Natural Science Report, Ochanomizu University, Vol. 19, No. 2, 1968
お茶の水女子大学 自然科学報告 第19巻 第2号

On Conformal Killing Tensor

Toyoko Kashiwada (柏田豊子)

Department of Mathematics, Faculty of Science,
Ochanomizu University, Tokyo

(*Received September 10, 1968*)

Introduction. Recently, S. Tachibana has introduced the notion of conformal Killing tensor of degree 2 in a Riemannian space. A skew-symmetric tensor field u_{ab} is called a conformal Killing tensor if it satisfies

$$\nabla_c u_{ab} + \nabla_a u_{cb} = 2\rho_b g_{ca} - \rho_c g_{ab} - \rho_a g_{cb}\,.$$

He discussed such tensors and obtained, for instance, the following theorems.

THEOREM. *If there exists a conformal Killing tensor which takes any preassigned value at any point of an n (>3) dimensional Riemannian space, then the space is conformally flat.*

THEOREM. *In an n (>2) dimensional non-flat space of constant curvature, a conformal Killing tensor u_{cd} is decomposed in the form;*

$$u_{cd} = p_{cd} + q_{cd}\,,$$

where p_{cd} is a Killing tensor and q_{cd} is a closed conformal Killing tensor.

In this paper, we shall generalize them to the case of degree $r \geqq 2$.

The author wishes to express her hearty thanks to Prof. S. Tachibana for his kind suggestions.

§1. Conformal Killing tensor of degree r. Let M^n $(n>3)$ be an n-dimensional Riemannian space whose metric tensor is given by g_{ab}. We shall call a shew-symmetric tensor $u_{a_1\cdots a_r}$ a conformal Killing tensor of degree r if there exists a skew-symmetric tensor $\rho_{a_1\cdots a_{r-1}}$ such that

$$\begin{aligned}\nabla_b u_{a_1\cdots a_r} &+ \nabla_{a_1} u_{ba_2\cdots a_r} \\ &= 2\rho_{a_2\cdots a_r} g_{a_1 b} - \sum_{i=2}^{r}(-1)^i(\rho_{a_1\cdots\hat{a}_i\cdots a_r} g_{ba_i} + \rho_{ba_2\cdots\hat{a}_i\cdots a_r} g_{a_1 a_i})\,,\end{aligned} \tag{1.1}$$

where $\hat{a}_i$ means that a_i is omitted. This $\rho_{a_1\cdots a_{r-1}}$ is called the associated tensor field of $u_{a_1\cdots a_r}$.

Now consider a conformal Killing tensor $u_{a_1\cdots a_r}$, then we have

$$\nabla^b u_{ba_2\cdots a_r} = (n-r+1)\rho_{a_2\cdots a_r}\,. \tag{1.2}$$

By virtue of Ricci's identity and (1.2), we can get

$$\nabla^a \rho_{a\, a_3 \cdots a_r} = 0\,. \tag{1.3}$$

If we regard the components $u_{a_1\cdots a_r}$ of a skew-symmetric tensor as coefficients of the exterior differential form

$$u = \frac{1}{r\,!} u_{a_1\cdots a_r} dx^{a_1} \wedge \cdots \wedge dx^{a_r}\,,$$

it follows for a conformal Killing tensor $u_{a_1\cdots a_r}$ that

$$(du)_{ba_1\cdots a_r} = (r+1)(\nabla_b u_{a_1\cdots a_r} + \sum_{i=1}^{r} (-1)^i \rho_{a_1\cdots \hat{a}_i \cdots a_r} g_{a_i b})\,.$$

