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On a $(2m+1)$-dimensional Sasakian Space with Sectional Curvature $>(4m-3)/4m(2m-1)$[1)]

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For a compact Sasakian space, the following result was recently given by S. Tachibana and Y. Ogawa [**1**].

THEOREM 1.[2)] *Let M be a compact $(2m+1)$-dimensional Sasakian space. If any sectional curvature of M is larger than $1/2m$, the second Betti number vanishes; i.e. $b_2(M)=0$.*

In this paper, by making use of Berger's method [**2**] we shall get a little better result, namely,

THEOREM 2. *Let M be a compact $(2m+1)$-dimensional Sasakian space. If any sectional curvature of M is larger than $(4m-3)/4m(2m-1)$, then $b_2(M)=0$.*

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§1. Notations.[3)]

Let M be an n-dimensional Riemannian space. For each point P of M, let $T_p(M)$ be the tangent space of M at P and $\{x^\lambda\}$ $(\lambda=1,\cdots,n)$ be the local coordinates system around P. We denote Riemannian metric by $g_{\lambda\mu}$, the curvature tensor by $R_{\lambda\mu\nu}{}^{\omega}$ and Ricci tensor by $R_{\mu\nu}=R_{\mu\sigma\nu}{}^{\sigma}$. Choose an orthonormal basis $\{X_{(i)}\}$ of $T_p(M)$ and denote its dual basis by $\{e_{(i)}\}$. If we define $R_{\lambda\mu\nu\omega}=R_{\lambda\mu\nu}{}^{\sigma}g_{\sigma\omega}$, then the sectional curvature of the 2-plane spanned by $X_{(i)}$ and $X_{(j)}$ is given by

$$\rho(X_{(i)}, X_{(j)})=\rho(i,j)=R_{ijij}$$

with respect to the basis $\{X_{(i)}\}$.

For a p-form u, let denote its covariant derivative by ∇u, its exterior differential by du and its co-differential by δu. Let $\Delta u=(d\delta+\delta d)u$ be the Laplacian of u. If η denotes the volume element of M, the global inner product of two p-forms u and v is defined by

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1) The result in this paper was reported at the meeting of the mathematical society of Japan which was held in October, 1966.

2) We assume that $m\geqq 2$.

3) As to notations we follow Berger [**2**].

$$\langle u, v\rangle = \int_M (u, v)\eta$$

and the global norm $||u|| \geqq 0$ of u by

$$||u||^2 = \int_M |u|^2\eta ,$$

where (u, v) means the local inner product of u and v and $|u|^2 = (u, u)$.

As for a p-form u the following formulas are well known.

$$(1.1) \qquad \langle u, \Delta u\rangle = ||du||^2 + ||\delta u||^2 .$$

$$(1.2) \qquad \Delta(|u|^2)/2 = (u, \Delta u) - |\nabla u|^2 + Q_p(u)/2(p-1)!,$$

where

$$(1.3) \qquad Q_p(u) = (p-1)R_{\kappa\rho\sigma\tau}u^{\kappa\rho\lambda_3\cdots\lambda_p}u^{\sigma\tau}{}_{\lambda_3\cdots\lambda_p} - 2R_{\rho\sigma}u^{\rho\lambda_2\cdots\lambda_p}u^{\sigma}{}_{\lambda_2\cdots\lambda_p} .$$

In the following α is a 2-form and β is the 4-form defined by $\beta = (\alpha \wedge \alpha)/2$. We know that it is possible to find an orthonormal basis such that α is expressed as

$$\alpha = \alpha_{12}e_{(1)} \wedge e_{(2)} + \alpha_{34}e_{(3)} \wedge e_{(4)} + \cdots + \alpha_{2m-1\,2m}e_{(2m-1)} \wedge e_{(2m)}$$

where $m = [n/2]$.

