha \varphi_{\lambda_\alpha}{}^\rho \nabla_\rho u_{\lambda_1\cdots\lambda_p},$$

$$(1.5) \qquad (\gamma u)_{\lambda_0\cdots\lambda_p} = \sum_{\alpha\neq\beta} (-1)^\alpha \nabla_{\lambda_\alpha} \varphi_{\lambda_\beta}{}^\rho u_{\lambda_0\cdots\hat{\alpha}\cdots\overset{\beta}{\hat{\rho}}\cdots\lambda_p},$$

$$(1.6) \qquad (\mathfrak{D}u)_{\lambda_0\cdots\lambda_p} = \sum_{\alpha<\beta} (-1)^\alpha \varphi^{\rho\sigma} \nabla_\rho \varphi_{\lambda_\alpha\lambda_\beta} u_{\lambda_0\cdots\hat{\alpha}\cdots\overset{\beta}{\hat{\sigma}}\cdots\lambda_p},$$

and

$$(1.7) \qquad (Cu)_{\lambda_2\cdots\lambda_p} = \varphi^{\rho\sigma} \nabla_\rho u_{\sigma\lambda_2\cdots\lambda},$$

$$(1.8) \qquad (cu)_{\lambda_2\cdots\lambda_p} = \sum_{\alpha=2}^{p} \nabla^\rho \varphi^\sigma{}_{\lambda_\alpha} u_{\rho\lambda_2\cdots\overset{\alpha}{\hat{\sigma}}\cdots\lambda_p},$$

$$(1.9) \qquad (\vartheta u)_{\lambda_2\cdots\lambda_p} = \sum_{\alpha=2}^{p} \varphi_{\lambda_\alpha}{}^\rho \nabla^\tau \varphi_\rho{}^\sigma u_{\sigma\lambda_2\cdots\overset{\alpha}{\hat{\tau}}\cdots\lambda_p}.$$

For the forms u_0 and u_1 of order 0 and 1, we define

$$\gamma u_0 = \mathfrak{D}u_0 = 0, \quad Cu_0 = cu_0 = \vartheta u_0 = 0, \quad cu_1 = \vartheta u_1 = 0.$$

It is easily verified that the images by these operators of differential forms are also skew-symmetric forms. The formulas (1.4), (1.5), (1.8)

2) They can be defined in an almost Hermitian space, and it is seen that Propositions 1.1 and 1.4 hold good in that case.

and (1.9) become of the expressions:

$$(1.4)' \qquad (\Gamma u)_{\lambda_0\cdots\lambda_p}=\varphi_{\lambda_0}{}^{\rho}\nabla_\rho u_{\lambda_1\cdots\lambda_p}-\sum_{i=1}^{p}\varphi_{\lambda_i}{}^{\rho}\nabla_\rho u_{\lambda_1\cdots\overset{i}{\hat{\lambda}_0}\cdots\lambda_p},$$

$$(1.5)' \qquad (\gamma u)_{\lambda_0\cdots\lambda_p}=-\sum_{\alpha<\beta}(-1)^{\alpha}\nabla^{\rho}\varphi_{\lambda_\alpha\lambda_\beta}u_{\lambda_0\cdots\hat{\alpha}\cdots\overset{\beta}{\hat{\rho}}\cdots\lambda_p},$$

and

$$(1.8)' \qquad (cu)_{\lambda_2\cdots\lambda_p}=-\frac{1}{2}\sum_{\alpha=2}^{p}\nabla_{\lambda_\alpha}\varphi^{\rho\sigma}u_{\rho\lambda_2\cdots\overset{\alpha}{\hat{\sigma}}\cdots\lambda_p},$$

$$(1.9)' \qquad (\vartheta u)_{\lambda_2\cdots\lambda_p}=-\frac{1}{2}\sum_{\alpha=2}^{p}\varphi_{\lambda_\alpha}{}^{\rho}\nabla_\rho\varphi^{\sigma\tau}u_{\sigma\lambda_2\cdots\overset{\alpha}{\hat{\tau}}\cdots\lambda_p}.$$

About the relations between Γ, γ and C, c, we know the following propositions.

PROPOSITION 1.1. ([8]) *In an almost Kählerian space, the operator* Γ *is a skew-derivation and it holds good that*

$$(1.10) \qquad *\Gamma*=-C.$$

PROPOSITION 1.2. ([8]) *In an almost Kählerian space, the operator* γ *is a skew-derivation and it holds good that*

$$(1.11) \qquad *\gamma*=-c.$$

By the same calculation of the proof of Proposition 1.1, we can show the following proposition. However its proof is easily given by the formulas (2.5, 6) which are shown later.

PROPOSITION 1.3. *In an almost Kählerian space, the operator* $\mathfrak{D}$ *is a skew-derivation, and we have*

$$(1.12) \qquad *\mathfrak{D}*=-\vartheta.$$

Next we define the operators[2)] Φ and Ψ: $\mathfrak{F}^p\to\mathfrak{F}^p$ by

$$(1.13) \qquad (\Phi u)_{\lambda_1\cdots\lambda_p}=\sum_{i=1}^{p}\varphi_{\lambda_i}{}^{\rho}u_{\lambda_1\cdots\overset{i}{\hat{\rho}}\cdots\lambda_p},$$

$$(1.14) \qquad (\Psi u)_{\lambda_1\cdots\lambda_p}=\varphi_{\lambda_1}{}^{\rho_1}\cdots\varphi_{\lambda_p}{}^{\rho_p}u_{\rho_1\cdots\rho_p},$$

for any p-form $u=(u_{\lambda_1\cdots\lambda_p})$. On 0-form u, it is defined that

$$\Phi u=\Psi u=0.$$

The fact that the images Φu and Ψu are p-forms for a p-form u is evident. The operators Φ and Ψ have the following intrinsic expressions. Let $X_1, \cdots, X_p$ are p vector fields. Then we have for a p-form u

$$(\Phi u)(X_1, \cdots, X_p)=\sum_{i=1}^{p}u(X_1, \cdots, \varphi X_i, \cdots, X_p)$$

$$(\Psi u)(X_1, \cdots, X_p)=u(\varphi X_1, \cdots, \varphi X_p),$$

where φ denotes the endomorphism of the vector fields corresponding to the almost complex structure. From this point of view, it is naturally considered that the operator $H_r: \mathfrak{F}^p \to \mathfrak{F}^p$ $(0 \leqq r \leqq p)$ is defined by

