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Schroedinger Equations Soluble in Terms of Confluent Hypergeometric Functions

Nobuko Kubota, (窪田伸子) **Mitsuko Yamaguchi** (山口光子)
and Giiti Iwata (岩田義一)

Department of Physics, Faculty of Science,
Ochanomizu University, Tokyo

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Inclusive of well-known potentials, parabolic and Morse potentials, potentials for which Schroedinger equations are soluble in terms of confluent hypergeometric functions are determined together with their eigenvalues and eigenfunctions.

§ 1. The condition of solubility

There are some types of potential, for which the Schroedinger equations are soluble in terms of confluent hypergeometric functions, such as, a parabolic potential, a Teller-Poeschel potential[2] and a Morse potential.[1] In any of these potentials, the Schroedinger equation of a particle is reducible to a confluent hypergeometric equation, yielding eigenvalues linear or quadratic in a quantum number. If the question is reversed, one may be led to a problem to find out other potentials for which the Schroedinger equations are analytically soluble in terms of confluent hypergeometric functions or hypergeometric functions. In the present paper, only confluent hypergeometric functions are kept in sight. We set the Schroedinger equation for a particle of mass m moving in a potential field $U(x)$ as

$$\frac{d^2\phi}{dx^2}+\frac{2m}{\mathrm{h}^2}[E-U(x)]\phi=0 \tag{1}$$

or in its reduced form

$$\frac{d^2\phi}{dx^2}+[\epsilon-V(x)]\phi=0 \tag{2}$$

where $\epsilon=2mE/\mathrm{h}^2$, $V(x)=2mU(x)/\mathrm{h}^2$.

We seek the conditions that the equation (2) should be reduced to a confluent hypergeometric equation

$$t\frac{d^2w}{dt^2}+(c-t)\frac{dw}{dt}-aw=0 \tag{3}$$

by the change of variables,

$$\psi(x)=w(t)g(x)\,, \qquad f(x)=t \tag{4}$$

under the following assumptions

A) the parameter a is arbitrary,

B) both the potential $V(x)$ and the function $f(x)$ are independent of the parameter a, and

C) the parameter c as well as the function $g(x)$ may depend on the parameter a.

The change of variables (4) gives

$$\frac{d\psi}{dx}=w'(t)f'g+w(t)g'$$

$$\frac{d^2\psi}{dx^2}=w''(t)f'^2g+w'(t)(f''g+2f'g')+w(t)g''$$

and turns the equation (2) into the following equation

$$f'^2gw''(t)+(f''g+2f'g')w'(t)+[g''+(\epsilon-V)g]w(t)=0$$

which is to be identical to the equation (3) except for a multiplication factor. Hence we have

$$(f''g+2f'g')/f'^2g=(c-f)/f \tag{5}$$

$$[g''+(\epsilon-V)g]/f'^2g=-a/f\,. \tag{6}$$

The equation (5) may be rewritten as

$$\frac{g'}{g}=\frac{1}{2}\left(\frac{cf'}{f}-f'-\frac{f''}{f'}\right) \tag{7}$$

which may be integrated to give

$$\log g=\frac{1}{2}(c\log f-f-\log f')+\text{const.}$$

or

$$g=\text{const.}\,f^{c/2}e^{-f/2}/\sqrt{f'}\,. \tag{8}$$

The equation (6) may be rewritten as

$$V(x)-\epsilon=\frac{g''}{g}+\frac{af'^2}{f}=\left(\frac{g'}{g}\right)'+\left(\frac{g'}{g}\right)^2+\frac{af'^2}{f}\,.$$

The elimination of g'/g by way of (7) leads to

$$V(x)-\epsilon=\frac{1}{2}\left(\frac{cf'}{f}-f'-\frac{f''}{f'}\right)'$$

$$+\frac{1}{4}\left(\frac{cf'}{f}-f'-\frac{f''}{f'}\right)^2+\frac{af'^2}{f}\,. \tag{9}$$

Our task is to determine $V(x)$ and $f(x)$ from this equation (9) by virtue of the preceding assumptions A), B), C).

