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On the Invariants of Algebraically Compact Groups and Compact Abelian Groups

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We are concerned only with abelian groups. Terminology and notations mainly follow [5]. Throughout this paper, $\alpha \cdot G$ and G^{α} stand for a direct sum of α-copies of G and a direct product of α-copies of G respectively, where G denotes a group and α denotes a cardinal number. To avoid repetition, let it be agreed that "compact group" will always mean "compact Hausdorff abelian group".

A compact group is algebraically isomorphic with a product of finite cyclic groups, groups of p-adic integers and a divisible group (see 25.25 in [3] or Proposition 3.3 in [2]). We remark that this structure is not invariant, since Lemma 1 in this paper give an isomorphism

$$\prod_{n} Z(p^{n}) \cong \Delta_{p} \oplus \prod_{n} Z(p^{n})$$

where $Z(p^n)$ denotes the cyclic group of order p^n and Δ_p denotes the additive group of p-adic integers.

With the aid of Propositions 2.1 and 2.2 in [2], we shall give a complete set of invariants of a reduced algebraically compact group (see Theorem 3 in this paper). That is, all reduced algebraically compact groups can be characterized by the family of sets $\{\alpha_{p,n}\}$ such that $\alpha_{p,n}$ is a cardinal number for each prime p and non-negative integer n.

From the structure theorem of compact groups, it follows that a compact group is algebraically compact. We shall characterize reduced compact groups by the sets of invariants as stated above.

THEOREM 1. *Let α_p be a cardinal number for each prime p. Then* $\mathrm{Hom}(Q/Z, \sum_{p} \alpha_p \cdot Z(p^{\infty})$ *is a reduced torsion free algebraically compact group. Conversely, if G is a reduced torsion-free algebraically compact group, then the set of cardinal numbers $\{\alpha_p\}$ is uniquely determined such that G is isomorphic with the group stated above. A reduced torsion free algebraically compact group can be a compact group if and only if each α_p is finite or of type 2^{α} for some cardinal number α.*

PROOF. First part follows immediately from Prop. 2.1 in [2] and Theorem 2 in [5].

A reduced torsion-free algebraically compact group G can be compact if and only if $G\cong\prod_p \Delta_p^{\alpha'_p}$, where Δ_p denotes the additive group of p-adic integers.

$$\begin{aligned}\prod \Delta_p^{\alpha'_p} &\cong \prod_p \mathrm{Hom}(Z(p^\infty), Z(p^\infty)^{\alpha'_p}) \cong \prod_p \mathrm{Hom}(Z(p^\infty), (Z(p^\infty)^{\alpha'_p})_t) \\ &\cong \prod_p \mathrm{Hom}(Z(p^\infty), \alpha_p \cdot Z(p^\infty)) \cong \prod_p \mathrm{Hom}(Z(p^\infty), \sum_p \alpha_p \cdot Z(p^\infty)) \\ &\cong \mathrm{Hom}(Q/Z, \sum_p \alpha_p \cdot Z(p^\infty)),\end{aligned}$$

where $\alpha_p = \alpha'_p$ if α_p is finite, otherwise $\alpha_p = 2^{\alpha'_p}$. Thus, second part of our theorem has been proved.

From this theorem, we have a group such as $\mathrm{Hom}(Z(p^\infty), \aleph_0 \cdot Z(p^\infty))$, which is algebraically compact but can not be compact.

Before going further, we need to observe the structure of a group $\prod_n Z(p^n)^{\alpha_n}$.

LEMMA 1. *Let G be a compact group of type $\prod_n Z(p^n)^{\alpha_n}$. Then, G is adjusted (i. e. without torsion-free direct summand) if and only if $\alpha_n = 0$ for all but a finite number of n's, in other words, G is of bounded order.*

PROOF. The sufficiency follows immediately, since a torsion group is adjusted.