(1.15) $$(H_r u)_{\lambda_1 \cdots \lambda_p} = \frac{1}{r!(p-r)!} \sum \epsilon^{\mu_1 \cdots \mu_r \nu_1 \cdots \nu_{p-r}}_{\lambda_1 \cdots \cdots \cdots \lambda_p} \varphi_{\mu_1}{}^{\rho_1} \cdots \varphi_{\mu_r}{}^{\rho_r} u_{\rho_1 \cdots \rho_r \nu_1 \cdots \nu_{p-r}},$$

and $H_r u = 0$ if u is a 0-form. We have for any p-form u $(p \geqq 1)$

$$H_0 u = u, \quad H_1 u = \Phi u, \quad H_p u = \Psi u.$$

PROPOSITION 1.4. ([8]) *In an almost Kählerian space, the operator Φ is a derivation, and we have for a p-form u*

(1.16) $$* \Phi * u = (-1)^p \Phi u.$$

It is evident that if the almost Hermitian structure is Kählerian then the operator γ vanishes. The converse is also true. Since γ is a skew-derivation, it is sufficient for the integrability that γ vanishes on 0 and 1-forms. As for the operator $\mathfrak{D}$, the same argument is valid. We define the vector valued 2-forms F and N by

$$F = (\nabla^\rho \varphi_{\lambda\mu}), \qquad N = (2\varphi^{\rho\sigma} \nabla_\rho \varphi_{\lambda\mu}).$$

N is identical with the Nijenhuis tensor in an almost Kählerian space. Making use of the notation $\overline{\wedge}$ [3] introduced in [1], we have

PROPOSITION 1.5. *In an almost Kählerian space, we get*

(1.17) $$\gamma u = -u \overline{\wedge} F,$$

(1.18) $$\mathfrak{D} u = \frac{1}{2} u \overline{\wedge} N.$$

PROOF. We have for a p-form $u = (u_{\lambda_1 \cdots \lambda_p})$ by the definition of $\overline{\wedge}$

$$(u \overline{\wedge} F)_{\lambda_0 \cdots \lambda_p} = \frac{1}{2!(p-1)!} \epsilon^{\rho_1 \rho_2 \sigma_2 \cdots \sigma_p}_{\lambda_0 \cdots \cdots \cdots \lambda_p} \nabla^\tau \varphi_{\rho_1 \rho_2} u_{\tau \sigma_2 \cdots \sigma_p}$$

$$= \sum_{\alpha < \beta} (-1)^{\alpha+\beta-1} \nabla^\rho \varphi_{\lambda_\alpha \lambda_\beta} u_{\rho \lambda_0 \cdots \hat{\alpha} \cdots \hat{\beta} \cdots \lambda_p}$$

$$= \sum_{\alpha < \beta} (-1)^\alpha \nabla^\rho \varphi_{\lambda_\alpha \lambda_\beta} u_{\lambda_0 \cdots \hat{\alpha} \cdots \overset{\beta}{\rho} \cdots \lambda_p} = -(\gamma u)_{\lambda_0 \cdots \lambda_p}.$$

In the similar way, it holds that

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3) Let u is a p-form and Q is a vector valued q-form. Then $(p+q-1)$-form $i_Q u = u \overline{\wedge} Q$ is defined for vector fields $X_1, \cdots, X_{p+q-1}$ by

$$(u \overline{\wedge} Q)(X_1, \cdots, X_{p+q-1}) = \frac{1}{q!(p-1)!} \sum_\sigma \varepsilon(\sigma) u(Q(X_{\sigma_1} \cdots X_{\sigma_q}), X_{\sigma_{q+1}}, \cdots, X_{\sigma_{p+q-1}}),$$

where $\varepsilon(\sigma)$ denotes the sign of the permutation σ. It has the local expression as

$$(u \overline{\wedge} Q)_{\lambda_1 \cdots \lambda_{p+q-1}} = \frac{1}{q!(p-1)!} \sum \epsilon^{\rho_1 \cdots \rho_q \sigma_1 \cdots \sigma_{p-1}}_{\lambda_1 \cdots \cdots \cdots \lambda_{p+q-1}} Q_{\rho_1 \cdots \rho_q}{}^\tau u_{\tau \sigma_1 \cdots \sigma_{p-1}}.$$

$$(u\bar{\wedge}N)_{\lambda_0\cdots\lambda_p}=\frac{1}{2!(p-1)!}\epsilon^{\rho_1\rho_2\sigma_2\cdots\sigma_p}_{\lambda_0\cdots\cdots\cdots\lambda_p}N_{\rho_1\rho_2}{}^{\tau}u_{\tau\sigma_2\cdots\sigma_p}$$

$$=2\sum_{\alpha<\beta}(-1)^{\alpha}\varphi^{\rho\sigma}\nabla_{\rho}\varphi_{\lambda_\alpha\lambda_\beta}u_{\lambda_0\cdot\hat{\alpha}\cdots\overset{\beta}{\hat{\sigma}}\cdots\lambda_p}$$

$$=2(\mathfrak{D}u)_{\lambda_0\cdots\lambda_p}\,.$$

Since the operator $i_Q u=u\bar{\wedge}Q$ satisfies that if $i_Q=0$ then the vector valued q-form Q vanishes, we see from Proposition 1.5 that the following conditions are equivalent.

(1) $\gamma=0$ or $\mathfrak{D}=0$.
(2) $F=0$ or $N=0$.
(3) The structure is Kählerian.