In this paragraph we argue with respect to this basis; then the coeffcients of

$$\beta = (1/4!) \sum_{i,j,k,h} \beta_{ijkh}e_i \wedge e_j \wedge e_k \wedge e_h$$

are given by

$$\beta_{ijkh} = \alpha_{ij}\alpha_{kh} + \alpha_{ik}\alpha_{hj} + \alpha_{ih}\alpha_{jk} .$$

On the other hand we have

$$\begin{aligned}(\delta\beta)_{\lambda\mu\nu} &= -\nabla_\sigma\beta^\sigma{}_{\lambda\mu\nu} \\ &= -(\nabla_\sigma\alpha^\sigma{}_\lambda)\alpha_{\mu\nu} - (\nabla_\sigma\alpha^\sigma{}_\mu)\alpha_{\nu\lambda} - (\nabla_\sigma\alpha^\sigma{}_\nu)\alpha_{\lambda\mu} \\ &\quad - \alpha^\sigma{}_\lambda\nabla_\sigma\alpha_{\mu\nu} - \alpha^\sigma{}_\mu\nabla_\sigma\alpha_{\nu\lambda} - \alpha^\sigma{}_\nu\nabla_\sigma\alpha_{\lambda\mu} .\end{aligned}$$

Supposing α to be harmonic, each component of $\delta\beta$ is the sum of three terms at the most. Therefore we have

$$|\delta\beta|^2 \leqq 3\sum_{i<j<k} \{\alpha_{ii^*}{}^2(\nabla\alpha)_{i^*jk}{}^2 + \alpha_{jj^*}{}^2(\nabla\alpha)_{j^*ki}{}^2 + \alpha_{kk^*}{}^2(\nabla\alpha)_{k^*ij}{}^2\}$$

where $(\nabla\alpha)_{\lambda\mu\nu}$ are components of $\nabla\alpha$ with respect to the basis and $i^* = i+1$ etc.. Thus, taking account of

$$|\nabla\alpha|^2 = \sum_{i<j}\sum_k (\nabla\alpha)_{kij}{}^2 ,$$

$$|\alpha|^2 = \sum_{i<j} \alpha_{ij}{}^2 ,$$

we have

$$||\delta\beta||^2 \leqq 3\int_M |\alpha|^2|\nabla\alpha|^2\eta .$$

From the formula (1.2) we get

$$|\alpha|^2\Delta(|\alpha|^2)/2=-|\alpha|^2|\nabla\alpha|^2+|\alpha|^2Q_2(\alpha)/2\,.$$

Now we can see that $\int_M|\alpha|^2\Delta(|\alpha|^2)\eta\geqq 0$; in fact it is the result of the formulas

$$|\alpha|^2\Delta(|\alpha|^2)/2=\Delta(|\alpha|^4)/4+(d(|\alpha|^2))^2\,,$$

$$\int_M\Delta(|\alpha|^4)\eta=0\,,\qquad \int_M(d(|\alpha|^2))^2\eta\geqq 0\,.$$

Hence, we obtain

$$\|\delta\beta\|^2\leqq(3/2)\int_M|\alpha|^2Q_2(\alpha)\eta\,. \tag{1.4}$$

Likewise for the 4-form β, taking account of (1.1), (1.2) and

$$d\beta=0,\qquad \int_M\Delta(|\beta|^2)\eta=0$$

we have

$$\|\delta\beta\|^2=\|\nabla\beta\|^2-(1/12)\int_M Q_4(\beta)\eta\geqq-(1/12)\int_M Q_4(\beta)\eta\,. \tag{1.5}$$

Consequently, (1.4) and (1.5) lead us to the inequality

$$\int_M\{(3/2)|\alpha|^2Q_2(\alpha)+(1/12)Q_4(\beta)\}\eta\geqq 0\,. \tag{1.6}$$

§ 2. Sasakian space.[4)]

An n-dimensional Sasakian space M is a Riemannian space which admits a unit Killing vector field $Z=\eta^\lambda\partial/\partial x^\lambda$ satisfying

$$\nabla_\lambda\nabla_\mu\eta_\nu=\eta_\mu g_{\lambda\nu}-\eta_\nu g_{\lambda\mu}\,.$$

It is well known that n is necessarily odd ($n=2m+1$) and M is orientable. If we define the tensor fields $\varphi_{\lambda\mu}$ and $\varphi_\lambda{}^\mu$ by

$$\nabla_\lambda\eta_\mu=\varphi_{\lambda\mu}\,,\qquad \varphi_\lambda{}^\mu=g^{\mu\sigma}\varphi_{\lambda\sigma}\,,$$

then the following identities hold good.