Suppose $\alpha_{n_i} \neq 0$ for a infinite sequence of positive integers $n_1 < n_2 < \cdots$. There exists a pure exact sequence

$$0 \to \sum_i Z(p^{n_i}) \to \sum_i Z(p^{n_i}) \to Z(p^\infty) \to 0$$

(see for example Exercises 16 & 17 in [4]). Let G^* be denoted the Pontrjagin dual group of a locally compact Hausdorff abelian group G. Considering $\sum_i Z(p^{n_i})$ as a discrete group, we have $(\sum_i Z(p^{n_i}))^* \cong \prod_i Z(p^{n_i})$. By the duality theory (see for example § 24 in [3], and especially (24.46) (b) for the purity), we have a pure exact sequence

$$0 \to \Delta_p \to \prod_i Z(p^{n_i}) \to \prod_i Z(p^{n_i}) \to 0,$$

where Δ_p denotes the additive group of p-adic integers. Now we have an algebraic isomorphism

$$\prod_i Z(p^{n_i}) \cong \Delta_p \oplus \prod_i Z(p^{n_i}),$$

since Δ_p is algebraically compact. This implies that $\prod_n Z(p^n)^{\alpha_n}$ is not adjusted.

LEMMA 2. *There exists a function $f(\{\alpha_n\}) = \alpha$ such that*

$$\prod_n Z(p^n)^{\alpha_n} \cong \mathrm{Ext}(Q/Z, (\prod_n \alpha'_n \cdot Z(p^n))_t) \oplus \mathrm{Hom}(Z(p^\infty), \alpha \cdot Z(p^\infty)),$$

where $\alpha'_n = \alpha_n$ if α_n is finite, otherwise $\alpha'_n = 2^{\alpha_n}$.

PROOF. This is an immediate consequence of Props. 2.1 and 2.2 in [2].

From Lemma 1 it follows that $f(\{\alpha_n\})=0$ if and only if $\alpha=0$ for all but a finite number of n's. However, we know very little about this function so far. Even in the case $\alpha_n=1$ for all n, we know only $\aleph_0 \leqslant f(\{\alpha_n\}) \leqslant 2^{\aleph_0}$.

THEOREM 2. *Let $\alpha_{p,n}$ be a cardinal number for each prime p and a positive integer n. Then $\mathrm{Ext}(Q/Z, (\prod_{p,n} \alpha_{p,n} Z(p^n))_t)$ is an adjusted, reduced, algebraically compact group. Conversely, let G be a reduced algebraically compact group. If G is adjusted, then a set of cardinal numbers $\{\alpha_{p,n}\}$ is uniquely determined such that*

$$G \cong \mathrm{Ext}(Q/Z, (\prod_{p,n} \alpha_{p,n} \cdot Z(p^n))_t) .$$

An adjusted, reduced, algebraically compact group can be a compact group if and only if each $\alpha_{p,n}$ is finite or of type 2^α and moreover, for each p, $\alpha_{p,n}=0$ for all but a finite number of n's.

PROOF. First part follows immediately from Prop. 2.2 in [2] and Theorem 3 in [5].

Next, suppose G is an adjusted, reduced, algebraically compact group. By Theorem 3 in [5], G_t is closed. From Theorem 29.6 and Corollary 34.2 in [1], it follows that

$$G_t \cong \sum_p (\prod_n \alpha_{p,n} \cdot Z(p^n))_t$$

where $\{\alpha_{p,n}\}$ is a set of invariants of G_t. By Corollary 2 to Theorem 3 in [5], $\{\alpha_{p,n}\}$ is a set of invariants of G too. Since G is adjusted, $G \cong \mathrm{Ext}(Q/Z, G_t)$ by Prop. 2.2 in [2].

Suppose an adjusted, reduced, algebraically compact group is compact. That is,

$$\mathrm{Ext}(Q/Z, (\prod_{p,n} \alpha_{p,n} \cdot Z(p^n))_t) \cong \prod_{p,n} Z(p^n)^{\alpha'_{p,n}} .$$