Under the definitions and notations of [1], a derivation A can be decomposed uniquely into a sum of two derivations A_d and A_i. A_d is a derivation of type d_*, that is, it commutes with the derivation d, and A_i is one of type i_*, that is, it vanishes on $\mathfrak{F}^0$. Then we have

PROPOSITION 1.6. *In an almost Kählerian space we have*

(1.19) $\qquad \Phi=\Phi_i\,, \qquad \gamma=\gamma_i\,, \qquad \mathfrak{D}=\mathfrak{D}_i\,,$

(1.20) $\qquad \Gamma=\Gamma_d+\Gamma_i\,, \qquad \Gamma_d=\Gamma-\gamma\,, \qquad \Gamma_i=\gamma\,.$

PROOF. Since $\Gamma-\gamma$ is a (skew-)derivation, (1.20) is the result of Corollary 2.4 of [8], which asserts that $\Gamma-\gamma$ is (skew-) commutative with d. The other formulas are evident.

COROLLARY 1.7. *An almost Kählerian space is Kählerian if and only if the operator Γ commutes with the exterior derivative d, that is*

(1.21) $\qquad d\Gamma+\Gamma d=0\,.$

More precisely, sence $d\Gamma+\Gamma d$ is a derivation which commutes with d, it is sufficient for the operator $d\Gamma+\Gamma d$ to vanish that it vanishes on any scalar functions.

2. Some formulas in almost Kählerian spaces

In this section let M^n be an almost Kählerian space. We first consider the relations between operators Φ, Ψ and the other operators. Many proofs are omitted since they are merely simple straightforward calculations, though they are complicated in some degree.

PROPOSITION 2.1. *Let u be a p-form. Then we have*

(2.1) $\qquad \Phi\Psi u=\Psi\Phi u\,,$

(2.2) $\qquad \Psi^2 u=(-1)^p u\,.$

PROPOSITION 2.2. ([8])

(2.3) $$(d\Phi-\Phi d)u=(-\Gamma+\gamma)u\,,$$

(2.4) $$(\delta\Phi-\Phi\delta)u=(-C+c)u\,.$$

PROPOSITION 2.3.

(2.5) $$(\Gamma\Phi-\Phi\Gamma)u=(d+\mathfrak{D})u\,,$$

(2.6) $$(C\Phi-\Phi C)u=(\delta+\vartheta)u\,.$$

The formula (2.5) shows that $\mathfrak{D}$ is a skew-derivation since $\Gamma\Phi-\Phi\Gamma$ is evidently a skew-derivation. Making comparison (2.5) with (2.6), we can see that the formula (1.12) holds good from (1.10), (1.16) and (1.2).

PROPOSITION 2.4.

(2.7) $$(\gamma\Phi-\Phi\gamma)u=-3\mathfrak{D}u\,,$$

(2.8) $$(c\Phi-\Phi c)u=-3\vartheta u\,.$$

PROPOSITION 2.5.

(2.9) $$(\mathfrak{D}\Phi-\Phi\mathfrak{D})u=-3\gamma u\,,$$

(2.10) $$(\vartheta\Phi-\Phi\vartheta)u=-3cu\,.$$

As for Ψ, we obtain the following formulas.

PROPOSITION 2.6.

(2.11) $$\Psi d\Psi u=(-1)^p(\Gamma+\gamma)u\,,$$

(2.12) $$\Psi\delta\Psi u=(-1)^p(C+c)u\,.$$

PROPOSITION 2.7.

(2.13) $$\Psi\Gamma\Psi u=(-1)^{p-1}(d-\mathfrak{D})u\,,$$

(2.14) $$\Psi C\Psi u=(-1)^{p-1}(\delta-\vartheta)u\,.$$

PROPOSITION 2.8.

(2.15) $$\Psi\gamma\Psi u=(-1)^p\mathfrak{D}u\,,$$

(2.16) $$\Psi c\Psi u=(-1)^{p-1}\vartheta u\,.$$

PROPOSITION 2.9.

(2.17) $$\Psi\mathfrak{D}\Psi u=(-1)^{p-1}\gamma u\,,$$

(2.18) $$\Psi\vartheta\Psi u=(-1)^p cu\,.$$

Next we denote by L (resp. Λ) the exterior (resp. interior) product by the fundamental 2-form $\varphi=(\varphi_{\lambda\mu})$. The operators $L:\ \mathfrak{F}^p\to\mathfrak{F}^{p+2}$ and $\Lambda:\ \mathfrak{F}^p\to\mathfrak{F}^{p-2}$ are written as

(2.19) $$Lu=\varphi\wedge u\,,\qquad \Lambda u=(-1)^p*L*u$$

for a p-form u. Then we obtain that

(2.20) $$*\Lambda=\Lambda*\,,\qquad *L=L*\,,$$

$$Lu = (-1)^p * \Lambda * u . \tag{2.21}$$

Λ is trivial on 0 and 1-forms. Their local expressions are

$$(Lu)_{\rho\sigma\lambda_1\cdots\gamma_p} = \varphi_{\rho\sigma} u_{\lambda_1\cdots\lambda_p} - \sum_{i=1}^{p} \varphi_{\lambda_i\sigma} u_{\lambda_1\cdots \overset{i}{\hat{\rho}}\cdots\lambda_p} - \sum_{i=1}^{p} \varphi_{\rho\lambda_i} u_{\lambda_1\cdots \overset{i}{\hat{\sigma}}\cdots\lambda_p} + \sum_{i<j} \varphi_{\lambda_i\lambda_j} u_{\lambda_1\cdots \overset{i}{\hat{\rho}}\cdots \overset{j}{\hat{\sigma}}\cdots\lambda_p} , \tag{2.22}$$

$$(\Lambda u)_{\lambda_3\cdots\lambda_p} = \frac{1}{2} \varphi^{\rho\sigma} u_{\rho\sigma\lambda_3\cdots\lambda_p} . \tag{2.23}$$

The following proposition is well known in a $2m$-dimensional almost Kählerian space.