$$\begin{aligned}
&\varphi_\lambda{}^\sigma\varphi_\sigma{}^\mu=-\delta_\lambda^\mu+\eta_\lambda\eta^\mu,\qquad \varphi_\sigma{}^\lambda\eta^\sigma=0\,,\\
&R_{\lambda\mu\nu}{}^\sigma\eta_\sigma=\eta_\mu g_{\lambda\nu}-\eta_\lambda g_{\mu\nu}\,,\\
&R_{\lambda\mu\rho\sigma}\varphi_\nu{}^\rho\varphi_\omega{}^\sigma=R_{\lambda\mu\nu\omega}-\varphi_{\lambda\omega}\varphi_{\mu\nu}+\varphi_{\lambda\nu}\varphi_{\mu\omega}-g_{\lambda\nu}g_{\mu\omega}+g_{\lambda\omega}g_{\mu\nu}\,,\\
&R_{\lambda\sigma\nu\rho}\varphi_\mu{}^\sigma\varphi_\omega{}^\rho=R_{\omega\rho\mu\sigma}\varphi_\nu{}^\rho\varphi_\lambda{}^\sigma+2\{\varphi_{\lambda\omega}\varphi_{\mu\nu}-\varphi_{\lambda\nu}\varphi_{\omega\mu}\\
&\qquad\qquad+(g_{\omega\mu}-\eta_\omega\eta_\mu)g_{\lambda\nu}-(g_{\mu\nu}-\eta_\mu\eta_\nu)g_{\lambda\omega}\}\,.
\end{aligned}$$

For any point of M, we can take an orthonormal basis $\{X_{(1)}, X_{(1^*)}, \cdots,$

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4) See S. Tachibana and Y. Ogawa [**1**].

$X_{(m)}, X_{(m^*)}, X_{(2m+1)}\}$, where $X_{(i^*)}=(\varphi_\lambda{}^\mu X_{(i)}{}^\lambda)$ and $X_{(2m+1)}=Z$, and we shall call such a basis an adapted basis.

With respect to an adapted basis $\{X_{(\lambda)}\}$, we find

$$g_{\lambda\mu}=\begin{cases}1 & \lambda=\mu\,,\\ 0 & \lambda\neq\mu\,,\end{cases}\qquad \eta^\lambda=\eta_\lambda=(0,\,\cdots,\,0,\,1)\,,$$

$$\varphi_{\lambda\mu}=\varphi_\lambda{}^\mu=\begin{cases}1 & \lambda=i\,,\quad \mu=i^*\,,\\ -1 & \lambda=i^*\,,\quad \mu=i\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

Moreover if we put $\rho(\lambda,\mu)=\rho(X_{(\lambda)}, X_{(\mu)})$, then the following equations are the direct result of (2.1).

$$\rho(\lambda,\,2m+1)=1\,,\qquad \lambda\neq 2m+1\,, \tag{2.2}$$

$$\rho(i^*,\,j^*)=\rho(i,\,j)\,, \tag{2.3}$$

$$\rho(i,\,j^*)=\rho(i^*,\,j)\,, \tag{2.4}$$

$$R_{ii^*jj^*}=\rho(i,\,j)+\rho(i,\,j^*)-2\,,\qquad i\neq j\,. \tag{2.5}$$

We call a skew-symmetric tensor $u_{\lambda\mu}$ to be pure if it has the properties

$$\eta^\sigma u_{\sigma\mu}=0\,,\qquad \varphi_\lambda{}^\rho\varphi_\mu{}^\sigma u_{\rho\sigma}=-u_{\lambda\mu}\,,$$

and to be hybrid if

$$\eta^\sigma u_{\sigma\mu}=0\,,\qquad \varphi_\lambda{}^\rho\varphi_\mu{}^\sigma u_{\rho\sigma}=u_{\lambda\mu}\,.$$

§ 3. Harmonic 2-form in a Sasakian space.

Our proof of Theorem 2 will be based on the following three propositions. Henceforth, we consider only a compact Sasakian space M.

PROPOSITION 1. (Tachibana [**3**]) *In a $(2m+1)$-dimensional compact Sasakian space, any harmonic p-form $u=(1/p\,!)u_{\lambda_1\cdots\lambda_p}dx^{\lambda_1}\wedge\cdots\wedge dx^{\lambda_p}$ satisfies $\eta^\sigma u_{\sigma\lambda_2\cdots\lambda_p}=0$, if $p\leqq m$. As the result of this, we can get $\varphi^{\rho\sigma}u_{\rho\sigma\lambda_3\cdots\lambda_p}=0$.*

Generally, for the structure tensor $\varphi_\lambda{}^\mu$, a p-form u which satisfies $\varphi^{\rho\sigma}u_{\rho\sigma\lambda_3\cdots\lambda_p}=0$ is called an effective form after the case of Kaehler manifold. Proposition 1 implies that any harmonic p-form in a compact Sasakian space is effective if $p\leqq m$.

