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Empty Spaces of Staeckel

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Spaces are determined such that they satisfy Einstein's field equations for empty spaces with or without cosmological terms and that the equation of geodesic there is integrable by separation of variables. A new space with cosmological terms is obtained which may be represented as the intersection of two cylindrical surfaces in a six dimensional euclidian space.

§ 1. Introduction

Several solutions of Einstein's field equations for empty spaces have been obtained by Schwarzschild,[1] Kasner,[2] Tolman,[1] Chou,[3] Wyman,[4] Nariai,[5] and others, under the assumption of some symmetric properties for the line element. It is an idea to determine canonical forms of the line element of the empty space for which the Hamilton-Jacobi equation of a geodesic is integrable by separation of variables and to solve Einstein's field equations for empty spaces with the canonical forms thus determined. In § 2, § 3, § 4 are determined the canonical forms, in § 5 empty spaces without cosmological terms, in § 6 empty spaces with cosmological terms, which are shown to be reducible to 4 spaces by change of coordinates in § 7. Only one new space with cosmological terms is obtained, which may be represented as the intersection of two cylindrical surfaces in a six dimensional euclidian space.

§ 2. Fundamental equations

If we assume the line element of a space to be $ds^2=\sum h_i dx_i^2$ $(i=1, 2, 3, 4)$, the condition of Levi-Civita for separability gives two sets of equations

$$\left.\begin{aligned} \frac{1}{2}\frac{\partial^2\lambda_i}{\partial x_i\partial x_j}+\frac{\partial\lambda_i}{\partial x_j}\frac{\partial\lambda_j}{\partial x_i}=0 \\ \frac{1}{2}\frac{\partial^2\lambda_j}{\partial x_i\partial x_j}+\frac{\partial\lambda_j}{\partial x_i}\frac{\partial\lambda_i}{\partial x_j}=0 \end{aligned}\right\} \quad i\neq j \tag{1}$$

$$\frac{1}{2}\frac{\partial^2\lambda_i}{\partial x_j\partial x_k}-\frac{\partial\lambda_i}{\partial x_j}\frac{\partial\lambda_i}{\partial x_k}+\frac{\partial\lambda_i}{\partial x_j}\frac{\partial\lambda_j}{\partial x_k}+\frac{\partial\lambda_i}{\partial x_k}\frac{\partial\lambda_k}{\partial x_j}=0 \qquad i, j, k \neq \tag{2}$$

where $\lambda_i=1/2\cdot\log h_i$ and the notation $i, j, k\neq$ means that any two among i, j, k are not equal.

The conditions for the space to be empty are grouped into two sets of equations by virtue of (1), (2),

$$R_{kl}=-\frac{1}{2}\frac{\partial^2(\lambda_i+\lambda_j)}{\partial x_k\partial x_l}=0\,, \qquad i, j, k, l\neq \tag{3}$$

$$R_{ii}=h_i\sum_k\frac{1}{h_k}\frac{\partial^2\lambda_i}{\partial x_k^2}+\sum_k\left(\frac{\partial\lambda_k}{\partial x_i}\right)^2-2\left(\frac{\partial\lambda_i}{\partial x_i}\right)^2=\Lambda h_i \tag{4}$$

Λ being a cosmological constant.

By the way, the conditions for the space to be flat are grouped into two sets of equations

$$\frac{\partial^2\lambda_i}{\partial x_k\partial x_l}=0\,, \qquad i, k, l\neq \tag{5}$$

$$\frac{\partial}{\partial x_j}\left(\frac{1}{\sqrt{h_j}}\frac{\partial\sqrt{h_i}}{\partial x_j}\right)+\frac{\partial}{\partial x_i}\left(\frac{1}{\sqrt{h_i}}\frac{\partial\sqrt{h_j}}{\partial x_i}\right)+\frac{1}{h_k}\frac{\partial\sqrt{h_i}}{\partial x_k}\frac{\partial\sqrt{h_i}}{\partial x_k}+\frac{1}{h_l}\frac{\partial\sqrt{h_j}}{\partial x_l}\frac{\partial\sqrt{h_j}}{\partial x_l}=0 \qquad i, j, k, l\neq \tag{6}$$

3 sets of equations (1), (2), (5) determine 18 canonical forms of the h's given in a previous paper[6]. Canonical forms of the h's for empty spaces are to be determined by (1), (2), (3).