We know that either $\alpha_{p,n}$ is finite or of type 2^α. Moreover, for each p, $\alpha_{p,n}=0$ for all but a finite number of n's by Lemma 1. Conversely if $\{\alpha_{p,n}\}$ satisfies such conditions as above, then

$$(\prod_{p,n} \alpha_{p,n} \cdot Z(p^n))_t \cong \sum_p (\prod_n \alpha_{p,n} \cdot Z(p^n))_t$$
$$\cong \sum_p \prod_n \alpha_{p,n} \cdot Z(p^n) \cong \sum_p \prod_n Z(p^n)^{\alpha'_{p,n}}$$

where $\prod_n Z(p^n)^{\alpha'_{p,n}}$ is a p-group of bounded order. Hence

$$\mathrm{Ext}(Q/Z, (\prod_{p,n} \alpha_{p,n} \cdot Z(p^n))_t) \cong \prod_p \mathrm{Ext}(Z(p^\infty), \prod_n Z(p^n)^{\alpha'_{p,n}})$$
$$\cong \prod_{p,n} Z(p^n)^{\alpha'_{p,n}} .$$

Thus our proof has been completed.

We note that $\aleph_0 \cdot Z(p)$ is the smallest (in a sense) algebraically compact group which can not be made into a compact group. We also show a simple example of *adjusted*, compact group which is not a torsion group. Let $p_1, p_2, \cdots$ be a infinite sequence of different primes. Then $\prod_i Z(p_i)$ is a desired group.

THEOREM 3. *Let $\alpha_{p,n}$ be a cardinal number for each prime p and non-negative integer n. Then*

$$\mathrm{Ext}(Q/Z, \sum_p (\prod_{n\geqq 1} \alpha_{p,n} \cdot Z(p^n))_t) \oplus \mathrm{Hom}(Q/Z, \sum_p \alpha_{p,0} \cdot Z(p^\infty))$$

is a reduced algebraically compact group. Conversely, if G is a reduced algebraically compact group, then the set of cardinal numbers $\{\alpha_{p,n}\}$ is uniquely determined such that G is isomorphic with the group stated above. A reduced algebraically compact group can be a compact group if and only if $\{\alpha_{p,n}\}$ satisfies following conditions:

For each prime p and $n \geqslant 1$, $\alpha_{p,n}$ is finite or of type 2^α. Moreover for each p, there exists a cardinal number α_p such that α_p is finite or of type 2^α and satisfies following relation, $\alpha_{p,0} = f(\{\alpha'_{p,n}\}) + \alpha_p$ where f is the function in Lemma 1 and $\alpha'_{p,n} = \alpha_{p,n}$ if $\alpha_{p,n}$ is finite, $\alpha_{p,n} = 2^{\alpha'_{p,n}}$ otherwise.

PROOF. First part follows immediately from Prop. 2.2 in [2] and Theorems 1 and 2.

Suppose a reduced group G is compact. Then

$$\begin{aligned} G &\cong \prod_{p,n} Z(p^n)^{\alpha'_{p,n}} \oplus \prod_p \Delta_p^{\alpha'_p} \\ &\cong \prod_p \mathrm{Ext}(Q/Z, (\prod \alpha_{p,n} \cdot Z(p^n))_t) \oplus \prod_p \mathrm{Hom}(Z(p^\infty), f(\{\alpha'_{p,n}\}) \cdot Z(p^\infty)) \\ &\qquad \oplus \prod_p \mathrm{Hom}(Z(p^\infty)), \alpha_p \cdot Z(p^\infty)) \\ &\cong \mathrm{Ext}(Q/Z, \sum_p (\prod_{n\geqq 1} \alpha_{p,n} \cdot Z(p^n))_t) \\ &\qquad \oplus \mathrm{Hom}(Q/Z, \sum_p (f(\{\alpha'_{p,n}\}) + \alpha_p) \cdot Z(p^\infty)), \end{aligned}$$

where $\alpha_{p,n} = \alpha'_{p,n}$ if $\alpha'_{p,n}$ is finite and $\alpha_{p,n} = 2^{\alpha'_{p,n}}$ otherwise, and $\alpha_p = \alpha'_d$ if α'_p is finite and $\alpha_p = 2^{\alpha'_p}$ otherwise. From the first part of this theorem, we have the relation $\alpha_{p,0} = f(\{\alpha'_{p,n}\}) + \alpha_p$.

Conversely, if $\{\alpha_{p,n}\}$ satisfies the conditions as stated above, apparently G can be compact. Thus our assertions have been proved.

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