PROPOSITION 2.10. *We have for a p-form u,*

$$(dL - Ld)u = 0 , \qquad (\delta\Lambda - \Lambda\delta)u = 0 , \tag{2.24}$$

$$(\Lambda L - L\Lambda)u = (m-p)u . \tag{2.25}$$

PROPOSITION 2.11.

$$(\delta L - L\delta)u = (\Gamma + \delta)u , \tag{2.26}$$

$$(d\Lambda - \Lambda d)u = -(C + c)u . \tag{2.27}$$

As a corollary, we can obtain

$$(\delta\Gamma + \Gamma\delta)u = -(\delta\gamma + \gamma\delta)u , \tag{2.28}$$

$$(dC + Cd)u = -(dc + cd)u . \tag{2.29}$$

PROPOSITION 2.12.

$$(L\Phi - \Phi L)u = 0 , \qquad (L\Psi - \Psi L)u = 0 , \tag{2.30}$$

$$(\Lambda\Phi - \Phi\Lambda)u = 0 , \qquad (\Lambda\Psi - \Psi\Lambda)u = 0 . \tag{2.31}$$

Since the formulas in Proposition 2.12 contain no differentiation they hold good in almost Hermitian spaces. In almost Kählerian spaces, we have

$$d\varphi = \delta\varphi = 0 .$$

Corresponding to these equations, the following is valid.

LEMMA 2.13. *In an almost Kählerian space, we have*

$$\Gamma\varphi = C\varphi = 0 , \tag{2.32}$$

$$\gamma\varphi = c\varphi = 0 , \tag{2.33}$$

$$\mathfrak{D}\varphi = \vartheta\varphi = 0 . \tag{2.34}$$

PROOF. As the tensor $\nabla_\lambda \varphi_{\mu\nu}$ is pure with respect to λ, μ, ν, we get

$$\varphi_\lambda{}^\rho \nabla_\mu \varphi_{\nu\rho} = \varphi_\mu{}^\rho \nabla_\rho \varphi_{\nu\lambda} .$$

We denote by $\mathfrak{S}$ (λ, μ, ν) a cyclic sum of the indices λ, μ, ν. As the

2-form φ is closed, we have

$$(\Gamma\varphi)_{\lambda\mu\nu}=\mathfrak{S}_{(\lambda,\mu,\nu)}\varphi_\lambda{}^\rho\nabla_\rho\varphi_{\mu\nu}$$
$$=-\mathfrak{S}\varphi_\mu{}^\rho\nabla_\rho\varphi_{\nu\lambda}-\mathfrak{S}\varphi_\nu{}^\rho\nabla_\rho\varphi_{\lambda\mu}$$
$$=-2(\Gamma\varphi)_{\lambda\mu\nu}\,,$$

which means $\Gamma\varphi=0$. As for $C\varphi$, since $\varphi_{\lambda\mu}\varphi^{\lambda\mu}=n$ holds good, we obtain that

$$(C\varphi)_\lambda=\varphi^{\rho\sigma}\nabla_\rho\varphi_{\sigma\lambda}$$
$$=-\frac{1}{2}\varphi^{\rho\sigma}\nabla_\lambda\varphi_{\rho\sigma}=0\,.$$

Substituting $\Gamma\varphi=C\varphi=0$ into (2.3, 4) and (2.5, 6), and taking account of $\Phi\varphi=0$, we see that (2.33) and (2.34) are true.

PROPOSITION 2.14.

$$(\Gamma L-L\Gamma)u=0\,, \tag{2.35}$$

$$(C\Lambda-\Lambda C)u=0\,. \tag{2.36}$$

PROOF. Since Γ is a skew-derivation, it holds that

$$\Gamma Lu=\Gamma(\varphi\wedge u)$$
$$=\Gamma(\varphi)\wedge u+\varphi\wedge\Gamma u=L\Gamma u$$

for any p-form u. The formula (2.36) is only dual to (2.35).

The following two propositions can be proved by the same way.

PROPOSITION 2.15.

$$(\gamma L-L\gamma)u=0\,, \tag{2.37}$$

$$(c\Lambda-\Lambda c)u=0\,. \tag{2.38}$$

PROPOSITION 2.16.

$$(\mathfrak{D}L-L\mathfrak{D})u=0\,, \tag{2.39}$$

$$(\vartheta\Lambda-\Lambda\vartheta)u=0\,. \tag{2.40}$$

Moreover we can calculate the following relations.

PROPOSITION 2.17.

$$(\Gamma\Lambda-\Lambda\Gamma)u=(\delta-\vartheta)u\,, \tag{2.41}$$

$$(CL-LC)u=(d-\mathfrak{D})u\,. \tag{2.42}$$

PROPOSITION 2.18.

$$(\gamma\Lambda-\Lambda\gamma)u=\vartheta u\,, \tag{3.43}$$

$$(cL-Lc)u=-\mathfrak{D}u\,. \tag{2.44}$$

PROPOSITION 2.19.

(2.45) $$(\mathfrak{D}\Lambda-\Lambda\mathfrak{D})u=-cu\,,$$

(2.46) $$(\vartheta L-L\vartheta)u=\gamma u\,.$$

3. Formulas related to the Laplacian operator

Let M^n be an almost Kählerian space. We denote by $R_{\lambda\mu\nu\omega}$ and $R_{\lambda\mu}$ the Riemannian curvature tensor and the Ricci tensor. We put

(3.1) $$S_{\lambda\mu}=\varphi_\lambda{}^\rho R_{\rho\mu}+\frac{1}{2}\varphi^{\rho\sigma}R_{\rho\sigma\lambda\mu}\,,$$

(3.2) $$S_{\lambda\mu\nu\omega}=\varphi_\lambda{}^\rho R_{\rho\mu\nu\omega}-\varphi_\mu{}^\rho R_{\rho\lambda\nu\omega}\,.$$

Then $S_{\lambda\mu}$ and $S_{\lambda\mu\nu\omega}$ are skew-symmetric with respect to the indices λ, μ and ν, ω. It is well known that in a Kählerian space, $S_{\lambda\mu}=S_{\lambda\mu\nu\omega}=0$ hold good. Conversely if $S_{\lambda\mu}=0$ in an almost Kählerian space, then it is Kählerian [5]. It is evident that $S_{\lambda\mu\nu\omega}=0$ means $S_{\lambda\mu}=0$. Making use of these tensors $S_{\lambda\mu}$ and $S_{\lambda\mu\nu\omega}$, we can obtain the following formulas.

PROPOSITION 3.1.

