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On a Problem Concerning Coincidence of Tangent Planes

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1. We shall be concerned with the following problem in this paper.

PROBLEM. *Let D be a domain in the Euclidean plane $\boldsymbol{R}_2$, K a non-empty connected subset of D, and $f(x, y)$ a real-valued function defined and continuously differentiable on D.*

Consider the tangent planes to the surface $z=f(x, y)$ at those surface-points whose orthogonal projections on the xy-plane belong to the set K. If these tangent planes are all perpendicular to the z-axis, then they are identical.

This problem can be solved affirmatively in two cases (Theorems 1 and 2) in each of which K is a continuous curve fulfilling a certain condition. The second theorem is an extension of the first.

Before dealing with them, we shall examine the problem and introduce notation and some definitions.

Suppose $(a, b)\in D$ and write $f(a, b)=c$. The tangent plane at the surface-point (a, b, c) exists and is expressed by

$$z-c=f_x(a, b)(x-a)+f_y(a, b)(y-b)\ ,$$

since f is continuously differentiable. This plane is perpendicular to the z-axis if and only if $f_x(a, b)=f_y(a, b)=0$, in which case its equation reduces to the form $z=c$. We can therefore replace the hypothesis of the problem by the condition that $f_x(x, y)=f_y(x, y)=0$ at all the points (x, y) of K, and the conclusion by identical constancy of $f(x, y)$ on K.

2. In the rest of this paper, the term "*function*" will always mean a finite real-valued one, unless another meaning is implied by the context.

NOTATION. Given a pair of functions $\mu(x)$, $\nu(x)$ defined on the real line $\boldsymbol{R}_1$ and given a set $E\subset\boldsymbol{R}_1$, we shall write "$\mu\prec\nu$ *on* E" if to each point x of E there correspond positive numbers δ and M such that the inequality $0<h<\delta$ implies

$$|\mu(x+h)-\mu(x)|\leqq M|\nu(x+h)-\nu(x)|\ .$$

In particular, "$\mu\prec\nu$ on $\boldsymbol{R}_1$" will simply be written $\mu\prec\nu$.

In the above definition, replace "$0<h<\delta$" by "$0<h<\delta$ and $x+h\in E$"; then we shall write $\mu\prec_E\nu$.

CONVENTION.

$$\frac{a}{0}=\begin{cases} +\infty & (0<a<+\infty) \\ 0 & (a=0) \\ -\infty & (0>a>-\infty) \end{cases}$$

DEFINITIONS (cf. [1], p. 108). (i) Suppose given on the real line $\boldsymbol{R}_1$ two functions $F(t)$, $U(t)$ and let t_0 be a point such that the function U is not identically constant in any closed interval whose left-hand extremity is t_0.

Let us consider the upper limit and the lower limit of the ratio $[F(t)-F(t_0)]/[U(t)-U(t_0)]$ as t tends to t_0 from the right by values other than those for which $F(t)-F(t_0)=U(t)-U(t_0)=0$. These two limits will be called, respectively, *right-hand upper derivate* $\overline{F}_U^+(t_0)$ and *right-hand lower derivate* $\underline{F}_U^+(t_0)$ of the function F at the point t_0 *with respect to the function* U.

(ii) Similarly, given on $\boldsymbol{R}_1$ two functions $F(t)$, $U(t)$ and given a set $E\subset \boldsymbol{R}_1$, let t_0 be a right-hand accumulation point of E such that the function U is not identically constant in any intersection $E\cap I$, where I is an arbitrary closed interval with left-hand extremity t_0.

Let us consider the upper limit and the lower limit of the ratio $[F(t)-F(t_0)]/[U(t)-U(t_0)]$ as t tends to t_0 from the right by values belonging to E and other than those for which $F(t)-F(t_0)=U(t)-U(t_0)=0$. These two limits will be termed, respectively, *right-hand upper derivate* $\overline{F}_{U,E}^+(t_0)$ and *right-hand lower derivate* $\underline{F}_{U,E}^+(t_0)$ of the function F at t_0, *with respect to the function* U and *relative to the set* E.

3. We are now in a position to state and prove the following

THEOREM 1. *Let D be a domain in $\boldsymbol{R}_2$ and let $f(x, y)$ be a real-valued function defined and continuously differentiable on D.*

Suppose given in D a parametric curve C whose equations are $x=\varphi(t)$ and $y=\psi(t)$, where φ and ψ are continuous functions on $\boldsymbol{R}_1$ admitting a finite real-valued function $U(t)$ such that $\varphi\prec U$ and $\psi\prec U$ simultaneously.