Applying to this the same method as in [2], we have

$$\begin{aligned} & r\nabla_b \nabla_c u_{a_1\cdots a_r} + \sum_{i=1}^{r} R_{a_i cb}{}^e u_{a_1\cdots e\cdots a_r} - \sum_{i<j} R_{a_i a_j b}{}^e u_{a_1\cdots e\cdots c\cdots a_r} \\ & \quad + r\sum_{i=1}^{r} (-1)^i \tau_{ba_1\cdots \hat{a}_i \cdots a_r} g_{a_i c} + \sum_{i=1}^{r} (-1)^i \tau_{ca_1\cdots \hat{a}_i \cdots a_r} g_{a_i b} \\ & \quad - \sum_{j=1}^{r} \sum_{i(\neq j)} (-1)^i \tau_{a_j a_1\cdots \hat{a}_i \cdots c \cdots a_r} g_{a_i b} - \sum_{i=1}^{r} (-1)^i \tau_{a_i a_1 \cdots \hat{a}_i \cdots a_r} g_{cb} \\ & \quad = 0\,, \end{aligned} \tag{1.4}$$

where $\tau_{a_1\cdots a_r} = \nabla_{a_1} \rho_{a_2\cdots a_r}$ and the indices e and c in $u_{a_1\cdots e\cdots c\cdots a_r}$ appear at the i-th and j-th position respectively. Interchanging the indices b and c and subtracting the equation from (1.4), we have

$$\begin{aligned} & -(r-1)\sum_{i=1}^{r} R_{bca_i}{}^e u_{a_1\cdots e\cdots a_r} - \sum_{i<j} R_{a_i a_j b}{}^e u_{a_1\cdots e\cdots c\cdots a_r} + \sum_{i<j} R_{a_i a_j c}{}^e u_{a_1\cdots e\cdots b\cdots a_r} \\ & \quad + \sum_{i=1}^{r} (-1)^i \Big\{ (r-1)\tau_{ba_1\cdots \hat{a}_i \cdots a_r} + \sum_{j\neq i} \tau_{a_j a_1 \cdots \hat{a}_i \cdots b \cdots a_r} \Big\} g_{a_i c} \\ & \quad - \sum_{i=1}^{r} (-1)^i \Big\{ (r-1)\tau_{ca_1\cdots \hat{a}_i \cdots a_r} + \sum_{j\neq i} \tau_{a_j a_1 \cdots \hat{a}_i \cdots c \cdots a_r} \Big\} g_{a_i b} = 0\,. \end{aligned} \tag{1.5}$$

Transvecting (1.5) with g^{ca_1}, it follows that

$$\begin{aligned} & (r-1)\tau_{ba_2\cdots a_r} + \sum_{i=2}^{r} \tau_{a_i a_2 \cdots b \cdots a_r} \\ & \quad = \frac{-1}{n-r} \Big\{ (r-1) R_b{}^e u_{ea_2\cdots a_r} + (r-2) \sum_{i=2}^{r} R_b{}^c{}_{a_i}{}^e u_{ca_2\cdots e\cdots a_r} \\ & \qquad + \sum_{i=2}^{r} R_{a_i}{}^e u_{ea_2\cdots b\cdots a_r} - \sum_{2\leqq i<j} R_{a_i a_j}{}^{ce} u_{ca_2\cdots e\cdots b\cdots a_r} \Big\}\,, \end{aligned} \tag{1.6}$$

where $R_b{}^e$ means Ricci tensor.

Substituting (1.6) into (1.5), we obtain

$$-(r-1)\sum_{i=1}^{r} R_{bca_i}{}^{e}u_{a_1\cdots e\cdots a_r}-\sum_{i<j} R_{a_ia_jb}{}^{e}u_{a_1\cdots e\cdots c\cdots a_r}+\sum_{i<j} R_{a_ia_jc}{}^{e}u_{a_1\cdots e\cdots b\cdots a_r}$$

$$-\frac{1}{n-r}\sum_{i=1}^{r}(-1)^i\Big\{(r-1)R_b{}^{e}u_{ea_1\cdots\hat{a}_i\cdots a_r}+(r-2)\sum_{i\neq j} R_b{}^{d}{}_{a_i}{}^{e}u_{da_1\cdots e\cdots\hat{a}_i\cdots a_r}$$

$$(1.7)\qquad +\sum_{j\neq i} R_{a_j}{}^{e}u_{ea_1\cdots b\cdots\hat{a}_i\cdots a_r}-\sum_{k<j,k,j\neq i} R_{a_ka_j}{}^{de}u_{da_1\cdots e\cdots b\cdots\hat{a}_i\cdots a_r}\Big\}g_{a_ic}$$