PROPOSITION 2. (S. Tachibana and Y. Ogawa [**1**]) *Any harmonic p-form $u=(1/2)u_{\lambda\mu}dx^\lambda\wedge dx^\mu$ in M can be decomposed in the form*

$$u_{\lambda\mu}=\zeta_{\lambda\mu}+\xi_{\lambda\mu}$$

where $\zeta_{\lambda\mu}$ is a pure harmonic tensor and $\xi_{\lambda\mu}$ is a hybrid harmonic tensor.

PROPOSITION 3. (S. Tachibana and Y. Ogawa [**1**]) *If M has positive sectional curvature, then there exists no pure harmonic 2-form other than zero.*

In a compact orientable Riemannian space, the second Betti number is the number of linearly independent harmonic 2-forms. By the aid of the above three propositions, it is sufficient to prove the following satement:

" *Under the assumption in Theorem 2, there does not exist an (effective) hybrid harmonic 2-form other than zero.*"

§ 4. Proof of Theorem 2.

Let α be an (effective) hybrid harmonic 2-form. We can find an adapted basis at any point such that if $\{e_{(1)}, e_{(1^*)}, \cdots, e_{(m)}, e_{(m^*)}, e_{(2m+1)}\}$ is the dual basis, the components of α are all zero except α_{ii^*}[5]: that is

$$\alpha = \sum_{i=1}^{m} u_i e_i \wedge e_{i^*}, \qquad |\alpha|^2 = \sum_{i=1}^{m} u_i^2 .$$

Moreover α being effective, we get $\sum_{i=1}^{m} u_i = 0$. Let β be the 4-form defined by $\beta = (\alpha \wedge \alpha)/2$, then we have

$$\beta = \sum_{i<j} u_i u_j e_{(i)} \wedge e_{(i^*)} \wedge e_{(j)} \wedge e_{(j^*)}$$

and

$$|\beta|^2 = \sum_{i<j} u_i^2 u_j^2 .$$

In the sequel, we shall investigate the inequality (1.6).

$$\begin{aligned} Q_2(\alpha) &= \sum_{\lambda,\mu,\nu,\omega} R_{\lambda\mu\nu\omega}\alpha_{\lambda\mu}\alpha_{\nu\omega} - 2\sum_{\lambda,\mu,\nu} R_{\lambda\mu}\alpha_{\lambda\nu}\alpha_{\mu\nu} \\ &= 2\sum_{k=1}^{m}\Big(2\sum_{\substack{s=1\\ s\neq k}}^{m} R_{kk^*ss^*}\alpha_{kk^*}\alpha_{ss^*} + 2R_{kk^*kk^*}\alpha_{kk^*}^2 \\ &\quad - R_{kk^*}\alpha_{kk^*}^2 - R_{k^*k^*}\alpha_{kk^*}^2\Big) \\ &= 2\sum_{k=1}^{m}\left\{2\sum_{\substack{s=1\\ s\neq k}}^{m} R_{kk^*ss^*}u_k u_s + 2\rho_{kk^*}u_k^2 - (R_{kk} + R_{k^*k^*})u_k^2\right\}. \end{aligned}$$

Taking account of (2.2)$\sim$(2.5), we get

$$Q_2(\alpha)/2 = -\sum_{k\neq s}(\rho_{ks} + \rho_{ks^*})(u_k - u_s)^2 - 4\sum_{k\neq s} u_k u_s - 2\sum_{k} u_k^2 .$$

If we assume that every sectional curvature is not less than a positive number δ, then we have by the equality (3.1) and $\sum_{k\neq s} u_k u_s = -|\alpha|^2$

$$\begin{aligned} Q_2(\alpha)/2 &\leqq -2\delta\sum_{k\neq s}(u_k - u_s)^2 + 2\sum_{k} u_k^2 \\ &= -2\delta\left\{2(m-1)\sum_{k} u_k^2 - 2\sum_{k\neq s} u_k u_s\right\} + 2\sum_{k} u_k^2 \\ &= -2(2m\delta - 1)|\alpha|^2 . \end{aligned}$$