If $f_x(x, y)=f_y(x, y)=0$ at every point (x, y) of the curve C, then $f(x, y)$ is identically constant along C.

REMARK. Throughout this paper, by neighbourhood of a point t of $\boldsymbol{R}_1$, we shall understand any half-open interval of the form $[t, t+\varepsilon)$, where $\varepsilon>0$.

PROOF. Let us write $F(t)=f(\varphi(t), \psi(t))$ for brevity. Consider the points t of $\boldsymbol{R}_1$ such that each t has no neighbourhood on which the function U is constant, and denote by T_1 the set of them. Writing further $T_2=\boldsymbol{R}_1-T_1$, we shall treat the two sets T_1 and T_2 separately.

(i) For each point t of $\boldsymbol{R}_1$, we have

$$F(t+h)-F(t)=f_x(x', y')[\varphi(t+h)-\varphi(t)]+f_y(x', y')[\psi(t+h)-\psi(t)],$$

whenever $|h|$ is so small that the segment connecting the pair of points $(\varphi(t), \psi(t))$ and $(\varphi(t+h), \psi(t+h))$ lies in D. The point (x', y') is suitably chosen on this segment.

On the other hand, there are, by hypothesis, positive numbers δ and M such that $0<h<\delta$ implies the inequalities

$$|\varphi(t+h)-\varphi(t)| \leqq M|U(t+h)-U(t)|,$$
$$|\psi(t+h)-\psi(t)| \leqq M|U(t+h)-U(t)|.$$

It follows that, for fixed $t \in T_1$ and for sufficiently small $h>0$ (cf. our convention on $a/0$),

$$\left|\frac{F(t+h)-F(t)}{U(t+h)-U(t)}\right| \leqq M(|f_x(x', y')|+|f_y(x', y')|).$$

But the right-hand side of this inequality tends to zero with h, since both f_x, f_y are continuous and since $f_x=f_y=0$ along the curve C. Hence $\overline{F}{}^+_U(t)=\underline{F}_U(t)=0$ at each point t of the set T_1. By Theorem 5.5 of [1] (p. 274), we conclude that $|F[T_1]|=0$. (As in [1], $|F[T_1]|$ denotes the outer Lebesgue measure of the set $F[T_1]$, while this latter means the image of the set T_1 under the mapping F. Similarly in what follows.)

(ii) Take up now the points t of the set $T_2=\boldsymbol{R}_1-T_1$. Each t has a neighbourhood $V(t)$ on which the function U is identically constant. As in (i), we can associate with t positive numbers δ and M such that $0<h<\delta$ implies the relations

$$|\varphi(t+h)-\varphi(t)| \leqq M|U(t+h)-U(t)|,$$
$$|\psi(t+h)-\psi(t)| \leqq M|U(t+h)-U(t)|.$$

We may and do assume that $V(t) \subset [t, t+\delta)$. The composed function $F(t)=f(\varphi(t), \psi(t))$ is thus constant on $V(t)$.

Let $\{r_n\}_{n=1,2,\cdots}$ be a sequence of all rational numbers. For each n, we denote by W_n the union of the neighbourhoods $V(t)$ for all the points t of T_2 such that $r_n \in V(t)$. As we readily see, $F(t)$ is constant on this set W_n. But plainly $T_2 \subset \bigcup_n W_n$, and it turns out that the set $F[T_2]$ is at most countable.

(iii) We deduce from the conclusions of (i) and (ii) that

$$|F[\boldsymbol{R}_1]|=|F[T_1 \cup T_2]| \leqq |F[T_1]|+|F[T_2]|=0.$$

Hence $|F[\boldsymbol{R}_1]|=0$, and this together with the continuity of F requires that F is constant over $\boldsymbol{R}_1$. In other words, $f(x, y)$ is identically constant along the curve C; and the proof is complete.

NOTE 1. In the above theorem, we can replace the hypothesis "$\varphi \prec U$ and $\psi \prec U$" by "$\varphi \prec U$ and $\psi \prec U$ on $\boldsymbol{R}_1-N$", where N is a countable set. In fact, $F[N]$ is countable and consequently $|F[N]|=0$.

4. In order to establish the second of our theorems, we shall require the following lemma, whose proof would be an almost verbal repetition of that for Theorem 5.5 of [1] (p. 274).