$$+\frac{1}{n-r}\sum_{i=1}^{r}(-1)^i\Big\{(r-1)R_c{}^{e}u_{ea_1\cdots\hat{a}_i\cdots a_r}+(r-2)\sum_{i\neq j} R_c{}^{d}{}_{a_i}{}^{e}u_{da_1\cdots e\cdots\hat{a}_i\cdots a_r}$$

$$+\sum_{j\neq i} R_{a_j}{}^{e}u_{ea_1\cdots c\cdots\hat{a}_i\cdots a_r}-\sum_{k<j,k,j\neq i} R_{a_ka_j}{}^{de}u_{da_1\cdots e\cdots c\cdots\hat{a}_i\cdots a_r}\Big\}g_{a_ib}=0\,.$$

Now we put

$$\delta^{j_1\cdots j_r}_{i_1\cdots i_r}=\sum_{\sigma\in\mathfrak{S}}\operatorname{sign}\sigma\,\delta^{\sigma(1)}_{i_1}\cdots\delta^{\sigma(r)}_{i_r}$$

where $\mathfrak{S}=\{\text{permutation } \sigma\mid\sigma=\binom{j_1\cdots j_r}{\sigma(1)\cdots\sigma(r)}\}$, and

$$B_{bca_1\cdots a_r}{}^{l_1\cdots l_r}=(r-1)\sum_{i=1}^{r} R_{bca_i}{}^{e}\delta^{l_1\cdots l_i\cdots l_r}_{a_1\cdots e\cdots a_r}+\sum_{i<j}(R_{a_ia_jb}{}^{e}\delta^{l_1\cdots l_i\cdots l_j\cdots l_r}_{a_1\cdots e\cdots c\cdots a_r}-R_{a_ia_jc}{}^{e}\delta^{l_1\cdots l_i\cdots l_j\cdots l_r}_{a_1\cdots e\cdots b\cdots a_r})$$

$$+\frac{1}{n-r}\sum_{i=1}^{r}(-1)^i\Big\{(r-1)R_b{}^{e}\delta^{l_1l_2\cdots\cdots l_r}_{ea_1\cdots\hat{a}_i\cdots a_r}+(r-2)\sum_{j\neq i}R_b{}^{d}{}_{a_i}{}^{e}\delta^{l_1l_2\cdots\cdots l_r}_{da_1\cdots e\cdots\hat{a}_i\cdots a_r}$$

$$+\sum_{j\neq i}R_{a_j}{}^{e}\delta^{l_1l_2\cdots\cdots l_r}_{ea_1\cdots b\cdots\hat{a}_i\cdots a_r}-\sum_{\substack{k<j\\k,j\neq i}}R_{a_ka_j}{}^{de}\delta^{l_1l_2\cdots\cdots l_r}_{da_1\cdots e\cdots b\cdots\hat{a}_i\cdots a_r}\Big\}g_{a_ic}$$

$$-\frac{1}{n-r}\sum_{i=1}^{r}(-1)^i\Big\{(r-1)R_c{}^{e}\delta^{l_1l_2\cdots\cdots l_r}_{ea_1\cdots\hat{a}_i\cdots a_r}+(r-2)\sum_{j\neq i}R_c{}^{d}{}_{a_i}{}^{e}\delta^{l_1l_2\cdots\cdots l_r}_{da_1\cdots e\cdots\hat{a}_i\cdots a_r}$$

$$+\sum_{j\neq i}R_{a_j}{}^{e}\delta^{l_1l_2\cdots\cdots l_r}_{ea_1\cdots c\cdots\hat{a}_i\cdots a_r}-\sum_{\substack{k<j\\k,j\neq i}}R_{a_ka_j}{}^{de}\delta^{l_1l_2\cdots\cdots l_r}_{da_1\cdots e\cdots c\cdots\hat{a}_i\cdots a_r}\Big\}g_{a_ib}\,,$$

so as to write (1.7) as

$$(1.8)\qquad \sum_{(l_1\cdots l_r)}B_{bca_1\cdots a_r}{}^{l_1\cdots l_r}u_{l_1\cdots l_r}=0$$

where $\sum_{(l_1\cdots l_r)}$ means the sum arised from combinations of indices $l_1,\cdots,l_r$.

