Natural Science Report, Ochanomizu University, Vol. 18, No. 1, 1967
お茶の水女子大学 自然科学報告 第18巻 第1号

Some Remarks on the Existential Theorem of Potential Theory

Hisako Hayakawa (早 川 ヒ サ 子)

Department of Mathematics, Faculty of Science,
Ochanomizu University

(*Received March 16, 1967*)

§ 1. Preliminaries.

Let K be a compact convex set in a Euclidian space and $\mathfrak{R}(K)$ be the family of all closed convex (non-empty) subsets of K. Kakutani's fixed-point theorem[1] asserts that for any upper semicontinuous point-to-set transformation φ from K into $\mathfrak{R}(K)$, there exists a point $x_0 \in K$ such that $x_0 \in \varphi(x_0)$, in which upper semicontinuity means that $\lim_n x_n = x_0$, and $\lim_n y_n = y_0$ $(y_n \in \varphi(x_n))$ imply $y_0 \in \varphi(x_0)$. Prof. K. Fan[2] has generalised this theorem to the case of, not only Euclidian but general locally convex topological vector space E. A point-to-set transformation φ is called upper semicontinuous, if, for any point $x_0 \in K$ and any open set U containing $\varphi(x_0)$, there exists a neighbourhood W of x_0 such that $\varphi(x) \subset U$ for all $x \in W$. For metric spaces both two definitions of the upper semicontinuity of φ is equivalent to each other.

In the present note, by applying this Fan's theorem, we shall prove a lemma useful to the potential theory in case of non-symmetric kernel.

§ 2. Fundamental lemma and its proof.

LEMMA. *Let K be a compact convex subset of a locally convex topological vector space E and $A(x, y)$ be a real-valued continuous function defined on $K \times K$ such that for any point x of K, the mapping $y \to A(x, y)$ is convex. Then there exists a point x_0 in K such that*

$$A(x_0, y) \geqq A(x_0, x_0)$$

for any y in K.

PROOF. Let $\mathfrak{R}(K)$ be the family of all closed convex (non-empty) subsets of K. We set, for any x in K,

$$\varphi(x) = \left\{ y \in K ;\ \inf_{z \in K} A(x, z) = A(x, y) \right\} .$$

1) See [3].
2) See [2].

Since K is a compact and the mapping $y \to A(x, y)$ is continuous, $\varphi(x)$ is a non-empty compact subset of K and since, by the assumption of our lemma, the mapping $y \to A(x, y)$ is convex and K is convex, $\varphi(x)$ is a convex subset of K. Hence φ is a mapping from K into $\mathfrak{R}(K)$.

Moreover we shall show that φ is upper semicontinuous. Let x be any point in K and U be open set containing the set $\varphi(x)$, then $K \cap U^c = K_1$[3] is compact and $K_1 \cap \varphi(x) = \phi$. Hence $\inf_{y \in K_1} A(x, y) = \lambda_1 > \lambda_0 = \inf_{y \in K} A(x, y)$. Writing $\lambda_1 - \lambda_0 = \varepsilon > 0$ and $V = \{y \in K;\ A(x, y) - \lambda_0 < \varepsilon\}$, we have $V \subset U$. Since $A(z, y)$ is a continuous function defined on $K \times K$, we can choose a neighbourhood $W(x)$ of x such that

$$z \in W(x),\ y \in K \Rightarrow |A(x, y) - A(z, y)| < \varepsilon/2 \tag{1}$$

Especially $A(z, y) < A(x, y) + \varepsilon/2$, hence

$$\inf_{y \in K} A(z, y) \leqq \lambda_0 + \varepsilon/2 \tag{2}$$

for any z in $W(x)$. For any point z in $W(x)$ and any point y in $\varphi(x)$, we have, from (1) and (2),

$$A(x, y) - \lambda_0 \leqq A(x, y) - A(z, y) + A(z, y) - \lambda_0 < \varepsilon/2 + \varepsilon/2 = \varepsilon\,.$$

Hence $\varphi(z) \subset V \subset U$ for any z in $W(x)$. This shows that the mapping φ is upper semicontinuous.