On the other hand, as we have

5) See S. Tachibana and Y. Ogawa [**1**].

$$Q_4(\beta)=3\sum_{\lambda,\mu,\nu,\omega,\rho,\sigma} R_{\lambda\mu\nu\omega}\beta_{\lambda\mu\rho\sigma}\beta_{\nu\omega\rho\sigma}-2\sum_{\lambda,\mu,\rho,\sigma,\tau} R_{\lambda\mu}\beta_{\lambda\rho\sigma\tau}\beta_{\mu\rho\sigma\tau}$$

$$=3\times 8\Big\{\sum_{k,s,i\neq} R_{kk^*ss^*}\beta_{kk^*ii^*}\beta_{ss^*ii^*}+\sum_{k\neq i} R_{kk^*kk^*}\beta_{kk^*ii^*}{}^2$$

$$+\sum_{k\neq i} R_{ki^*ki^*}\beta_{ki^*k^*i}{}^2+\sum_{k\neq i} R_{kiki}\beta_{kik^*i^*}{}^2\Big\}$$

$$-2\times 6\sum_{k\neq i}(R_{kk}+R_{k^*k^*})\beta_{kk^*ii^*}{}^2\,,$$

we can get

$$Q_4(\beta)/12=-\sum_{k,s,i\neq}(\rho_{ks}+\rho_{ks^*})(u_k-u_s)^2u_i^2$$

$$-4\sum_{k,s,i\neq}u_ku_su_i^2-2\sum_{k,s}u_k^2u_s^2$$

and hence we have

$$Q_4(\beta)/12\leqq -2\delta\sum_{k,s,i\neq}(u_k-u_s)^2u_i^2-4\sum_{k,s,i\neq}u_ku_su_i^2-\sum_{k\neq s}u_k^2u_s^2\,. \tag{4.1}$$

In order to estimate the right hand side we shall prove the following.

LEMMA. $\sum_{k,s,i\neq}u_ku_su_i^2=|\alpha|^4-4|\beta|^2$,

$$|\beta|\leqq\sqrt{(m-1)/2m}\,|\alpha|^2\,.$$

PROOF. We set $\sum_{k,s,i\neq}u_ku_su_i^2=E$. Making use of (3.1) and $\sum_{l\neq s}u_ku_s=-|\alpha|^2$, we get

$$|\alpha|^4=(\sum_{k,s}u_ku_s)^2$$

$$=2\sum_{k\neq s}u_k^2u_s^2+\sum_{k,l,s\neq}(u_k+u_l+u_s+\sum_{t\ k,l,s}u_t)+E$$

$$=4|\beta|^2+E.$$

Therefore we obtain the first equality. For the second we have

$$|\alpha|^4=(\sum_k u_k^2)^2$$

$$=\sum_{k=1}^m u_k^4+2\sum_{k<s}u_k^2u_s^2$$

$$\geqq\{2/(m-1)+2\}\sum_{k<s}u_k^2u_s^2$$

$$=2m|\beta|^2/(m-1)\,.$$

q. e. d.

By virtue of Lemma and (4.1) we can see that the following inequality holds good.

$$(1/12)Q_4(\beta)\leqq -2\delta\{4(m-2)|\beta|^2-2(|\alpha|^4-4|\beta|^2)\}$$

$$-4(|\alpha|^4-4|\beta|^2)-4|\beta|^2$$

$$=-4(2m\delta-3)|\beta|^2-4(1-\delta)|\alpha|^4\,.$$

Consequently we obtain

$$(3/2)|\alpha|^2 Q_2(\alpha)+(1/12)Q_4(\beta)\leqq 2[\{1-(6m+2)\delta\}|\alpha|^4-2(2m\delta-3)|\beta|^2].$$

Now we consider the real-valued function f:

$$f(\alpha)=\{1-(6m+2)\delta\}|\alpha|^4-2(2m\delta-3)|\beta|^2.$$

Taking account of $|\beta|\leqq\sqrt{(m-1)/2m}\,|\alpha|^2$ and (1.6), we conclude that if $\delta>(4m-3)/4m(2m-1)$, then $f\leqq 0$, and $f=0$ if and only if $|\alpha|=0$; i.e. $\alpha=0$. This completes the proof of Theorem 2.

REMARK. After we had prepared this paper, M. Moskal and S. Tanno reported that the second Betti number of a compact Sasakian space with strictly positive sectional curvature is zero.

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