THEOREM 1. *For any point P of an n dimensional Riemannian space and any skew-symmetric constants $C_{a_1\cdots a_r}$, if there exists (locally) a conformal Killing tensor $u_{a_1\cdots a_r}$ of degree r $(2\leq r<n-1)$ satisfying $u_{a_1\cdots a_r}(P)=C_{a_1\cdots a_r}$, then the space is conformally flat.*

PROOF. Under the assumption of the theorem, by virtue of (1.8) we have $B_{bca_1\cdots a_r}{}^{l_1\cdots l_r}=0$. Contractinng this with respect to a_2 and $l_2,\cdots,a_r$ and l_r, we get

$$R_{bca}{}^{l}+\frac{1}{n-2}(R_{ab}\delta_{c}{}^{l}-R_{ac}\delta_{b}{}^{l}+R_{c}{}^{l}g_{ab}-R_{b}{}^{l}g_{ac})$$
$$-\frac{R}{(n-1)(n-2)}(\delta_{c}{}^{l}g_{ab}-\delta_{b}{}^{l}g_{ac})=0$$

on taking account of $n-1>r\geqq 2$, where R denotes the scaler curvature.

This equation means that the space is conformally flat. Q. E. D.

§ 2. A sufficient condition to be conformal Killing tensor.

Let $u_{a_1\cdots a_r}$ be a conformal Killing tensor, then we have

$$\nabla_b\nabla_c u_{a_1\cdots a_r}+\nabla_b\nabla_{a_1}u_{ca_2\cdots a_r}$$
$$=2\tau_{ba_2\cdots a_r}g_{a_1c}-\sum_{i=2}(-1)^i(\tau_{ba_1\cdots\hat{a}_i\cdots a_r}g_{ca_i}+\tau_{bca_2\cdots\hat{a}_i\cdots a_r}g_{a_1a_i})\,. \tag{2.1}$$

By interchanging indices b, c, a, as $b\rightarrow c\rightarrow a\rightarrow b$ and $b\rightarrow a\rightarrow c\rightarrow b$ in this equation and subtracting the latter equation from the sum of (2.1) and the former, we can get

$$2\nabla_b\nabla_c u_{a_1\cdots a_r}+\sum_{i=1}^{r}R_{bca_i}{}^{e}u_{a_1\cdots e\cdots a_r}-(R_{ba_1c}{}^{e}+R_{ca_1b}{}^{e})u_{ea_2\cdots a_r}$$
$$-\sum_{i=2}(R_{ba_1a_i}{}^{e}u_{ca_2\cdots e\cdots a_r}+R_{ca_1a_i}{}^{e}u_{ba_2\cdots e\cdots a_r})$$
$$=2\tau_{ba_2\cdots a_r}g_{a_1c}+2\tau_{ca_2\cdots a_r}g_{a_1b}-2\tau_{a_1\cdots a_r}g_{bc}-\sum_{i=2}(-1)^i(\tau_{bca_2\cdots\hat{a}_i\cdots a_r}+\tau_{cba_2\cdots\hat{a}_i\cdots a_i})g_{a_1a_i}$$
$$-\sum_{i=2}(-1)^i(\tau_{ba_1\cdots\hat{a}_i\cdots a_r}-\tau_{a_1ba_2\cdots\hat{a}_i\cdots a_r})g_{ca_i}-\sum_{i=2}(-1)^i(\tau_{ca_1\cdots\hat{a}_i\cdots a_r}-\tau_{a_1ca_2\cdots\hat{a}_i\cdots a_r})g_{ba_i}. \tag{2.2}$$