A general solution to (1), (2) is, as has been obtained by Staeckel,

$$h_i=\Delta/A_i\,, \qquad i=1, 2, 3, 4$$

$$\Delta=\begin{vmatrix} a_1(x_1) & a_2(x_2) & a_3(x_3) & a_4(x_4) \\ b_1(x_1) & b_2(x_2) & b_3(x_3) & b_4(x_4) \\ c_1(x_1) & c_2(x_2) & c_3(x_3) & c_4(x_4) \\ d_1(x_1) & d_2(x_2) & d_3(x_3) & d_4(x_4) \end{vmatrix} \tag{7}$$

$A_i=$ cofactor of a_i in Δ

$a_i(x_i)$, $b_i(x_i)$, $c_i(x_i)$, $d_i(x_i)$ being dependent on only x_i for any i. Hence we have

$$2\lambda_i=\log\Delta-\log A_i\,, \qquad i=1, 2, 3, 4.$$

Putting

$$2 \log \Delta - \sum_i \log A_i = S \tag{8}$$

we have the conditions to be satisfied by S from (3)

$$\frac{\partial^2 S}{\partial x_k \partial x_l} = 0\,, \qquad k \neq l \tag{9}$$

because of the relation

$$\begin{aligned} 2(\lambda_i + \lambda_j) &= 2 \log \Delta - \log A_i - \log A_j \\ &= S + \log A_k + \log A_l\,, \qquad i, j, k, l \neq \end{aligned}$$

A_k, A_l being independent of x_k, x_l respectively.

Hence we get

$$S = \sum_i \log s_i \tag{10}$$

s_i, being any function of x_i only. Multiplying elements of the i-th column of Δ by s_i, we can reduce S to nothing. So we can assume $S=0$ or

$$2 \log \Delta = \sum_i \log A_i \qquad \text{or} \qquad \Delta^2 = A_1 A_2 A_3 A_4 \tag{11}$$

and set out to obtain canonical forms of the h's which are now to be determined by (11).

§ 3. Soluions of the equation (11)

To solve the functional equation (11) is a toilsome task. Since solutions to (11) are ramified into numerous degenerate cases, here is presented only a non-degenerated case.

We assume d_1, d_2, d_3, d_4 in Δ not to be zero and multiply each element of the i-th column of the determinant Δ by $1/d_i$, and change the variables so as to get $d_i dx_i^2 = dx'^2_i$.

We have then

$$\Delta = d_1 d_2 d_3 d_4 \begin{vmatrix} a_1' & a_2' & a_3' & a_4' \\ b_1' & b_2' & b_3' & b_4' \\ c_1' & c_2' & c_3' & c_4' \\ 1 & 1 & 1 & 1 \end{vmatrix}\,, \qquad A_1 = d_2 d_3 d_4 \begin{vmatrix} b_2' & b_3' & b_4' \\ c_2' & c_3' & c_4' \\ 1 & 1 & 1 \end{vmatrix}\,, \ \text{etc.}$$

and put

$$\Delta = d_1 d_2 d_3 d_4 \Delta'\,, \qquad A_1 = d_2 d_3 d_4 A_1'\,, \quad \text{etc.}$$

where $a_i' = a_i/d_i$ etc.

The equation (11) becomes then

$$\Delta'^2 = d_1 d_2 d_3 d_4 A_1' A_2' A_3' A_4'$$

and

$$h_i = d_i h_i', \qquad h_i' = \Delta' / A_i' .$$

By the change of variables, h_i' may be regarded as h_i in a new coordinate so that Δ, A_i may be replaced by Δ', A_i' respectively. Hence we can assume from the start

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ 1 & 1 & 1 & 1 \end{vmatrix}, \qquad A_1 = \begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ 1 & 1 & 1 \end{vmatrix}, \quad \text{etc.}$$

and seek a solution to

$$\Delta^2 = A_1 A_2 A_3 A_4 .$$

From a theorem on the determinant we have

$$(c_3 - c_4)\Delta = A_1 B_2 - A_2 B_1$$

B_1, B_2 being cofactors of b_1, b_2 in Δ. Hence we have

$$\left(\frac{B_2}{A_2} - \frac{B_1}{A_1} \right)^2 = (c_3 - c_4)^2 \frac{A_3 A_4}{A_1 A_2} \tag{12}$$

For a moment we consider x_3, x_4 = constants and x_1, x_2 = variables. This equation (12) is then of the form

$$(g_1 - g_2)^2 = f_1 f_2 (r_1 - r_2)(s_1 - s_2) \tag{13}$$

g_i, f_i, r_i, s_i being functions of x_i, since A_1, B_1 and A_2, B_2 are independent of x_2 and of x_1 respectively. A general solution to (13) is for $g_1' g_2' \neq 0$

$$\left.\begin{aligned} \frac{1}{g_1 - g_0} &= \frac{p_0}{r_1 - r_0} + p_0', & \frac{1}{g_2 - g_0} &= \frac{p_0}{r_2 - r_0} + p_0' \\ \frac{1}{s_1 - s_0} &= \frac{q_0}{r_1 - r_0} + q_0', & \frac{1}{s_2 - s_0} &= \frac{q_0}{r_2 - r_0} + q_0' \end{aligned}\right\} \tag{14}$$

where g_0, r_0, s_0, p_0, p_0', q_0, q_0' are any constants independent of x_1, x_2.