LEMMA. *If U and F are a pair of finite functions defined on $\boldsymbol{R}_1$ and if $\overline{F}^+_{U,E}(t)=\underline{F}^+_{U,E}(t)=0$ at each point t of a set $E\subset \boldsymbol{R}_1$, then $|F[E]|=0$.*

NOTE 2. If, in the above lemma, we merely require the hypothesis $\overline{F}^+_{U,E}(t)=\underline{F}^+_{U,E}(t)=0$ to hold at the points t of a set $A\subset E$, then we can at least conclude that $|F[A]|=0$.

This may be seen immediately by considering the intersections $A\frown E_{n,k}$ instead of the sets $E_{n,k}$ that would appear in the proof of the lemma.

THEOREM 2. *Let D be a domain in $\boldsymbol{R}_2$, and $f(x, y)$ a real-valued function defined and continuously differentiable on D. Let further C be a parametric curve $x=\varphi(t)$, $y=\psi(t)$ contained in D such that both φ and ψ are continuous functions defined on $\boldsymbol{R}_1$.*

Suppose that to each positive integer n there correspond a set $E_n\subset \boldsymbol{R}_1$ and a finite real-valued function $U_n(t)$ defined on $\boldsymbol{R}_1$, in such a manner that

$$\varphi\prec_{E_n}U_n\,,\qquad \psi\prec_{E_n}U_n\,,\qquad \boldsymbol{R}_1=\bigcup_n E_n.$$

If $f_x(x, y)=f_y(x, y)=0$ at every point (x, y) of the curve C, then $f(x, y)$ is identically constant along C.

PROOF. Let us write $F(t)=f(\varphi(t), \psi(t))$ for brevity. It is sufficient to prove that $|F[E_n]|=0$ for each n. Let us suppose that every point of E_n is a right-hand accumulation point of E_n, since the set of the points isolated on the right is at most countable. To simplify the notation, let us write $E_n=E$ and $U_n=U$, keeping n fixed. We shall argue by decomposing E into a pair of sets T_1 and T_2.

(i) Let T_1 be the set of the points $t\in E$ for each of which the function U is not identically constant on any intersection $E\frown V(t)$, where $V(t)$ is an arbitrary neighbourhood of t. Just as in the proof of Theorem 1, the conditions $\varphi\prec_E U$ and $\psi\prec_E U$ together ensure that $\overline{F}^+_{U,E}(t)=\underline{F}^+_{U,E}(t)=0$ for every point t of T_1. We therefore have $|F[T_1]|=0$ in virtue of our Lemma (cf. Note 2).

(ii) Let T_2 be the set of the points $t\in E$ each of which has a neighbourhood $V(t)$ such that the function U is identically constant on the intersection $E\frown V(t)$. By hypothesis, there exist for each $t\in T_2$ positive numbers δ and M such that the conditions $0<h<\delta$ and $t+h\in E$ together imply the inequalities

$$|\varphi(t+h)-\varphi(t)|\leqq M|\,U(t+h)-U(t)|\,,$$
$$|\psi(t+h)-\psi(t)|\leqq M|\,U(t+h)-U(t)|\,.$$

We may suppose δ to satisfy the further condition $V(t)\subset[t, t+\delta)$, so that the function $F(t)=f(\varphi(t), \psi(t))$ is constant on $E\frown V(t)$.

For each positive integer m and each integer i, let us consider the closed interval $\left[\frac{i}{m}, \frac{i+1}{m}\right]$. We denote by $\alpha_{m,i}$ any fixed point of $E \cap \left[\frac{i}{m}, \frac{i+1}{m}\right]$ if this intersection is not empty, and we set $\alpha_{m,i}=0$ in the opposite case. Since the points $\alpha_{m,i}$ are countable in their totality, we can rearrange them in a simple sequence $\{\alpha_j\}_{j=1,2,\cdots}$. For each j, let us denote by W_j the union of the sets $E \cap V(t)$ for all the points $t \in T_2$ such that $\alpha_j \in E \cap V(t)$; then F is constant on W_j as we readily see.

On the other hand we can attach to each point $t \in T_2$ an interval $\left[\frac{i}{m}, \frac{i+1}{m}\right]$ contained in $V(t)$ and having a point or points in common with E, because t is a right-hand accumulation point of E. Hence there is a point α_j in $E \cap V(t)$. It follows by definition of the set W_j that $T_2 \subset \bigcup_j W_j$. Since, moreover, the function F is constant on every W_j, we conclude that the set $F[T_2]$ is at most countable.