(3.3) $$\begin{aligned}(d\Gamma u+\Gamma du)_{\lambda_1\cdots\lambda_{p+2}}=&\sum_{i<j}(-1)^{i+j}\nabla^\rho\varphi_{\lambda_i\lambda_j}\nabla_\rho u_{\lambda_1\cdots\hat{i}\cdots\hat{j}\cdots\lambda_{p+2}}\\&+\sum_{i<j,\ k\neq i,j}(-1)^{i+j}S_{\lambda_i\lambda_j\lambda_k}{}^\rho u_{\lambda_1\cdots\hat{i}\cdots\hat{j}\cdots\overset{k}{\hat{\rho}}\cdots\lambda_{p+2}}\,,\end{aligned}$$

(3.4) $$\begin{aligned}(\delta Cu+C\delta u)_{\lambda_3\cdots\lambda_p}=&\frac{1}{2}\nabla^\rho\varphi^{\sigma\tau}\nabla_\rho u_{\sigma\tau\lambda_3\cdots\lambda_p}\\&+S^{\tau\omega}u_{\tau\omega\lambda_3\cdots\lambda_p}+\frac{1}{2}\sum_{i=3}^{p}S_{\lambda_i}{}^{\rho\sigma\tau}u_{\sigma\tau\lambda_3\cdots\overset{i}{\hat{\rho}}\cdots\lambda_p}\,.\end{aligned}$$

PROPOSITION 3.2.

(3.5) $$\begin{aligned}(\delta\Gamma u+\Gamma\delta u)_{\lambda_1\cdots\lambda_p}=&\sum_{i=1}^{p}\nabla^\rho\varphi_{\lambda_i}{}^\sigma\nabla_\sigma u_{\lambda_1\cdots\overset{i}{\hat{\rho}}\cdots\lambda_p}\\&+\sum_{i=1}^{p}S_{\lambda_i}{}^\rho u_{\lambda_1\cdots\overset{i}{\hat{\rho}}\cdots\lambda_p}+\frac{1}{2}\sum_{i<j}S_{\lambda_i\lambda_j}{}^{\rho\sigma}u_{\lambda_1\cdots\overset{i}{\hat{\rho}}\cdots\overset{j}{\hat{\sigma}}\cdots\lambda_p}\,,\end{aligned}$$

(3.6) $$\begin{aligned}(dCu+Cdu)_{\lambda_1\cdots\lambda_p}=&\sum_{i=1}^{p}\nabla_{\lambda_i}\varphi^{\rho\sigma}\nabla_\rho u_{\lambda_1\cdots\overset{i}{\hat{\sigma}}\cdots\lambda_p}\\&+\frac{1}{2}\sum_{i<j}S_{\lambda_i\lambda_j}{}^{\rho\sigma}u_{\lambda_1\cdots\overset{i}{\hat{\rho}}\cdots\overset{j}{\hat{\sigma}}\cdots\lambda_p}\,.\end{aligned}$$

Next we look for the relations with respect to the Laplacian operator $\Delta=d\delta+\delta d$.

PROPOSITION 3.3.

(3.7) $$\frac{1}{2}(\Delta L-L\Delta)u=(d\Gamma+\Gamma d)u\,,$$

(3.8) $$\frac{1}{2}(\Delta\Lambda-\Lambda\Delta)u=-(\delta C+C\delta)u\,.$$

PROOF. By virtue of Propositions 2.10 and 2.11, we have

$$(d\delta L - Ld\delta)u = d(\Gamma + \gamma)u\,,$$

$$(\delta dL - L\delta d)u = (\Gamma + \gamma)du\,.$$

Adding side by side, and taking account of $d\Gamma + \Gamma d = d\gamma + \gamma d$, we get

$$(\varDelta L - L\varDelta)u = 2(d\Gamma + \Gamma d)u\,.$$

The formula (3.8) can be similarly proved.

PROPOSITION 3.4.

$$-\frac{1}{2}(\varDelta\Phi - \Phi\varDelta)u = (dC + Cd + \delta\Gamma + \Gamma\delta)u\,. \tag{3.9}$$

PROOF. We use Proposition 2.2. Then we have

$$\delta d\Phi u - (\Phi\delta - C + c)du = -\delta(\Gamma - \gamma)u\,,$$

$$d\delta\Phi u - (\Phi d - \Gamma + \gamma)\delta u = -d(C - c)u\,.$$

Hence taking consideration of (2.28, 29), we see that

$$\begin{aligned}(-\varDelta\Phi + \Phi\varDelta)u &= (\delta\Gamma + \Gamma\delta - \delta\gamma - \gamma\delta)u + (dC + Cd - dc - cd)u \\ &= 2(\delta\Gamma + \Gamma\gamma + dC + Cd)u\,.\end{aligned}$$

PROPOSITION 3.5. *In an almost Kählerian space, we have*

$$\begin{aligned}(\Gamma Cu + C\Gamma u)_{\lambda_1\cdots\lambda_p} = (\varDelta u)_{\lambda_1\cdots\lambda_p} &+ \sum_{i=1}^{p} \varphi_{\lambda_i}{}^{\sigma}\nabla^{\rho}\varphi_{\sigma}{}^{\tau}\nabla_{\rho}u_{\lambda_1\cdots\overset{i}{\hat{\tau}}\cdots\lambda_p} \\ &- \sum_{i=1}^{p} \varphi_{\lambda_i}{}^{\rho}S_{\rho}{}^{\sigma}u_{\lambda_1\cdots\overset{i}{\hat{\sigma}}\cdots\lambda_p} - \sum_{i\neq j} S_{\lambda_i}{}^{\rho}{}_{\lambda_j}{}^{\sigma}\varphi_{\rho}{}^{\tau}u_{\lambda_1\cdots\overset{i}{\hat{\sigma}}\cdots\overset{j}{\hat{\tau}}\cdots\lambda_p}\,,\end{aligned} \tag{3.10}$$

for any p-form $u = (u_{\lambda_1\cdots\lambda_p})$.

THEOREM 3.6. ([3]) *An almost Kählerian space is Kählerian if and only if*

$$(\varDelta L - L\varDelta)f = 0$$

holds good for any scalar function f.

PROOF. It is an easy result of Proposition 3.3 and Corollary 1.7.