Comparing (13) with (12), we may put

$$r_1 = \frac{b_1 - b_3}{c_1 - c_3}, \quad r_2 = \frac{b_2 - b_3}{c_2 - c_3}, \quad s_1 = \frac{b_1 - b_4}{c_1 - c_4}, \quad s_2 = \frac{b_2 - b_4}{c_2 - c_4} \tag{15}$$

and get the following relations

$$\frac{1}{\dfrac{b_1 - b_4}{c_1 - c_4} - s_{34}} = \frac{p_{34}}{\dfrac{b_1 - b_3}{c_1 - c_3} - r_{34}} + q_{34} \tag{16}$$

$$\frac{1}{\dfrac{b_2-b_4}{c_2-c_4}-s_{34}}=\frac{p_{34}}{\dfrac{b_2-b_3}{c_2-c_3}-r_{34}}+q_{34} \tag{16}$$

p_{34}, q_{34}, r_{34}, s_{34} being some functions of x_3, x_4. We can rearrange the first equation of (15) as

$$\begin{aligned}\alpha(b_1-b_3)(b_1-b_4)+\beta(b_1-b_3)(c_1-c_4)+\gamma(b_1-b_4)(c_1-c_3)&\\ +\delta(c_1-c_3)(c_1-c_4)=0&\end{aligned} \tag{17}$$

where α, β, γ, δ are some functions of x_3, x_4. If we put $x_3=\text{const}$, $x_4=\text{const}$, we see that the point (b_1, c_1) lies on a certain quadratic curve

$$hb_1^2+kb_1c_1+lc_1^2+pb_1+qc_1+r=0\,, \qquad h, k, l, \cdots=\text{const}. \tag{18}$$

Because the two equations (16) and (17) are to be compatible, we have the following relations for any values of x_3, x_4,

$$\left.\begin{aligned}&\alpha=h\sigma\,, \quad \beta+\gamma=k\sigma\,, \quad \delta=l\sigma\,,\\ &-\alpha(b_3+b_4)-\beta c_4-\gamma c_3=p\sigma\\ &-\delta(c_3+c_4)-\beta b_3-\gamma b_4=q\sigma\\ &\alpha b_3b_4+\beta b_3c_4+\gamma b_4c_3+\delta c_3c_4=r\sigma\end{aligned}\right\} \tag{19}$$

σ being some function of x_3, x_4.

Eliminating α, γ, δ from (19), we have

$$\left.\begin{aligned}&\sigma\{p+h(b_3+b_4)+kc_3\}=\beta(c_3-c_4)\\ &\sigma\{q+l(c_3+c_4)+kb_4\}=\beta(b_4-b_3)\\ &\sigma\{r-hb_3b_4-kb_4c_3-lc_3c_4\}=\beta(b_3c_4-b_4c_3)\end{aligned}\right\} \tag{20}$$

Eliminating β/σ from the first and the second, we have

$$hb_3^2+kb_3c_3+lc_3^2+pb_3+qc_3=hb_4^2+kb_4c_4+lc_4^2+pb_4+qc_4\,.$$

The left member is independent of x_4 and the right member is independent of x_3. Hence both members are equal to a constant, say t. Multiplying the first of (20) by b_3, the second by c_3, the third by 1 and adding them, we have $t=-r$ so that every point (b_i, c_i) $i=1, 2, 3, 4$ lies on the same curve

$$hb_i^2+kb_ic_i+lc_i^2+pb_i+qc_i+r=0 \tag{21}$$

Therefore we can express b_i, c_i as rational functions of a certain function u_i of x_i

$$b_i=\frac{a''u_i^2+b''u_i+c''}{au_i^2+bu_i+c}\,, \qquad c_i=\frac{a'u_i^2+b'u_i+c'}{au_i^2+bu_i+c}$$

so we have

$$\left.\begin{aligned} &A_i = D\cdot\Delta(u)\cdot(au_i^2+bu_i+c)\cdot\prod_{k\neq i}(u_i-u_k)\Big/\prod_{j=1}^{4}(au_j^2+bu_j+c)\\ &D=\begin{vmatrix} a'' & b'' & c''\\ a' & b' & c'\\ 1 & 1 & 1\end{vmatrix},\qquad \Delta(u)=\begin{vmatrix} u_1^3 & u_2^3 & u_3^3 & u_4^3\\ u_1^2 & u_2^2 & u_3^2 & u_4^2\\ u_1 & u_2 & u_3 & u_4\\ 1 & 1 & 1 & 1\end{vmatrix}\\ &\Delta = D^2\Delta(u)\Big/\prod_k(au_k^2+bu_k+c)^{3/2}\end{aligned}\right\}\qquad(22)$$

But, if we put

$$g_1=B_2/A_2\,,\qquad g_2=B_1/A_1$$

in the solution of (13), we see that a_1 is a rational function of b_1, c_1. Therefore Δ must be rational in u_1, so $au_1^2+bu_1+c$ must be a perfect square. Hence we can put

$$b_1=a''u_1^2+b''u_1+c''\,,\qquad a=b=0$$

$$c_1=a'u_1^2+b'u_1+c'\,,\qquad c=1$$

regarding a rational function of the old u_1 as a new u_1.

We have then

$$\left.\begin{aligned} &\Delta=D^2\cdot\Delta(u)\\ &A_i=D\cdot\Delta(u)\Big/\prod_{k\neq i}(u_i-u_k)\\ &h_i=D\cdot\prod_{k\neq i}(u_i-u_k)\,.\end{aligned}\right\}\qquad(23)$$

This form of the h_i's is nothing but the canonical form (R) in flat spaces except for an irrelevant numerical factor D.