It is to be noted that the wave function $\psi(x)$ is given as

$$\psi(x)=\text{const.}\, f^{c/2}e^{-f/2}f'^{-1/2}{}_1F_1(a,\ c,\ f(x))\ , \tag{10}$$

since the solution of the confluent hypergeometric equation is given by ${}_1F_1(a,\ c,\ t)$ and $g(x)$ is given by (8).

For the sake of simplicity, we put

$$V(x)-\epsilon=R(x)+\frac{c}{2}Q(x)+\frac{c^2}{4}P(x)+aS(x) \tag{11}$$

where

$$P=\frac{f'^2}{f^2}\ ,\quad Q=-\frac{1+f}{f^2}f'^2\ ,\quad R=-\frac{1}{2}\left(f'+\frac{f''}{f'}\right)'+\frac{1}{4}\left(f'+\frac{f''}{f'}\right)^2,$$

and

$$S=f'^2/f\ . \tag{12}$$

According as S' is zero or not, the path forward is divided.

In § 2, we take the case where $S'=0$, and in § 3 the case where $S'\neq 0$.

§ 2. Case 1

When S' is zero, S is a constant, so one may put

$$f'^2/f=4k^2$$

then one gets

$$f'/\sqrt{f}=2k \qquad \text{or} \qquad f=(kx)^2\ . \tag{13}$$

Hence one has

$$V(x)-\epsilon=k^2\left[(kx)^2+\frac{d}{(kx)^2}-2c+4a\right]$$

where

$$d=\left(c-\frac{1}{2}\right)\left(c-\frac{3}{2}\right)\ . \tag{14}$$

Therefore one gets immediately

$$V(x)=k^2[(kx)^2+d(kx)^{-2}] \tag{15}$$

and

$$\epsilon=k^2(2c-4a)\ .$$

The potential is to be independent of the parameter a, so that d, hence c, is a constant independent of the parameter a.

The case $d=0$ gives a well-known parabolic potential. For the potential to be a sort of well, d must be non-negative.

For $d>0$, the relation (14) gives

$$c=1+\sqrt{d+\frac{1}{4}} \quad \text{or} \quad c=1-\sqrt{d+\frac{1}{4}}$$

while, for the wave function

$$\psi(x)=\text{const.}\ e^{-(kx)^2/2}x^{c-1/2}{}_1F_1(a,\ c,\ k^2x^2) \tag{16}$$

to be finite over the range $x \geqq 0$ of the variable x, one has the following conditions

$$c \geqq 1/2 \quad \text{at} \quad x=0$$

and $a=0$ or negative integer at $x=\infty$. Therefore one has

$$c=1+\sqrt{d+\frac{1}{4}}$$

and

$$\epsilon=k^2(2+\sqrt{4+d}-4a) \qquad a=0,\ -1,\ -2,\ \cdots. \tag{17}$$

The number of discrete energy levels is infinite.

§ 3. Case 2

When S' is not zero identically, one sees that for a particular value of x, there must be a relation

$$a=\frac{c^2}{4}\alpha+\frac{c}{2}\beta+\gamma \tag{18}$$

α, β, γ being constants independent of c. In other words, a must be quadratic in c. Since $V(x)$ is independent of c, and ϵ is independent of x, the expression

$$V(x)-\epsilon=c^2(P+\alpha S)/4+c(Q+\beta S)/2+R+\gamma S \tag{19}$$

must satisfy the condition

$$\frac{\partial^2}{\partial c \partial x}(V(x)-\epsilon)=c(P+\alpha S)'/2+(Q+\beta S)'/2=0 \tag{20}$$

for any value of x or c. Hence

$$(P+\alpha S)'=0 \quad \text{or} \quad (1+\alpha f)f'^2/f^2=\text{const.}$$

$$(Q+\beta S)'=0 \quad \text{or} \quad [1+(1-\beta)f]f'^2/f^2=\text{const.}$$

For these two relations to be compatible, one is lead to get

$$\beta=1-\alpha. \tag{21}$$

One may set then

$$(1+\alpha f)f'^2/f^2=4k^2$$

or

$$\sqrt{1+\alpha f}\, f'/f=2k\,, \qquad k>0\,. \tag{22}$$

The change of dependent variable from f to θ by $\alpha f=\theta^2-1$ gives the equation

$$\theta'=k(\theta^2-1)/\theta^2 \tag{23}$$

which is integrated to give

$$\theta+\frac{1}{2}\log\frac{\theta-1}{\theta+1}=kx\,. \tag{24}$$

One sees then that $x\to\infty$ as $\theta\to+\infty$ and $x\to-\infty$ as $\theta\to1+$.