Transvecting (2.2) with g^{bc}, we have

$$\nabla^b\nabla_b u_{a_1\cdots a_r}+R_{a_1}{}^{e}u_{ea_2\cdots a_r}-\sum_{i=2}^{r}R^{b}{}_{a_1a_i}{}^{e}u_{ba_2\cdots e\cdots a_r}$$
$$+(n-r-1)\tau_{a_1\cdots a_r}+\sum_{i=2}^{r}(-1)^i\tau_{a_ia_1\cdots\hat{a}_i\cdots a_r}=0 \tag{2.3}$$

and as the skew-symmetric part of (2.3) we get

$$r\nabla^b\nabla_b u_{a_1\cdots c_r}+\sum_{i=1}^{r}R_{a_i}{}^{e}u_{a_1\cdots e\cdots a_r}+\sum_{i<j}R_{a_ia_j}{}^{ec}u_{a_1\cdots e\cdots c\cdots a_r}$$
$$=-(2r-n)\sum_{i=1}^{r}(-1)^i\tau_{a_ia_1\cdots\hat{a}_i\cdots a_r}\,, \tag{2.4}$$

which is a consequence of (1.4) too.

In the following we shall show that (2.3) or (2.4) is sufficient for a skew-symmetric tensor $u_{a_1\cdots a_r}$ to be a conformal Killing in a compact space.

Now we put

(2.5) $$A_{ba_1\cdots a_r}=\nabla_b u_{a_1\cdots a_r}+\nabla_{a_1}u_{ba_2\cdots a_r}-2\rho_{a_2\cdots a_r}g_{a_1 b}$$
$$+\sum_{i=2}^{r}(-1)^i(\rho_{a_1\cdots\hat{a}_i\cdots a_r}g_{ba_i}+\rho_{ba_2\cdots\hat{a}_i\cdots a_r}g_{a_1a_i})$$

for a skew-symmetric tensor $u_{a_1\cdots a_r}$, where $\rho_{a_2\cdots a_r}$ is given by

$$(n-r+1)\rho_{a_2\cdots a_r}=\nabla^b u_{ba_2\cdots a_r}\,.$$

By some computations we have

$$u^{a_1\cdots a_r}\nabla^b A_{ba_1\cdots a_r}=u^{a_1\cdots a_r}\Big\{\nabla^b\nabla_b u_{a_1\cdots a_r}+R_{a_1}{}^e u_{ea_2\cdots a_r}-\sum_{i=2}^{r}R^b{}_{a_1a_i}{}^e u_{ba_2\cdots e\cdots a_r}$$
$$+(n-r-1)\tau_{a_1\cdots a_r}+\sum_{i=2}^{r}(-1)^i\tau_{a_ia_1\cdots\hat{a}_i\cdots a_r}\Big\}\,,$$
$$A_{ba_1\cdots a_r}\nabla^b u^{a_1\cdots a_r}=\frac{1}{2}A_{ba_1\cdots a_r}A^{ba_1\cdots a_r}\,.$$

If the space in consideration is compact orientable, applying Green's theorem to $\nabla^b(A_{ba_1\cdots a_r}u^{a_1\cdots a_r})$, we can get tne following

THEOREM 2. *In a compact orientable Riemannian space M^n, the following integral formula is valid for any skew-symmetric tensor field* $u_{a_1\cdots a_r}$;

$$\int_{M^n}\Big\{u^{a_1\cdots a_r}(\nabla^b\nabla_b u_{a_1\cdots a_r}+R_{a_1}{}^e u_{ea_2\cdots a_r}-\sum_{i=2}^{r}R^b{}_{a_1a_i}{}^e u_{ba_2\cdots e\cdots a_r}$$
$$-(n-r-1)\tau_{a_1\cdots a_r}-\sum_{i=2}^{r}(-1)^i\tau_{a_ia_1\cdots\hat{a}_i\cdots a_r})+\frac{1}{2}A_{ba_1\cdots a_r}A^{ba_1\cdots a_r}\Big\}d\sigma=0\,,$$

where $d\sigma$ means the volume element of M^n and $A_{ba_1\cdots a_r}$ is given by (2.5).

THEOREM 3. *In a compact Riemannian space a necessary and sufficient condition for a skew-symmetric tensor field $u_{a_1\cdots a_r}$ to be a conformal Killing tensor is* (2.3) (*or* (2.4)).