In scanning each of degenerate cases where some factors vanish, we have the same 18 canonical forms in empty spaces as those in flat spaces. This result might be reached by a shorter approach.

§ 4. 18 Canonical forms

(A) $h_1=\rho_1\sigma_1\tau_1\,,\quad h_2=\rho_1\,,\quad h_2=\sigma_1\,,\quad h_4=\tau_1$

(B) $h_1=\sigma_1^2\tau_1\,,\quad h_2=\sigma_1\sigma_2\,,\quad h_3=\sigma_1\sigma_2\,,\quad h_4=\tau_1$

(C) $h_1=\sigma_1^3\,,\quad h_2=\sigma_1\sigma_2\tau_2\,,\quad h_3=\sigma_1\sigma_2\,,\quad h_4=\sigma_1\tau_2$

(D) $h_1=\sigma_1-\sigma_2\,,\quad h_2=\sigma_2-\sigma_1\,,\quad h_3=1\,,\quad h_4=1$

(E) $h_1=\sigma_1(\sigma_1-\sigma_2)\,,\quad h_2=\sigma_2(\sigma_2-\sigma_1)\,,\quad h_3=\sigma_1\sigma_2\,,\quad h_4=1$

(F) $h_1=\sigma_1(\sigma_1+1)(\sigma_1-\sigma_2)\,,\quad h_2=\sigma_2(\sigma_2+1)(\sigma_2-\sigma_1)\,,$
$h_3=\sigma_1\sigma_2\,,\quad h_4=(\sigma_1+1)(\sigma_2+1)$

(G) $h_1=\sigma_1^3\,,\quad h_2=\sigma_1\sigma_2^2\,,\quad h_3=\sigma_1\sigma_2\sigma_3\,,\quad h_4=\sigma_1\sigma_2\sigma_3$

(H) $h_1=\sigma_1^2\tau_1\,,\quad h_2=\sigma_1(\sigma_2-\sigma_3)\,,\quad h_3=\sigma_1(\sigma_3-\sigma_2)\,,\quad h_4=\tau_1$

(I) $h_1=\sigma_1-\sigma_2\,,\quad h_2=\sigma_2-\sigma_1\,,\quad h_3=\sigma_3\,,\quad h_4=\sigma_3$

(J) $h_1=\sigma_1^2(\sigma_1-\sigma_2)\,,\quad h_2=\sigma_2^2(\sigma_2-\sigma_1)\,,\quad h_3=\sigma_1\sigma_2\sigma_3\,,\quad h_4=\sigma_1\sigma_2\sigma_3$

(K) $h_1=\sigma_1^3\,,\quad h_2=\sigma_1\sigma_2(\sigma_2-\sigma_3)\,,\quad h_3=\sigma_1\sigma_3(\sigma_3-\sigma_2)\,,\quad h_4=\sigma_1\sigma_2\sigma_3$

(L) $h_1=(\sigma_1-\sigma_2)(\sigma_1-\sigma_3)\,,\quad h_2=(\sigma_2-\sigma_1)(\sigma_2-\sigma_3)\,,$
$h_3=(\sigma_3-\sigma_1)(\sigma_3-\sigma_2)\,,\quad h_4=1$

(M) $h_1=\sigma_1(\sigma_1-\sigma_2)(\sigma_1-\sigma_3)\,,\quad h_2=\sigma_2(\sigma_2-\sigma_1)(\sigma_2-\sigma_3)\,,$
$h_3=\sigma_3(\sigma_3-\sigma_1)(\sigma_3-\sigma_2)\,,\quad h_4=\sigma_1\sigma_2\sigma_3$

(N) $h_1=\sigma_1-\sigma_2\,,\quad h_2=\sigma_2-\sigma_1\,,\quad h_3=\sigma_3-\sigma_4\,,\quad h_4=\sigma_4-\sigma_3$

(O) $h_1=\sigma_1^3\,,\quad h_2=\sigma_1\sigma_2^2\,,\quad h_3=\sigma_1\sigma_2(\sigma_3-\sigma_4)\,,\quad h_4=\sigma_1\sigma_2(\sigma_4-\sigma_3)$

(P) $h_1=\sigma_1^2(\sigma_1-\sigma_2)\,,\quad h_2=\sigma_2^2(\sigma_2-\sigma_1)\,,\quad h_3=\sigma_1\sigma_2(\sigma_3-\sigma_4)\,,\quad h_4=\sigma_1\sigma_2(\sigma_4-\sigma_3)$

(Q) $h_i=\sigma_4(\sigma_i-\sigma_j)(\sigma_i-\sigma_k)\,,\quad h_4=\sigma_4^3\,,\qquad j,k\neq i$

(R) $h_i=(\sigma_i-\sigma_j)(\sigma_i-\sigma_k)(\sigma_i-\sigma_l)\,,\qquad i,j,k,l\neq$

§ 5. Empty spaces without cosmological terms

The h_i's put in canonical forms are to be determined by (16) for flat spaces, by (4) for empty spaces without cosmological terms. It is a disappointment to find that the h_i's for empty spaces with $\Lambda=0$ coincide with the h's for flat spaces in almost all cases. The difference is found only in cases (A), (B), (H).