So the whole range of x, $-\infty<x<\infty$, corresponds to the range of θ, $\theta>1$. One gets further

$$V(x)-\epsilon=k^2\Big[\frac{\theta^2}{\alpha^2}+\Big(\frac{3}{4}+\frac{1}{\alpha^2}-\frac{4\gamma}{\alpha}\Big)\frac{1}{\theta^2}$$
$$+\frac{3}{2}\frac{1}{\theta^4}-\frac{5}{4}\frac{1}{\theta^6}+c^2-2c+\frac{4\gamma}{\alpha}-\frac{2}{\alpha^2}\Big]\,.$$

Instead of a, c is to be regarded as a parameter here. Since one may add additive constant terms to the potential, one may put

$$V(x)=k^2\Big[\frac{\theta^2}{\alpha^2}+\Big(\frac{3}{4}+\frac{1}{\alpha^2}-\frac{4\gamma}{\alpha}\Big)\frac{1}{\theta^2}$$
$$+\frac{3}{2}\frac{1}{\theta^4}-\frac{5}{4}\frac{1}{\theta^6}+\frac{4\gamma}{\alpha}-\frac{2}{\alpha^2}-1\Big] \tag{25}$$

$$\epsilon=-k^2(c-1)^2\,. \tag{26}$$

As is seen,

$$\theta\to+\infty \qquad \text{as} \qquad x\to+\infty$$

so that the convergence of the wave function for $x\to\infty$ requires that $\alpha>0$ and a be 0 or a negative integer. Further one sees that

$$\theta\sim1+2e^{2(kx-1)} \qquad \text{and} \qquad f\sim4e^{2(kx-1)}/\alpha \qquad \text{as} \qquad x\to-\infty$$

so that $\psi\sim f^{c/2}f'^{-1/2}\sim\exp[(c-1)(kx-1)]$, therefore the convergence of the wave function for $x\to-\infty$ requires $c-1\geqq0$.

Since $\beta=1-\alpha$ by (21), a is expressed as a polynomial of degree 2 in c

$$a=\alpha c^2/4+(1-\alpha)c/2+\gamma\,. \tag{27}$$

Hence one gets

$$c=\alpha-1+\sqrt{\{(\alpha-1)^2-4\alpha(\gamma-a)\}} \tag{28}$$

with the restriction that

$$a=0,\ -1,\ -2,\ -3,\ \cdots$$

$$c \geqq 1$$

$$(\alpha-1)^2-4\alpha(\gamma-a) \geqq 0\ .$$

The number of discrete energy levels is finite.

Finally we show that the limiting case $\alpha=0$ gives a Morse potential. In this case, we have from (22)

$$f'/f=2k$$

so that we get

$$f=Ae^{2kx}\ , \qquad A=\text{const.}$$

and

$$V(x)-\epsilon=k^2[A^2e^{4kx}+4\gamma Ae^{2kx}+(c-1)^2]\ .$$

Therefore we may get

$$V(x)=k^2[A^2e^{4kx}+4\gamma Ae^{2kx}]$$

$$\epsilon=-k^2(c-1)^2=-k^2(2a-2\gamma-1)^2$$

$$a=0,\ -1,\ -2,\ \cdots.$$

Summing up, we obtained two new potentials (15) and (25) for which the Schroedinger equations are analytically soluble in terms of confluent hypergeometric functions.

It is to be added that the confluent hypergeometric function of Whittaker W[3] that satisfies the differential equation

$$\frac{d^2W}{dx^2}+\left\{-\frac{1}{4}+\frac{k}{x}+\frac{1/4-m^2}{x^2}\right\}W=0$$

gives the Coulomb potential plus a repulsive potential proportional to the inverse square of the distance.

References

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2) Teller, E. and Poeschel, G.: Z. Phys. **83** (1933) 143.
3) Whittaker, E.T. and Watson, G.N.: Modern Analysis, p. 337, Cambridge, 1935.