§ **3.** Non-existence of conformal Killing tensor.[1] In a compact orientable Riemannian space, the following integral formula is vaild for any skew-symmetric tensor $u_{a_1\cdots a_r}$;

(3.1) $$\int_{M^n}\{F(u_{a_1\cdots a_r})+\nabla_b u_{a_1\cdots a_r}\nabla^{a_1}u^{ba_2\cdots a_r}-\nabla_b u^b{}_{a_2\cdots a_r}\nabla_{a_1}u^{a_1a_2\cdots a_r}\}d\sigma=0$$

where we put

$$F(u_{a_1\cdots a_r})=u^{ba_2\cdots a_r}u_{ca_2\cdots a_r}R_b{}^c+\frac{r-1}{2}u^{bca_3\cdots a_r}u_{dea_3\cdots a_r}R_{bc}{}^{de}\,.$$

If $u_{a_1\cdots a_r}$ is a conformal Killing tensor, then it follows that

$$\nabla_b u_{a_1\cdots a_r}\nabla^{a_1}u^{ba_2\cdots a_r}=-\nabla_b u_{a_1\cdots c_r}\nabla^b u^{a_1\cdots a_r}+(n-r+1)(r+1)\rho_{a_2\cdots a_r}\rho^{a_2\cdots a_r}\,.$$

1) Cf. [3].

Substituting this into (3.1), we get

$$\int_{M^n}\{F(u_{a_1\cdots a_r})-\nabla_b u_{a_1\cdots a_r}\nabla^b u^{a_1\cdots a_r}-(n-r+1)(n-2r)\rho_{a_2\cdots a_r}\rho^{a_2\cdots a_r}\}d\sigma=0\,.$$

Then we obtain the following theorem.

THEOREM 4. *In a compact Riemannian spaec M^n, there exists no conformal Killing tensor of degree r $\left(\leqq \frac{n}{2}\right)$ which satisfies*

$$F(u_{a_1\cdots a_r})\leqq 0$$

unless we have

$$\nabla_b u_{a_1\cdots a_r}=0\,.$$

Especially, if $F(u_{a_1\cdots a_r})$ is negative definite, then there exists no conformal Killing tensor of degree r other than the zero tensor.

COROLLARY 1. *In a compact Riemannian space of negative constant curvature, there exists no conformal Killing tensor of degree r $\left(\leqq \frac{n}{2}\right)$ other than zero tensor.*

COROLLARY 2.[2)] *In a conformally flat compact Riemannian space, if Ricci quadratic form $R_{ij}u^iu^j$ is negative definite, then there exists no conformal Killing tensor of degree r $\left(\leqq \frac{n}{2}\right)$ other than zero tensor.*

§ 4. Conformal Killing tensor in a space of constant curvature.

In this section, we assume that the space is of constant curvature. Then, from (1.6) we get that

$$(r-1)\nabla_b\rho_{a_2\cdots a_r}+\sum_{i=2}\nabla_{a_i}\rho_{a_2\cdots b\cdots a_r}=0\,.$$

Interchanging b and a_2 and adding these equations, we have that

$$\nabla_b\rho_{a_2\cdots a_r}+\nabla_{a_2}\rho_{ba_3\cdots a_r}=0\,.$$

Thus we obtain the following:

LEMMA 1. *In a space of constant curvature, an associated tensor of conformal Killing tensor of degree r is a Killing tensor.*

Let us consider a Killing tensor $v_{a_1\cdots a_{r-1}}$ in this space, then we have[3)]

$$\nabla_b\nabla_{a_1}v_{a_2\cdots a_r}=\frac{R}{n(n-1)}\sum_{i=1}^{r}(-1)^i g_{a_ib}v_{a_1\cdots\hat{a}_i\cdots a_r}\,.$$

2) This corollary is as same as Theorem 4.2 of [3] though the definition of conformal Killing is unlike.

3) Cf. [2].