We show at first the above coincidence with the case (R) of canonical form, where the h's have the following form

$$h_i=\prod_{k\neq i}(\sigma_i-\sigma_k)\,,\qquad i=1,2,3\ 4.$$

Inserting these expressions in (6) for $i=1$, $j=2$, we have for a flat space

$$\begin{aligned}&(\sigma_1-\sigma_3)(\sigma_1-\sigma_4)\left\{2\sigma_2''-\sigma_2'^2\left(\frac{2}{\sigma_2-\sigma_1}+\frac{1}{\sigma_2-\sigma_3}+\frac{1}{\sigma_2-\sigma_4}\right)\right\}\\&+(\sigma_2-\sigma_3)(\sigma_2-\sigma_4)\left\{2\sigma_1''-\sigma_1'^2\left(\frac{2}{\sigma_1-\sigma_2}+\frac{1}{\sigma_1-\sigma_3}+\frac{1}{\sigma_1-\sigma_4}\right)\right\}\\&+\frac{(\sigma_1-\sigma_2)^2(\sigma_1-\sigma_4)(\sigma_2-\sigma_4)}{(\sigma_1-\sigma_3)(\sigma_2-\sigma_3)(\sigma_3-\sigma_4)}\sigma_3'^2+\frac{(\sigma_1-\sigma_2)^2(\sigma_1-\sigma_3)(\sigma_2-\sigma_3)}{(\sigma_1-\sigma_4)(\sigma_2-\sigma_4)(\sigma_4-\sigma_3)}\sigma_4'^2=0\end{aligned}\tag{24}$$

where $\sigma_3'=d\sigma_3/dx_3$ etc. Putting $x_1=$constant, $x_2=$constant, $x_3=$constant, we see that $\sigma_4'^2$ is a polynomial of degree 4 in σ_4. So we can put

$$\sigma_4'^2=a_0\sigma_4^4+a_1\sigma_4^3+a_2\sigma_4^2+a_3\sigma_4+a_4\,. \tag{25}$$

We see quite similarly that $\sigma_1'^2$, $\sigma_2'^2$, $\sigma_3'^2$ are also polynomials of degree 4 in σ_1, σ_2, σ_3 respectively. Inserting these expressions into (24), we find that the above 4 polynomials have the same coefficients, that is,

$$\sigma_i'^2=a_0\sigma_i^4+a_1\sigma_i^3+a_2\sigma_i^2+a_3\sigma_i+a_4 \qquad i=1,\,2,\,3,\,4. \tag{26}$$

If we insert (26) into (4) with $\Lambda=0$ for $i=1$, we have for an empty space without cosmological terms

$$\begin{aligned}2R_{11}=&\left(\frac{1}{\sigma_1-\sigma_2}+\frac{1}{\sigma_1-\sigma_3}+\frac{1}{\sigma_1-\sigma_4}\right)\sigma_1''\\
&-\left(\frac{1}{(\sigma_1-\sigma_2)^2}+\frac{1}{(\sigma_1-\sigma_3)^2}+\frac{1}{(\sigma_1-\sigma_4)^2}+\frac{1}{(\sigma_1-\sigma_2)(\sigma_1-\sigma_3)}\right.\\
&\left.+\frac{1}{(\sigma_1-\sigma_2)(\sigma_1-\sigma_4)}+\frac{1}{(\sigma_1-\sigma_3)(\sigma_1-\sigma_4)}\right)\sigma_1'^2\\
&-\frac{(\sigma_1-\sigma_3)(\sigma_1-\sigma_4)}{(\sigma_2-\sigma_3)(\sigma_2-\sigma_4)}\left(\frac{\sigma_2''}{\sigma_2-\sigma_1}-\frac{\sigma_2'^2}{(\sigma_2-\sigma_1)^2}\right)\\
&-\frac{(\sigma_1-\sigma_2)(\sigma_1-\sigma_4)}{(\sigma_3-\sigma_2)(\sigma_3-\sigma_4)}\left(\frac{\sigma_3''}{\sigma_3-\sigma_1}-\frac{\sigma_3'^2}{(\sigma_3-\sigma_1)^2}\right)\\
&-\frac{(\sigma_1-\sigma_2)(\sigma_1-\sigma_3)}{(\sigma_4-\sigma_2)(\sigma_4-\sigma_3)}\left(\frac{\sigma_4''}{\sigma_4-\sigma_1}-\frac{\sigma_4'^2}{(\sigma_4-\sigma_1)^2}\right)\end{aligned}$$

To eliminate σ_3'' we form an expression

$$2R_{11}(\sigma_2-\sigma_4)+2R_{22}(\sigma_1-\sigma_4)=\sigma_3'^2\frac{(\sigma_1-\sigma_2)^2(\sigma_1-\sigma_4)(\sigma_2-\sigma_4)}{(\sigma_3-\sigma_4)(\sigma_3-\sigma_1)^2(\sigma_3-\sigma_2)^2}+F(\sigma_3)$$

where $F(\sigma_3)(\sigma_3-\sigma_4)(\sigma_3-\sigma_1)^2(\sigma_3-\sigma_2)^2$ is a polynomial of degree 5 in σ_3. Hence we see that $\sigma_3'^2$ is a polynomial of degree 5 in σ_3 and quite similarly that $\sigma_1'^2$, $\sigma_2'^2$, $\sigma_4'^2$ are polynomials of degree 5 in σ_1, σ_2, σ_4, respectively.