THEOREM 3.7. *An almost Kählerian space is Kählerian if and only if*

$$(\delta C + C\delta)u = 0$$

or equivalently

$$(\varDelta\Lambda - \Lambda\varDelta)u = 0$$

holds good for any 2-form u.

PROOF. By virtue of the formula (3.4), we have for a 2-form $u = (u_{\lambda\mu})$

$$(\delta C + C\delta)u = \frac{1}{2}\nabla^{\rho}\varphi^{\sigma\tau}\nabla_{\rho}u_{\sigma\tau} + S^{\rho\sigma}u_{\rho\sigma}\,.$$

Clearly the vanishing of this right hand side for any $u_{\lambda\mu}$ means $S^{\rho\sigma}=0$ since $S_{\lambda\mu}$ is skew-symmetric. Hence the almost Kählerian structure is Kählerian.

THEOREM 3.8. *An almost Kählerian space is Kählerian if and only if*

$$(\Delta\Phi-\Phi\Delta)u=0$$

holds good for any 1-form u.

PROOF. Making use of Proposition 3.2, we have

$$(\delta\Gamma u+\Gamma\delta u+dCu+Cdu)_\lambda=\nabla^\rho\varphi_\lambda{}^\sigma\nabla_\rho u_\sigma+S_\lambda{}^\rho u_\rho$$

for a 1-form $u=(u_\lambda)$. The left hand side is equal to $(\Phi\Delta u-\Delta\Phi u)/2$, and thus the right hand side is zero for any u_λ, from which we can easily conclude that $S_{\lambda\mu}=0$.

Now we assume that the almost Kählerian space M^n is compact. We denote by $\langle u, v\rangle$ and (u, v) the local and global inner products of p-forms u and v. Then we have

$$(u, v)=\int_{M^n}u\wedge *v=\int_{M^n}\langle u, v\rangle dV$$

where dV is the volume element of M^n and

$$\langle u, v\rangle=\frac{1}{p!}u_{\lambda_1\cdots\lambda_p}v^{\lambda_1\cdots\lambda_p}\,.$$

It is a fundamental and well known fact that

$$(du, v)=(u, \delta v) \tag{3.11}$$

is valid for a p-form u and a $(p+1)$-form v. A local calculation shows that

$$\langle Lu, v\rangle=\langle u, \Lambda v\rangle$$

holds good for a p-form u and a $(p+2)$-form v. Hence we have

$$(Lu, v)=(u, \Lambda v)\,. \tag{3.12}$$

PROPOSITION 3.9. *In a compact almost Kählerian space, we have*

$$\langle H_r u, v\rangle=(-1)^r\langle u, H_r v\rangle$$

for p-forms u and v, and hence

$$(H_r u, v)=(-1)^r(u, H_r v)\,. \tag{3.13}$$

COROLLARY 3.10.

$$(\Phi u, v)=-(u, \Phi v)\,. \tag{3.14}$$

THEOREM 3.11. *In a compact almost Kählerian space, we see that*

$$(\Gamma u, v)=(u, Cv) \tag{3.15}$$

is true for a p-form u and a $(p+1)$-form v.

PROOF. We have

$$\langle \Gamma u, v\rangle = \nabla_\rho[\sum_{\alpha=0}^{p}(-1)^\alpha \varphi_{\lambda_\alpha}{}^\rho u_{\lambda_0\cdot\hat{\alpha}\cdot\lambda_p} v^{\lambda_1\cdots\lambda_p}]/(p+1)!$$
$$-\sum_{\alpha=0}^{p}\varphi^{\lambda_\alpha\rho}u^{\lambda_0\cdots\hat{\alpha}\cdots\lambda_p}\nabla_\rho v_{\lambda_\alpha\lambda_0\cdot\hat{\alpha}\cdot\lambda_p}/(p+1)!$$

in an almost semi-Kählerian space. The first term in the right hand side is zero when we integrate it on M^n. As for the second term, we see

$$-\sum_{\alpha=0}^{p}\varphi^{\lambda_\alpha\rho}u^{\lambda_0\cdot\hat{\alpha}\cdot\lambda_p}\nabla_\rho v_{\lambda_\alpha\lambda_0\cdot\hat{\alpha}\cdot\lambda_p} = (p+1)u^{\lambda_1\cdots\lambda_p}\varphi^{\rho\sigma}\nabla_\rho v_{\sigma\lambda_1\cdots\lambda_p},$$

therefore it becomes $\langle u, Cv\rangle$, and the theorem is proved.

PROPOSITION 3.12. *In a compact almost Kählerian space, we have*

(3.16) $$(\gamma u, v) = (u, cv)$$

(3.17) $$(\mathfrak{D}u, v) = (u, \vartheta v)$$

hold good for a p-form u and a $(p+1)$-form v.

4. Covariant pseudo analytic forms on a compact Kählerian space

We know that in a Kählerian space M^n, it holds that

$$\gamma = \mathfrak{D} = 0, \quad c = \vartheta = 0, \quad S_{\lambda\mu\nu\omega} = S_{\lambda\mu} = 0.$$

Therefore the formulas in § 2 and § 3 can be slightly simplified. We make the list of them in the following.

PROPOSITION 4.1. *In a Kählerian space, we have*

(4.1) $$d\Phi - \Phi d = -\Gamma, \qquad \delta\Phi - \Phi\delta = -C,$$

(4.2) $$\Gamma\Phi - \Phi\Gamma = d, \qquad C\Phi - \Phi C = \delta,$$

(4.3) $$\delta L - L\delta = \Gamma, \qquad d\Lambda - \Lambda d = -C,$$

(4.4) $$\Gamma\Lambda - \Lambda\Gamma = \delta, \qquad CL - LC = -d,$$

(4.5) $$d\Gamma + \Gamma d = 0, \qquad \delta C + C\delta = 0,$$

(4.6) $$\delta\Gamma + \Gamma\delta = 0, \qquad dC + Cd = 0.$$

By virtue of Proposition 3.5, we see that

(4.7) $$\Gamma C + C\Gamma = \Delta$$

holds good in a Kählerian space. Since Γ and C are adjoint operators in a compact case, we have