By K. Fan's theorem, there exists a point x_0 in K such that $\varphi(x_0) \ni x_0$. From the definition of φ,

$$A(x_0, y) \geqq A(x_0, x_0)$$

for any y in K. Hence the lemma is proved.

§ 3. Applications of the lemma to potential theory.

1. Let X be a locally compact Hausdorff space and $G(x, y)$ be a strictly positive lower semicontinuous function on the product space $X \times X$ of X. Such a function $G(x, y)$ is called a kernel on X. Especially, if $G(x, y)$ is a finite positive continuous function, $G(x, y)$ is called a finitely continuous kernel. If $G(x, y) = G(y, x)$ holds for any x and y in X, $G(x, y)$ is called symmetric. The adjoint kernel $\check{G}(x, y)$ of a kernel $G(x, y)$ is defined by $\check{G}(x, y) = G(y, x)$. Whenever we speak of a measure on X, we mean a Radon measure on X. The potential $G_\mu(x)$ of a measure μ relative to a kernel $G(x, y)$ is defined by

$$G_\mu(x) = \int G(x, y)\,d\mu(y)\,.$$

In the theory of potentials with non-symmetric kernel, Prof. Kishi has proved the following fundamental existence theorem ;

---

3) U^c denotes the complementary set of U in E.

Assume that the adjoint kernel $\check{G}(x, y)$ *of* $G(x, y)$ *satisfies the continuity principle. Given a non-empty separable compact subset* K *of* X *and a strictly positive finite upper semicontinuous function* f *on* K. *Then there exists a positive measure* μ *with support* S_μ *in* K *such that*

$$G_\mu(x) \geqq f(x)$$

G-p.p.p. on K *and*

$$G_\mu(x) \leqq f(x)$$

everywhere on S_μ.

A property is said to hold G-p.p.p. on a subset Y of X, if the property holds on Y except a set E which does not contain any compact support S_ν of a positive measure $\nu \neq 0$ with finite G-energy $\int G_\nu(x) d\nu(x)$.

Prof. M. Nakai has proved in [5] Kishi's theorem without assuming the condition of separability. He began with a finitely continuous kernel and showed that Kishi's theorem is ture for a general kernel by means of the continuity principle of the adjoint kernel. We shall give a more direct proof of the above theorem in case of a finitely continuous kernel.

THEOREM 1. *Let* X *be a locally compact Hausdorff space, and* $G(x, y)$ *be a strictly positive continuous function on* $X \times X$. *Given a non-empty compact subset* K *of* X *and a strictly positive continuous function* f *on* K. *Then there exists a positive measure* μ *on* K *such that for any positive measure* ν *on* K,

$$\iint G(x, y) d\mu(y) d\nu(x) \geqq \int f(x) d\nu(x)$$

and

$$\int G(x, y) d\mu(y) \leqq f(x)$$

for any point x *in the support of* μ.

PROOF. Without loss of generality we may assume $f(x)$ is a constant 1 for any $x \in K$. Let M be the family of all measures on K. M becomes a locally convex topological vector space with the vague topology. Let M^1 be the all positive measures on K with the total measure 1. It is well known that M^1 is a compact convex subset of M. For any pair (μ, ν) of $M \times M$, if we define

$$A(\mu, \nu) = \iint G(x, y) d\mu(y) d\nu(x),$$

then $A(\mu, \nu)$ is a bilinear form on $M \times M$ and a continuous function on $M^1 \times M^1$. From our lemma, there exists a measure $\mu_0 \in M^1$ such that

$$A(\mu_0, \nu) \geqq A(\mu_0, \mu_0) \qquad (\text{for any } \nu \in M^1).$$

Since $A(\mu_0, \mu_0)>0$, we may put $\mu_1=\mu_0/A(\mu_0, \mu_0)$, and we get $A(\mu_1, \nu)\geqq 1$ for any $\nu\in M^1$. Hence for any positive measure ν on K, we have

$$A(\mu_1, \nu)\geqq\int d\nu .$$

From this and $A(\mu_1, \mu_1)=\int d\mu_1$, we get $G_{\mu_1}(x)=1$ for any x in the support of μ_1. Hence μ_1 has all properties required.