Replacing $\sigma_i'^2$ in R_{11} by respective polynomials, we find the same result (26) to our disappointment. Empty, not flat spaces appear only in 3 canonical forms (A), (B), (H).

(A) In this case we have the Kasner solution a little generalized

$$ds^2=dx_1^2+x_1^{2\alpha}dx_2^2+x_1^{2\beta}dx_3^2+x_1^{2\gamma}dx_4^2$$

α, β, γ being constants subjected to the conditions

$$\alpha+\beta+\gamma=1\,,\qquad \alpha^2+\beta^2+\gamma^2=1\,.$$

(B) In this case we have the Schwarzschild solution in a spherical coordinate

$$ds^2 = \frac{d\sigma_1^2}{1-c/\sigma_1} + \sigma_1^2(dx_2^2 + \sin^2 x_2 dx_3^2) + \left(1 - \frac{c}{\sigma_1}\right) dx_4^2 .$$

(H) In this case we have the same Schwarzschild solution in another coordinate

$$ds^2 = \frac{d\sigma_1^2}{1-c/\sigma_1} + \sigma_1^2(\wp(x_2) - \wp(x_3))(dx_2^2 - dx_3^2) + \left(1 - \frac{c}{\sigma_1}\right) dx_4^2$$

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3 , \qquad g_2,\ g_3 : \text{ arbitrary.}$$

§ 6. Empty spaces with cosmological terms

Solutions of Einstein's gravitational field equations with cosmological terms, $R_{ik} = \Lambda g_{ik}$ are obtained in terms of elementary, elliptic and hyperelliptic functions with the above canonical forms except for (D), (E), (L), (Q). With the canonical forms (D), (E), (L), a component of curvature R_{44} vanishes identically.

With the canonical form (Q), we have a relation

$$R_{11}/h_1 = f(\sigma_1, \sigma_2, \sigma_3)/\sigma_4 = \Lambda , \qquad f = \text{some function of } \sigma_1,\ \sigma_2,\ \sigma_3$$

whence we deduce $\sigma_4 = \text{const.}$, so that this case reduces to the case (L). For other cases solutions are listed as follows. Since the calculations are straightforward and tedius they are omitted.

(A) $$ds^2 = (1 - 3\Lambda x_1^2/4)^{-2} dx_1^2 + (1 - 3\Lambda x_1^2/4)^{-2/3}(x_1^{2\alpha} dx_2^2 + x_1^{2\beta} dx_3^2 + x_1^{2\gamma} dx_4^2)$$

$$\alpha + \beta + 1 = 1 , \quad \alpha^2 + \beta^2 + \gamma^2 = 1 .$$

(B) $$ds^2 = \left(1 - \frac{l}{x_1} - \frac{\Lambda}{3} x_1^2\right)^{-1} dx_1^2 + x_1^2(dx_2^2 + dx_3^2)$$

$$+ \left(1 - \frac{l}{x_1} - \frac{\Lambda}{3} x_1^2\right) dx_4^2$$

Schwarzschild's solution

(C) $$ds^2 = (1 - \Lambda r^2/3)^{-1} dr^2 + r^2(du_2^2 + \sin^2 u_2 du_3^2 + \cos^2 u_2 du_4^2)$$

de Sitter's solution

(F) $$ds^2 = 3/\Lambda \cdot (\wp(u_1) - \wp(u_2))(du_1^2 - du_2^2) + (\wp(u_1) - e_1)(\wp(u_2) - e_1) dx_3^2$$

$$+ (\wp(u_1) - e_2)(\wp(u_2) - e_2) dx_4^2$$

$$g_2 = 4(3e_1^2 - 3e_1 + 1) , \quad g_3 = 4e_1(2e_1 - 1)(1 - e_1) , \quad e_2 = e_1 - 1$$

where $\wp(u)$ is the Weierstrass' elliptic function defined by the differential equation

$$\wp'^2(u) = 4\wp^3(u) - g_2\wp(u) - g_3$$

(G) $ds_2 = 3/\Lambda \cdot \{du_1^2 + \sinh^2 u_1(du_2^2 + \sin^2 u_2(du_3^2 + \sin^2 u_3 du_4^2))\}$ $\sim$(C)

this is equivalent to (C).

