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On Dirichlet Spaces with Homeomorphic Silov Boundaries

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§1. Preliminaries

Let C(X) be the set of all continuous **real**-valued functions on a compact Hausdorff space X and let B be a separating subspace of C(X) containing constant functions. In [1], Prof. Bauer has proved some necessary and sufficient conditions that any continuous function on the Silov boundary Γ of X with respect to B is extended to a function in B. Prof. Bear has called B a Dirichlet space when B satisfies the foregoing condition.

In the present note we shall prove some theorems with respect to two Dirichlet spaces with homeomorphic Silov boundaries.

§2. Definitions and the equivalent conditions

Let C(X) be the set of all continuous real-valued functions on X with uniform norm. We consider a linear separating subspace B of C(X) containing constant functions. We denote Γ the Silov boundary of X with respect to B. Let B* be the space, of all continuous linear functionals of B, as usual with W*-topology. The set of all W*-continuous linear functionals on B* are isomorphic to B under the natural correspondence. We may therefore regard B as the space of continuous linear functionals on B , or their restriction to subsets of B*.

We shall write

$$T_B=\{F\in B\ ;\ F(1)=\|F\|=1\} \quad \cdots\cdots\cdots\cdots (1)$$

and consider X as embedded in B*. Then evidently $X\subset T_B$. T_B is known to be the closed convex hull of X in B*. Let P(X) be the set of all probability measures on X. For each point x of X, we put

$$M_x=\{\mu\in P(X)\ ;\ \mu(u)=u(x),\ \forall u\in B\}\ .$$

We denote ε_x the point measure of x. A point x of X is called a point of Choquet boundary of X with respect to B when $M_x=\{\varepsilon_x\}$. Prof. Bauer has proved that the Choquet boundary is the set of all extreme points of T_B.

Now B will be called a Dirichlet space on X if $B|\Gamma = |C(\Gamma)$[1]. From (1) it is clear that T_B can be identified with the set of all probability measures on Γ if B is a Dirichlet space.

THEOREM 1. (Bauer)
Let X be a compact Hausdorff space and B be a separating subspace of C(X) containing constant functions. The following assertions are equivalent;

(a) B is a Dirichlet space on X,
(b) B is uniformly closed and a lattice in natural order,
(c) for each point $x \in X$ there exists a unique probability measure on Γ representing x,
(d) T_B is a simplex in the sense of Choquet [5] and the set of all extreme points of T_B is closed.

Here we shall say that a measure μ on Γ represents a point $x \in X$ if

$$u(x) = \int_\Gamma u d\mu \qquad (\forall u \in B).$$

Several necessary and sufficient conditions for a convex compact subset of a topological vector space to be a simplex have been discovered.

Example. Let X be the unit closed disk in $\mathbf{R}^2$ and H be the set of all functions, continuous on X and harmonic in the unit open disk. Then H is a Dirichlet space.

THEOREM 2. (Bauer)
For each convex compact subset X of a topological vector space E, the following three assertions are equivalent;

(a) X is a simplex and the set X_e, of all extreme points of X, is closed,
(b) for each x in X there exists a unique probability measure μ_x on $\overline{X}_e$ such that x is the barycenter of μ_x.
(c) the space of all restrictions on X of continuous affine functions is a Dirichlet space on X.

§ 3. A main theorem

THEOREM 3.
Let B_i ($i=1, 2$) be Dirichlet spaces on compact Hausdorff spaces on X_i. If their Silov boundaries Γ_1 and Γ_2 are homeomorphic, then T_{B_1} and T_{B_2} are both topologically and affinely isomorphic.