Thus it follows that

$$\nabla_b\nabla_{a_1}v_{a_2\cdots a_r}+\nabla_{a_1}\nabla_b v_{a_2\cdots a_r}$$
$$=\frac{-R}{n(n-1)}\left\{2g_{a_1b}v_{a_2\cdots a_r}-\sum_{i=2}^{r}(-1)^i(g_{a_ib}v_{a_1\cdots\hat{a}_i\cdots a_r}+g_{a_ia_1}v_{ba_2\cdots\hat{a}_i\cdots a_r})\right\}$$

and hence we get

LEMMA 2. *In a space of constant curvature, the covariant derivative* $\nabla_{a_1}v_{a_2\cdots a_r}$ *of a Killing tensor* $v_{a_1\cdots a_{r-1}}$ *is a conformal Killing tensor whose associated tensor is given by* $\dfrac{-R}{n(n-1)}v_{a_1\cdots a_{r-1}}$.

LEMMA 3. *In a space of constant curvature, if a Killing tensor of degree* $r(<n)$ *is closed then it is a zero tensor.*

PROOF. For a Killing tensor, we have

$$(du)_{ba_1\cdots a_r}=(r+1)\nabla_b u_{a_1\cdots a_r}$$

which vanish by the assumption. Then we have

$$\sum_{i=1}^{r}R_{bca_i}{}^{e}u_{a_1\cdots e\cdots a_r}=0\,.$$

As the space is of constant curvature, simple computations give us $u_{ba_2\cdots a_r}=0$. Q. E. D.

From these lemmas we can obtain

THEOREM 5. *In a space of constant curvature with* $R\neq 0$, *a conformal Killing tensor* $u_{a_1\cdots a_r}$ *of degree* $r(<n)$ *is uniquely decomposed in the form:*

$$u_{a_1\cdots a_r}=p_{a_1\cdots a_r}+q_{a_1\cdots a_r}\,, \tag{4.1}$$

where $p_{a_1\cdots a_r}$ *is a Killing tensor and* $q_{a_1\cdots a_r}$ *is a closed conformal Killing tensor. In this case* $q_{a_1\cdots a_r}$ *is the form*

$$q_{a_1\cdots a_r}=-\frac{n(n-1)}{R}\nabla_{a_1}\rho_{a_2\cdots a_r}\,,$$

where $\rho_{a_2\cdots a_r}$ *is the associated tensor of* $u_{a_1\cdots a_r}$.

Conversely if $p_{a_1\cdots a_r}$ *and* $\rho_{a_1\cdots a_{r-1}}$ *are Killing tensors then* $u_{a_1\cdots a_r}$ *given by* (4.1) *is a conformal Killing tensor.*

PROOF. As $u_{a_1\cdots a_r}$ is a conformal Killing tensor, its associated tensor $\rho_{a_1\cdots a_{r-1}}$ is a Killing tensor and then $\nabla_{a_1}\rho_{a_2\cdots a_r}$ is a conformal Killing tensor whose associated tensor is given by $\dfrac{-R}{n(n-1)}\rho_{a_1\cdots a_{r-1}}$.

By virture of these facts, if we put

$$p_{a_1\cdots a_r}=u_{a_1\cdots a_r}+\frac{n(n-1)}{R}\nabla_{a_1}\rho_{a_2\cdots a_r},$$

simple computations give us

$$\nabla_b p_{a_1\cdots a_r}+\nabla_{a_1}p_{ba_2\cdots a_r}=0,$$

which shows that $p_{a_1\cdots a_r}$ is a Killing tensor. As $q_{a_1\cdots a_r}=\frac{-n(n-1)}{R}\nabla_{a_1}\rho_{a_2\cdots a_r}$ is a closed conformal Killing tensor, the decomposition is shown. The uniquness follows from LEMMA 3. Converse is trivial. Q. E. D.

Bibliography

[1] S. Tachibana: On conformal Killing tensor in a Riemannian space, to appear.
[2] S. Tachibana and T. Kashiwada: On the integrability of Killing-Yano's equation, to appear.
[3] K. Yano and S. Bochner: Curvature and Betti numbers, Ann. of Math. Stud. 32, 1953.