(H) $$ds^2 = \left(1 - \frac{l}{x_1} - \frac{\Lambda}{3}x_1^2\right)^{-1} dx_1^2 + x_1^2 d\Sigma^2 + \left(1 - \frac{l}{x_1} - \frac{\Lambda}{3}x_1^2\right) dx_4^2 \quad \sim\text{(B)}$$

$$d\Sigma^2 = (\wp(x_2) - \wp(x_3))(dx_2^2 - dx_3^2)$$

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3, \qquad g_2,\ g_3: \text{ arbitrary}$$

(I) $$ds^2 = 1/\Lambda \cdot (d\Sigma_1^2 + d\Sigma_2^2)$$

$$d\Sigma_1^2 = (\wp(x_1) - \wp(x_2))(dx_1^2 - dx_2^2), \quad d\Sigma_2^2 = \sinh^{-2} x_3 \cdot (dx_3^2 + dx_4^2)$$

(J) $$ds^2 = 3/\Lambda \cdot (\wp(u_1) - \wp(u_2))(du_1^2 - du_2^2) + (\wp(u_1) + l)(\wp(u_2) + l) d\Sigma^2 \quad \sim\text{(C)}$$

$l,\ g_2,\ g_3$ arbitrary

$$d\Sigma_2^2 = \sinh^{-2} x_3 (dx_3^2 + dx_4^2)$$

(K) $$ds^2 = (\Lambda u_1^2/3 - 1)^{-1} du_1^2 + u_1^2\{(\wp(u_2) - \wp(u_3))(du_2^2 - du_3^2) + (\wp(u_2) + l)(\wp(u_2) + l) du_4^2\} \quad \sim\text{(C)}$$

(M) $$ds^2 = \frac{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)}{\sigma_1 f(\sigma_1)} d\sigma_1^2 + \frac{(\sigma_2 - \sigma_1)(\sigma_2 - \sigma_3)}{\sigma_2 f(\sigma_2)} d\sigma_2^2 + \frac{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)}{\sigma_3 f(\sigma_3)} d\sigma_3^2 + \sigma_1\sigma_2\sigma_3 d\sigma_4^2 \quad \sim\text{(C)}$$

$$f(\sigma) = 4\Lambda/3 \cdot \sigma^3 + 2k\sigma^2 + 4l\sigma + 4m$$

(N) $$ds^2 = 1/\Lambda \cdot (d\Sigma_1^2 + d\Sigma_2^2) \quad \sim\text{(I)}$$

$$d\Sigma_1^2 = (\wp(x_1) - \wp(x_2))(dx_1^2 - dx_2^2)$$

$$d\Sigma_2^2 = (\wp(x_3) - \wp(x_4))(dx_3^2 - dx_4^2)$$

$$\wp'^2(x_i) = 4\wp^3(x_i) - g_2\wp(x_i) - g_3, \qquad i = 1,\ 2, \quad g_2,\ g_3: \text{ arbitrary}$$

$$\wp'^2(x_i) = 4\wp^3(x_i) - g_2'\wp(x_i) - g_3', \qquad i = 3,\ 4, \quad g_2',\ g_3': \text{ arbitrary}$$

(O) $$ds^2 = 3/\Lambda \cdot \{du_1^2 + \sin^2 u_1 du_2^2 + \sin^2 u_1 \sin^2 u_2 d\omega^2\} \quad \sim\text{(C)}$$

$$d\omega^2 = (\wp(u_3) - \wp(u_4))(du_3^2 - du_4^2)$$

(P) $$ds^2 = 3/\Lambda \cdot (\wp(u_1) - \wp(u_2))(du_1^2 - du_2^2) + (\wp(u_1) + l)(\wp(u_2) + l) d\omega^2$$

$$d\omega^2 = (\wp(u_3) - \wp(u_4))(du_3^2 - du_4^2) \quad \sim\text{(J)} \sim\text{(C)}$$

(R) $$ds^2 = \sum_{i=1}^{4} (\sigma_i - \sigma_j)(\sigma_i - \sigma_k)(\sigma_i - \sigma_l) d\sigma_i^2 / f(\sigma_i) \qquad i,\ j,\ k,\ l \neq \quad \sim\text{(C)}$$

$$f(\sigma) = 2\Lambda\sigma^5 + a_1\sigma^4 + a_2\sigma^3 + a_3\sigma^2 + a_4\sigma + a_5 \qquad a_j \text{ arbitrary}$$

(Q) $$ds^2 = \sigma_4 \sum_{i-1}^{3} (\sigma_i - \sigma_j)(\sigma_i - \sigma_k) dx_i^2 + \sigma_4^2 dx_4^2 \qquad i,\ j,\ k \neq \quad \sim\text{(C)}$$

§ 7. Use of sphero-conical coordinates

It is a further disappointment to find that out of the above 15 solutions survive only 4 solutions (A), (B), (C), (I), for other solutions are reducible to them by change of variables. The type (A) corresponds to the type (A) in § 6. The type (B) gives the Schwarzschild's exterior solution. The type (C) gives the line element of the de Sitter space, which may be represented as a spherical surface in a euclidian space of dimension 5. The types (F), (G), (J), (K), (M), (O), (P), (Q) and (R) also may be interpreted to represent the spherical surface, as will be shown later. The types (I) and (N) are interpreted as the intersection of two cylindrical surfaces $x_1^2+x_2^2+x_3^2=\text{const}$, $x_4^2+x_5^2+x_6^2=\text{const}$ in a euclidian space of dimension 6.