2. Let X be a Hilbert space and U be a convex closed subset of X. We put, for a point x in U,

$$V_x=\{y\in X\,;\ \exists\varepsilon>0,\ x+\varepsilon y\in U\} .$$

Namely V_x is the convex cone generated from the convex set $U-x=\{u-x\,;\ u\in U\}$.

Let $a(x, y)$ be a continuous bilinear from on $X\times X$. Suppose there exist a positive constant C sunh that

$$a(x, x)\geqq C\|x\|^2$$

for any $x\in X$. Let f be any point in X' (strongly dual of X) and $\langle f, x\rangle$ signify the value at x. Thcn, there exists a unique $x\in U$ such that

$$a(x, y)\geqq\langle f, y\rangle$$

for any $y\in V_x$.[4)]

We shall prove that in a locally convex topological vector space such a point x exists in U under an additional assumption that U is compact.

THEOREM 2. *Let K be a compact convex subset in a locally oonvex topological vector space X, $a(x, y)$ be a continuous bilinear from on $X\times X$ and f be a continuous linear functional on X, then there exists a point $x\in K$ such that*

$$a(x, y)\geqq f(y)$$

for any y in V_x, where $V_x=\{y\,;\ \exists\varepsilon>0,\ x+\varepsilon y\in K\}$.

PROOF. Put $A(x, y)=a(x, y)-f(y)$ and apply our lemma, then there exists $x\in K$ such that $A(x, z)\geqq A(x, x)$ for any $z\in K$. Hence we have

$$a(x, z)-f(z)\geqq a(x, x)-f(x) . \qquad (*)$$

Since the mapping $y\rightarrow a(x, y)$ and the function f are both linear, for any $\varepsilon>0$,

$$a(x, \varepsilon(z-x))\geqq f(\varepsilon(z-x)) .$$

That is, $a(x, y)\geqq f(y)$ holds for any $y\in V_x$.

4) See [6].

REMARK. More generally, if we assume only that for any $x \in X$ the mapping: $y \rightarrow a(x, y)$ is a convex continuous function and f is a concave continuous function on K, still (*) holds. From (*), we get $a(x_0, z) - a(x_0, x_0) \geqq f(z) - f(x_0)$ for any $z \in K$.

References

[1] Bourbaki, N. Intégration, Paris, (Hermann).
[2] Fan, K. Fixed-point and minimax theorems in locally convex topological linear spaces, Proc. Nat. Acad. Sci. U. S. A. **38** (1952), 121–126.
[3] Kakutani, S. A generalization of Brouwer's fixed point theorem, Duke Math. J. **8** (1941), 457–459.
[4] Kishi, M. Maximum principles in the potential theory, Nagoya Math. J. **23** (1963), 165–187.
[5] Nakai, M. On the fundamental existance theorem of Kishi, Nagoya Math. J. **23** (1963), 189–198.
[6] Stampacchia, G. Formes bilinéaires coercitives sur les ensembles convexes, C. R. Acad. Sci. Paris, **258** (1964), 4413–4416.

(Added in proof)

During the period of the annual meeting of the mathematical society of Japan, may 1967, Dr. Masayuki Ito of Nagoya University let the auther know that the similar proof of Theorem 1 basing on K. Fan's fixed point theorem had already been obtained by Prof. Kishi himself in Nagoya Mathematical Journal, Vol. 27.