THEOREM 4.2. *In a compact Kählerian space, it is necessary and sufficient for a p-form u to be harmonic that u satisfies the equations*

$$\Gamma u = Cu = 0.$$

Now we treat of the covariant analytic tensors. Let M^n be a compact Kählerian space. Then a pure tensor is called to be analytic if its covariant derivative is also pure. Let $\varphi=(\varphi_\lambda{}^\mu)$ be the complex structure. Yano-Ako [4] defined a covariant analytic tensor u of order p by the condition that u is pure and satisfies for any vector fields $X, Y_1, \cdots, Y_p$

$$(\theta(\varphi X)u-\theta(X)(u\circ\varphi))(Y_1, \cdots, Y_p)+ \\ +\sum_{\alpha=2}^{p} u(Y_1, \cdots, \varphi\theta(X)Y_\alpha, \cdots, Y_p) \\ -\sum_{\alpha=2}^{p} u(\varphi Y_1, \cdots, \theta(X)Y_\alpha, \cdots, Y_p)=0\,, \tag{4.8}$$

where $\theta(X)$ denotes the Lie derivative with respect to X and $u\circ\varphi$ is defined by

$$(u\circ\varphi)(X_1, X_2, \cdots, X_p)=u(\varphi X_1, X_2, \cdots, X_p)\,.$$

They remarked that if u is free from the condition of purity and is skew-symmetric, then the condition (4.8) becomes

$$i_\varphi du-2di_\varphi u=0\,. \tag{4.9}$$

If u is a 1-form, then (4.8) is equal to

$$\varphi_\lambda{}^\rho\nabla_\rho u_\mu-\varphi_\mu{}^\rho\nabla_\lambda u_\rho=0\,. \tag{4.8$'$}$$

Taking the skew part of (4.8)$'$, we obtain

$$\varphi_\lambda{}^\rho(\nabla_\rho u_\mu+\nabla_\mu u_\rho)-\varphi_\mu{}^\rho(\nabla_\rho u_\lambda+\nabla_\lambda u_\rho)=0\,, \tag{4.9$'$}$$

which are the same as (4.9) in the case of 1-form.

From the above consideration, we take a p-form u which is a skew-symmetric tensor of covariant order p and call it to be covariant pseudo analytic or simply pseudo analytic when it satisfies the equation (4.9). Since φ is a (1, 1)-tensor, we have

$$i_\varphi u=u\overline{\wedge}\varphi=\Phi u$$

for a p-form u. This implies that a p-form u is pseudo analytic if and only if it satisfies

$$\Phi du-2d\Phi u=0\,. \tag{4.10}$$

From (4.1), the equation is equivalent to

$$d\Phi u-\Gamma u=0 \tag{4.11}$$

or

$$\Phi du-2\Gamma u=0 \tag{4.12}$$

in a Kählerian space. We study the properties of such forms on M^n.

If we take a harmonic form u, then we have $\Gamma u=0$ and $du=0$ by virtue of Theorem 4.2. Hence (4.12) is trivially true and u is pseudo analytic. A closed form u is pseudo analytic if and only if Φu is again closed or it satisfies $\Gamma u=0$.

Suppose that a p-form u $(p\neq 0)$ is pure. If its covariant derivative ∇u is pure too, then we see that du is pure and that

$$Cu=\delta u=0\,, \qquad d\Phi u=p\Gamma u\,, \qquad \Phi du=(p+1)\Gamma u$$

hold good. Hence Γu is closed and coclosed from (4.6), which means that $p\Gamma u=d\Phi u$ is harmonic. Therefore $\Gamma u=0$ is valid. By virtue of Theorem 4.2, u must be harmonic. Thus we have

THEOREM 4.3. ([6]) *If the covariant derivative ∇u is pure for a pure p-form u in a compact Kählerian space, then u is effective harmonic and covariant pseudo analytic.*

THEOREM 4.4. *In a compact Kählerian space, let u be a covariant pseudo analytic p-form. Then $\Phi^k u$ is pseudo analytic for any positive integer k if and only if u satisfies the equation*

$$\Phi^2 du=-4du\,. \tag{4.13}$$

PROOF. We show by the induction. For a pseudo analytic form u, we have

$$d\Phi(\Phi u)-\Gamma(\Phi u)=-\frac{1}{2}(\Phi^2 u+4du)\,,$$

therefore (4.13) is equivalent to the fact that Φu is pseudo analytic. We assume that $u, \Phi u, \cdots, \Phi^{k-1}u$ $(k\geqq 1)$ are all pseudo analytic. Then we can obtain by a similar calculation that

$$d\Phi(\Phi^k u)-\Gamma(\Phi^k u)=-2^{k-2}\Phi^{k-1}(\Phi^2 du+4du)\,.$$

This shows that $\Phi^k u$ is pseudo analytic under the condition (4.13).

COROLLARY 4.5. *For a closed pseudo analytic form u, $\Phi^k u$ is pseudo analytic, too.*

Next we consider a p-form u satisfying the condition

$$\Phi^2 u=-p^2 u\,. \tag{4.14}$$

For a 1-form (4.14) is always valid, and if u is pure, then (4.14) is satisfied. From (4.1) and (4.2) we get

$$\Phi^2 du=-(p^2-1)du+2\Phi\Gamma u\,.$$

THEOREM 4.6. *In a compact Kählerian space, suppose that a p-form u $(p\geqq 2)$ satisfies (4.14). Then the necessary and sufficient condition for u to be covariant pseudo analytic is that*

$$d\Phi u=0\,.$$

PROOF. We have from (4.1) and (4.13)

$$\begin{aligned}(d\Phi u, d\Phi u)&=-(\Phi^2 du, du)-2(\Gamma u, \Phi du)+(\Gamma u, \Gamma u)\\&=(p^2-1)(du, du)+(\Gamma u, \Gamma u)\,.\end{aligned} \tag{4.15}$$

Therefore $d\Phi u=0$ means $du=0$ and $\Gamma u=0$ if $p\neq 1$, and u is pseudo analytic. Conversely, if u is pseudo analytic, then we have $d\Phi u=\Gamma u$. Then the above (4.15) becomes $(p^2-1)(du, du)=0$, hence u is a closed form. From the definition (4.10), we can conclude that $d\Phi u=0$.