In order to make clear the equivalence of several spaces to a de Sitter space we make use of a coordinate system which is akin to a confocal coordinate system and may be called sphero-conical coordinate system. Let $x_1^2+x_2^2+\cdots+x_n^2=\mu^2$ be a sphere in a euclidian space of dimension n. Let $\rho_1, \rho_2, \cdots, \rho_{n-1}$ be $n-1$ zeros of an equation

$$\frac{x_1^2}{a_1-\rho}+\frac{x_2^2}{a_2-\rho}+\cdots+\frac{x_n^2}{a_n-\rho}=\mu^2 \qquad a_1>a_2>\cdots>a_n>0$$

such that the ρ_i be between a_{i+1} and a_i. Then we have an identity

$$\frac{x_1^2}{a_1-\rho}+\cdots+\frac{x_n^2}{a_n-\rho}=-\mu^2\frac{(\rho-\rho_1)\cdots(\rho-\rho_{n-1})}{(\rho-a_1)\cdots(\rho-a_n)}.$$

Putting $\rho=a_i$ after multiplying both members by $a_i-\rho$, we have

$$x_i^2=\mu^2\prod_{j=1}^{n-1}(a_i-\rho_j)/\prod_{k\neq i}(a_i-a_k) \qquad i=1, 2, \cdots, n.$$

Hence the coordinates $x_1, x_2, \cdots, x_n$ are expressible with $\mu, \rho_1, \cdots, \rho_{n-1}$, though the correspondence is not uniformized. The uniformization is attained by a method similar to that used in a confocal coordinate system. The coordinate system $\mu, \rho_1, \cdots, \rho_{n-1}$ is an orthogonal curvilinear one. In this coordinate system, we have

$$ds^2=\sum_i dx_i^2=d\mu^2+\mu^2\sum_{j=1}^{n-1}\prod_{k\neq j}(\rho_j-\rho_k)d\rho_j^2/4\varphi(\rho_j)$$

where $\varphi(\rho)$ stands for the product $(\rho-a_1)\cdots(\rho-a_n)$. The derivation of the expression ds^2 is similar to that in a confocal coordinate system.

Put $n=5$, $\mu^2=-4/\Lambda$ and one gets the line element of the spherical surface $x_1^2+x_2^2+x_3^2+x_4^2+x_5^2=-4/\Lambda$ in a euclidian space of dimension 5

$$ds^2=\sum_{i=1}^{4}(\rho_i-\rho_j)(\rho_i-\rho_k)(\rho_i-\rho_l)d\rho_i^2/f(\rho_i)$$

$$f(\rho)=2\Lambda\prod_{i=1}^{5}(\rho-a_i)$$

which reduces to the line element in (R).

Let $a_4=a_5$, put

$$\frac{x_1^2}{a_1-\rho}+\frac{x_2^2}{a_2-\rho}+\frac{x_3^2}{a_3-\rho}+\frac{x_4^2+x_5^2}{a_4-\rho}$$

$$=-\mu^2\frac{(\rho-\rho_1)(\rho-\rho_2)(\rho-\rho_3)}{(\rho-a_1)(\rho-a_2))(\rho-a_3)(\rho-a_4)}$$

$$\frac{x_4^2}{b_4-\rho}+\frac{x_5^2}{b_5-\rho}=-(x_4^2+x_5^2)\frac{\rho-\rho_4}{(\rho-b_4)(\rho-b_5)}$$

and one gets the line element of the spherical surface in terms of 4 variables ρ_1, ρ_2, ρ_3, ρ_4. This line element reduces to the line element in (M).

Let $a_4=a_5$, $a_2=a_3$, put

$$\frac{x_1^2}{a_1-\rho}+\frac{x_2^2+x_3^2}{a_2-\rho}+\frac{x_4^2+x_5^2}{a_4-\rho}=-\mu^2\frac{(\rho-\rho_1)(\rho-\rho_2)}{(\rho-a_1)(\rho-a_2)(\rho-a_4)}$$

$$\frac{x_2^2}{b_2-\rho}+\frac{x_3^2}{b_3-\rho}=-\frac{(x_2^2+x_3^2)(\rho-\rho_3)}{(\rho-b_2)(\rho-b_3)}$$

$$\frac{x_4^2}{b_4-\rho}+\frac{x_5^2}{b_5-\rho}=-\frac{(x_4^2+x_5^2)(\rho-\rho_4)}{(\rho-b_4)(\rho-b_5)}$$

b_i being constants, and one gets the line element of the spherical surface that reduces to the line element in (F).

Repeating the same procedure as above, one gets the line elements in (C), (G), (J), (K), (O), (P), (Q). In these cases, the solutions represent the line element of the same spherical surface in different coordinate systems.

Therefore empty spaces of Staeckel with cosmological terms are only four in number.

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