(iii) By what has been proved in (i) and (ii), we have finally $|F[E]|=|F[T_1 \cup T_2]|=0$, which completes the proof in view of continuity of the function F.

NOTE 3. As in Theorem 1, we can replace in the above theorem the hypothesis "$\varphi <_{E_n} U_n$ and $\psi <_{E_n} U_n$" by "$\varphi <_{D_n} U_n$ and $\psi <_{D_n} U_n$", where each D_n is a set obtained from E_n by removing at most a countable infinity of points.

5. We have thus derived two theorems which solve in the affirmative the problem of § 1 under special conditions.

In this final section, we shall give an example to each of our theorems (as modified in accordance with Notes 1 and 3 respectively), in order to show what cases are at least included within their range of applicability.

EXAMPLE 1. The continuous functions $\varphi(t)$ and $\psi(t)$ are VBG$_*$ on $\boldsymbol{R}_1$.

On account of a theorem of [1] (p. 236), there exist bounded increasing functions U_1 and U_2 such that the extreme derivates $\overline{\varphi}_{U_1}(t)$, $\underline{\varphi}_{U_1}(t)$, $\overline{\psi}_{U_2}(t)$, $\underline{\psi}_{U_2}(t)$ are all finite at each point t of $\boldsymbol{R}_1$ except, perhaps, those of a countable set. Let us write $U=U_1+U_2$; then $U_1 < U$ and $U_2 < U$, since both U_1 and U_2 are increasing functions. It follows that $\varphi < U$ and $\psi < U$ at each point t of $\boldsymbol{R}_1$ except at most those of a countable set. The functions φ and ψ thus fulfil the condition of Theorem 1 when we modify this theorem in accordance with Note 1.

EXAMPLE 2. The continuous functions $\varphi(t)$ and $\psi(t)$ are VBG on $\boldsymbol{R}_1$.

By definition of VBG functions, there is a decomposition of $\boldsymbol{R}_1$ into a disjoint sequence of sets $E_1, E_2, \cdots$ such that both φ and ψ are VB on each E_n. We then deduce from Lemma (4.1) of [1] (p. 221) that there correspond to each $n=1, 2, \cdots$ two functions φ_n and ψ_n, of bounded variation over the whole line $\boldsymbol{R}_1$, coinciding on E_n with φ and ψ respectively.

Let $\mathrm{O}(\varphi_n; I)$, $\mathrm{V}(\varphi_n; I)$ and $\mathrm{V}_*(\varphi_n; I)$ denote severally the *oscillation* ([1], p. 60 and p. 96), *weak variation* (p. 221) and *strong variation* (p. 228) of φ_n on any closed interval I. It is clear that $\mathrm{O}(\varphi_n; I) \leqq \mathrm{V}(\varphi_n; I)$ for each n. We therefore get

$$\sum_k \mathrm{O}(\varphi_n; I_k) \leqq \sum_k \mathrm{V}(\varphi_n; I_k) \leqq \mathrm{V}(\varphi_n; \boldsymbol{R}_1),$$

provided $\{I_k\}$ is a sequence of non-overlapping intervals whose end-points belong to E_n. It results that $\mathrm{V}_*(\varphi_n; E_n) \leqq \mathrm{V}(\varphi_n; \boldsymbol{R}_1)$ for each n. But $\mathrm{V}(\varphi_n; \boldsymbol{R}_1)$ is finite since φ_n is VB on $\boldsymbol{R}_1$. Accordingly φ_n is VB_* on E_n; and the same is true of ψ_n, too.

By a theorem of [1] (p. 236) already utilized, we can attach to each $n=1, 2, \cdots$ a bounded increasing function U_n defined on $\boldsymbol{R}_1$ such that $\varphi_n \prec U_n$ and $\psi_n \prec U_n$ on some set $D_n \subset E_n$, where $E_n - D_n$ is at most countable. Recalling that $\varphi_n = \varphi$ and $\psi_n = \psi$ on E_n, we find that $\varphi \prec_{D_n} U_n$ and $\psi \prec_{D_n} U_n$. The functions φ and ψ thus fulfil the condition of Theorem 2 as modified in conformity to Note 3.

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There naturally arises the question: what kind of continuous curve $x = \varphi(t)$, $y = \psi(t)$ satisfies the respective condition imposed on the curve C in each of our theorems? We are unable to answer this at present.

Reference

[1] S. Saks: Theory of the integral, Warszawa-Lwów (1937).