We pursue covariant pseudo analytic 1-forms which are excluded in Theorem 4.6. For a 1-form u, (4.14) is valid and (4.15) is of the form

$$(d\Phi u, d\Phi u)=(\Gamma u, \Gamma u)\,. \tag{4.16}$$

Therefore for a 1-form u, $d\Phi u=0$ is equivalent to $\Gamma u=0$. We want to make up the integral formula for a 1-form to be analytic.

LEMMA 4.7. *In a compact Kählerian space, it holds that*

$$(dCu+\Gamma\delta u, \Phi u)=(\delta u, \delta u)-(Cu, Cu)$$

for a p-form u.

PROOF. By virtue of (4.6), (4.2) and (4.1), we have

$$\begin{aligned}(dCu, \Phi u)&=-(u, \delta\Gamma\Phi u)\\ &=-(u, \delta\Phi\Gamma u)-(u, \delta du)\\ &=-(u, \Phi\delta\Gamma u)+(u, C\Gamma u-\delta du)\\ &=(\Phi u, \delta\Gamma u)+(u, d\delta u-\Gamma Cu)\,.\end{aligned}$$

It follows easily that the lemma is true.

THEOREM 4.8. *For a 1-form u in a compact Kählerian space, we have the following integral formula.*

$$\frac{1}{2}(d\Phi u-\Gamma u, d\Phi u-\Gamma u)=(\delta u, \delta u)+(\Gamma u, \Gamma u)\,. \tag{4.17}$$

PROOF. We put $A=\Gamma u-d\Phi u$. A is a 2-form whose components are

$$A_{\lambda\mu}=\varphi_\lambda{}^\rho(\nabla_\rho u_\mu+\nabla_\mu u_\rho)-\varphi_\mu{}^\rho(\nabla_\rho u_\lambda+\nabla_\lambda u_\rho)\,.$$

Then we can show that

$$\begin{aligned}\nabla^\lambda A_{\lambda\mu}&=(dCu)_\mu+(\Gamma\delta u)_\mu+(\Phi\Delta u)_\mu\,,\\ \varphi^{\mu\rho}\nabla_\rho A_{\lambda\mu}&=(d\delta u)_\lambda-(\Gamma Cu)_\lambda+(\Delta u)_\lambda\,.\end{aligned}$$

Therefore we have

$$\begin{aligned}(A, A)&=\frac{1}{2}\int A_{\lambda\mu}A^{\lambda\mu}dV\\ &=\int \nabla^\lambda A_{\lambda\mu}\varphi^{\mu\sigma}u_\sigma dV+\int \varphi^{\mu\rho}\nabla_\rho A_{\lambda\mu}u^\lambda dV\\ &=(dCu+\Gamma\delta u, \Phi u)-(\Delta u, \Phi^2 u)+(d\delta u-\Gamma Cu+\Delta u, u)\,.\end{aligned}$$

Making use of Lemma 4.7 and (4.14), we get

$$\frac{1}{2}(A, A)=(d\delta u-\Gamma Cu+\Delta u, u)$$

$$=(d\delta u+C\Gamma u, u)=(\delta u, \delta u)+(\Gamma u, \Gamma u)\,.$$

THEOREM 4.9. *In a compact Kählerian space, a 1-form u is covariant pseudo analytic if and only if it satisfies*

$$\Gamma u=0\,,$$

and then u is coclosed.

PROOF. Necessity easily follows from (4.17). If a 1-form u satisfies $\Gamma u=0$, then $d\Phi u=0$ is valid from (4.16). Thus u is pseudo analytic and $\delta u=0$ is obtained from (4.17).

Lastly we see the integral formula for a p-form u to be pseudo analytic. We have

$$(d\Phi u-\Gamma u, d\Phi u-\Gamma u)=(d\Phi u, d\Phi u)+(\Gamma u, \Gamma u)-2(d\Phi u, \Gamma u)$$

and making use of (4.6) and (4.2)

$$(d\Phi u, \Gamma u)=-(dC\Phi u, u)=-(\Phi Cu+\delta u, \delta u)$$

$$=(Cu, \Phi\delta u)-(\delta u, \delta u)\,.$$

Hence we obtain

THEOREM 4.10. *In a compact Kählerian space, the formula*

$$(4.18)\qquad (d\Phi u-\Gamma u, d\Phi u-\Gamma u)+2(Cu, \Phi\delta u)=$$

$$2(\delta u, \delta u)+(\Gamma u, \Gamma u)+(d\Phi u, d\Phi u)$$

holds good for any p-form u.

From (4.18), we have the following two theorems.

THEOREM 4.11. *In a compact Kählerian space, let u be a coclosed p-form. Then u is covariant pseudo analytic if and only if it satisfies*

$$\Gamma u=d\Phi u=0\,.$$

THEOREM 4.12. *In a compact Kählerian space, let u be a p-form satisfying $Cu=0$. If u is covariant pseudo analytic, then u is harmonic.*

Bibliography

[1] Frölicher, A. and A. Nijenhuis: Theory of vector-valued forms, I; II. Proc. Kon. Ned. Akad. Wet. Amsterdam, 59 (1956), 339–350; 351–359.
[2] Frölicher, A. and A. Nijenhuis: Some new cohomology invariants for complex manifold, I; II. ibid., 59 (1956), 540–552; 553–564.
[3] Hsiung, C.C.: Structures and operators on almost Hermitian manifolds. Trans. Amer. Math. Soc., 122 (1966), 136–152.
[4] Yano, K. and M. Ako: On certain operators associated with tensor fields. Kôdai Math. Sem. Rep., 20 (1968), 414–436.
[5] Yano, K.: Differential geometry on complex and almost complex spaces. Pergamon Press, 1965.

[6] Tachibana S.: Analytic tensor and its generalization. Tôhoku Math. Jour., 12 (1960), 208-221.
[7] Ogawa, Y.: On C-Killing forms in a compact Sasakian space. Tôhoku Math. Jour., 19 (1967), 467-484.
[8] Ogawa, Y.: Operators on almost Hermitian manifolds. Jour. Differential Geometry, 4 (1970), 105-119.