PROOF. Since, for each i ($i=1, 2$), B_i is a Dirichlet space, T_{B_i} becomes a simplex and the set $(T_{B_i})_e$ of all extreme points is closed from

1) We denote by $B|\Gamma$ the set of all restrictions Γ of $u \in B$.

theorem 1. Therefore $(T_{B_i})_e=\Gamma_i$. T_{B_i} is a convex compact set in the topological vector space B_i^k and from theorem 2 for each $x_i\in T_{B_i}$, there exists a unique probability measure μ_{x_i} on Γ_i of which barycenter is x_i. Since the barycenter of any probability measure on Γ_i is contained in T_{B_i}, we can set a one-to-one correspondence φ_i between T_{B_i} and the set $P(\Gamma_i)$ of all probability measures on Γ_i. Then φ_i and φ_i^{-1} are continuous. It is clear that φ_i and φ_i^{-1} are affine mappings. Let the homeomorphic mapping from Γ_1 into Γ_2 be h. Further, for any $\mu\in P(\Gamma_1)$ let us put

$$(\phi(\mu))(g)=\int g\circ h d\mu \qquad (\forall g\in C(\Gamma_2)) .$$

Then $\phi(\mu)\in P(\Gamma_2)$, and ϕ is a one-to-one continuous linear mapping between $P(\Gamma_1)$ and $P(\Gamma_2)$. ϕ^{-1} is also a continuous linear mapping. We put $\varphi=\varphi_2^{-1}\circ\phi\circ\varphi_1$. Then both φ and φ^{-1} are continuous affine mappings.

COROLLARY.

Let X_i be a compact convex subset of topological vector space E_i, for each i $(i=1, 2)$, and A_i be the set of all restrictions on X of continuous affine functions in E_i. Assume that X_i is a simplex and the set of all extreme points of X_i is closed. Then X_1 and X_2 are both topologically and affinely isomorphic provided that their Silov boundaries Γ_1 and Γ_2 should be homeomorphic.

REMARK. Under the mere conditions of theorem 3, X_1 and X_2 in general are not necessarily homeomorphic. For example, in let H the same as the example in § 2. Let X_1 be the union of the disk $||x||\leqq 1/2$ and the unit circle. Let X_2 be the union of the circle $||x||=1/2$, the unit circle and the set of all points between the two circles. We put B_i all restrictions of H on X_i $(i=1, 2)$. Then each of the Silov boundaries of X_i is the unit circle. The Dirichlet space B_1 and B_2 satisfy the conditions of theorem 3, but X_1 and X_2 are not homeomorphic.

§ 4. Gleason parts of Dirichlet spaces

We decompose the convex set T_B into disjoint convex subsets, called parts or Gleason parts of T_B.

The relation $\sim$ is defined as follows; $x\sim y$ if and only if there exists a constant $c>1$ such that

$$1/c<u(x)/u(y)<c$$

for all positive $u\in B$. The equivalent classes of this relation are called parts or Gleason parts of T_B.

For any x in T_B, $F_x=\{x\}$ if x is an extreme point of T_B and $F_x=$ {the union of open interval in T_B containing x} if x is not an extreme point. F_x is called the minimal face containing x.

Prof. Bear has proved in [2] that two points $x_1, x_2 \in T_B$ lie in the same part of T_B if and only if $F_{x_1} = F_{x_2}$. Therefore each part T_B is convex.

THEOREM 4.

Under the same conditions of theorem 3, two points $x_1, y_1 \in T_{B_1}$ lie in the same part of T_{B_1} if and only if corresponding points $x_2, y_2 \in T_{B_2}$ lie in the same part of T_{B_2}.

PROOF. We assume $F_{x_2} \neq F_{y_2}$, then any open interval in T_{B_2} containing x_2 and any open interval in T_{B_2} containing y_2 do not intersect. Since the correspondence between T_{B_1} and T_{B_2} is affine from theorem 3, any open interval in T_{B_1} containing x_1 and any open interval in T_{B_1} containing y_1 do not intersect which means $F_{x_1} \neq F_{x_2}$. Similarly $F_{x_1} \neq F_{x_2}$ implies $F_{x_2} \neq F_{y_2}$.

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