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On the Bochner Curvature Tensor

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The projective curvature tensor of an n dimensional Riemannian space M^n is given by

$$W_{\lambda\mu\nu}{}^{\kappa}=R_{\lambda\mu\nu}{}^{\kappa}+\frac{1}{n-1}(R_{\lambda\nu}\delta_{\mu}{}^{\kappa}-R_{\mu\nu}\delta_{\lambda}^{\kappa})\,,$$

which is invariant under any projective correspondence, where $R_{\lambda\mu\nu}{}^{\kappa}$, $R_{\mu\nu}$ are the Riemannian curvature tensor, the Ricci tensor.

The conformal curvature tensor of M^n is given by

$$C_{\lambda\mu\nu}{}^{\kappa}=R_{\lambda\mu\nu}{}^{\kappa}+\frac{1}{n-2}(R_{\lambda\nu}\delta_{\mu}{}^{\kappa}-R_{\mu\nu}\delta_{\lambda}^{\kappa}+g_{\lambda\nu}R_{\mu}{}^{\kappa}-g_{\mu\nu}R_{\lambda}^{\kappa})$$
$$-\frac{R}{(n-1)(n-2)}(g_{\lambda\nu}\delta_{\mu}{}^{\kappa}-g_{\mu\nu}\delta_{\lambda}^{\nu})\,,$$

where $g_{\lambda\mu}$ is the Riemannian metric of M^n and $R_{\lambda}^{\kappa}=g^{\kappa\alpha}R_{\lambda\alpha}$, $R=g^{\lambda\mu}R_{\lambda\mu}$.

Let K^n be an n $(=2m)$ dimensional Kählerian space with the structure tensor $g_{\lambda\mu}$ and $\varphi_{\lambda}^{\kappa}$. It is known that the tensor

$$P_{\lambda\mu\nu}{}^{\kappa}=R_{\lambda\mu\nu}{}^{\kappa}+\frac{1}{n+2}(R_{\lambda\nu}\delta_{\mu}{}^{\kappa}-R_{\mu\nu}\delta_{\lambda}^{\kappa}+S_{\lambda\nu}\varphi_{\mu}{}^{\kappa}-S_{\mu\nu}\varphi_{\lambda}^{\kappa}+2S_{\lambda\mu}\varphi_{\nu}{}^{\kappa})\,,$$

called the holomorphically projective curvature tensor of K^n, is invariant under any holomorphically projective correspondence.[1] $P_{\lambda\mu\nu}{}^{\kappa}$ may be considered as the tensor corresponding to $W_{\lambda\mu\nu}{}^{\kappa}$. Under this situation it is natural to ask what tensor of K^n does correspond to $C_{\lambda\mu\nu}{}^{\kappa}$.

On the other hand, S. Bochner[2] has introduced a tensor in K^n given by

$$K_{jh^*lk^*}=R_{jh^*lk^*}-\frac{1}{m+2}(R_{h^*l}g_{jk^*}+R_{jh^*}g_{lk^*}+g_{h^*l}R_{jk^*}+g_{jh^*}R_{lk^*})$$
$$+\frac{R}{2(m+1)(m+2)}(g_{h^*l}g_{jk^*}+g_{jh^*}g_{lk^*})$$

with respect to complex local coordinates. It seems that this tensor is

1) S. Tachibana and S. Ishihara, [4].
2) S. Bochner, [2], K. Yano and S. Bochner, [8] p. 162.

the one which we now asked.

In this note at first we shall deal with this problem by giving components of this Bochner's curvature tensor with respect to real local coordinates and find identities they satisfy.

Recently M. Tani obtained the following

THEOREM.[3] *If an n ($\geqq 3$) dimensional compact orientable conformally flat Riemannian space of constant scalar curvature has positive definite Ricci form, then it is a space of constant curvature.*

For a Kählerian space with vanishing Bochner's curvature tensor we shall give a corresponding theorem.

An n $(=2m)$ dimensional Kählerian space K^n is a Riemannian space which admits a tensor field $\varphi_\lambda{}^\mu$ satisfying

$$\varphi_\alpha{}^\lambda \varphi_\mu{}^\alpha = -\delta_\mu{}^\lambda ,$$

$$\varphi_{\lambda\mu} = -\varphi_{\mu\lambda} , \qquad (\varphi_{\lambda\mu} = g_{\mu\alpha}\varphi_\lambda{}^\alpha) ,$$

$$\nabla_\nu \varphi_\lambda{}^\mu = 0 ,$$

where ∇_ν means the operator of covariant differentiation.

We denote by $R_{\lambda\mu\nu}{}^\kappa$ the Riemannian curvature tensor:

$$R_{\lambda\mu\nu}{}^\kappa = \partial_\lambda \left\{ {}^{\kappa}_{\mu\nu} \right\} - \partial_\mu \left\{ {}^{\kappa}_{\lambda\nu} \right\} + \left\{ {}^{\kappa}_{\lambda\alpha} \right\}\left\{ {}^{\alpha}_{\mu\nu} \right\} - \left\{ {}^{\kappa}_{\mu\alpha} \right\}\left\{ {}^{\alpha}_{\lambda\nu} \right\} ,$$ [4]

and by $R_{\mu\nu} = R_{\alpha\mu\nu}{}^\alpha$, $R = g^{\lambda\mu}R_{\lambda\mu}$ the Ricci tensor, the scalar curvature respectively.

It is well known that these tensors satisfy the following identities:

$$R_{\alpha\mu\nu}{}^\kappa \varphi_\lambda{}^\alpha = -R_{\lambda\alpha\nu}{}^\kappa \varphi_\mu{}^\alpha , \qquad R_{\lambda\mu\alpha}{}^\kappa \varphi_\nu{}^\alpha = R_{\lambda\mu\nu}{}^\alpha \varphi_\alpha{}^\kappa ,$$

$$\varphi_\lambda{}^\alpha R_{\alpha\mu} = -R_{\lambda\alpha}\varphi_\mu{}^\alpha , \qquad \varphi_\lambda{}^\alpha R_\alpha{}^\kappa = R_\lambda{}^\alpha \varphi_\alpha{}^\kappa ,$$

$$\nabla_\alpha R_{\lambda\mu\nu}{}^\alpha = \nabla_\lambda R_{\mu\nu} - \nabla_\mu R_{\lambda\nu} , \qquad \nabla_\lambda R = 2\nabla_\alpha R_\lambda{}^\alpha .$$

If we define a tensor $S_{\mu\nu}$ by

$$S_{\mu\nu} = \varphi_\mu{}^\alpha R_{\alpha\nu} ,$$

then we have

$$S_{\mu\nu} = -S_{\nu\mu} , \qquad \varphi_\lambda{}^\alpha S_{\alpha\nu} = -S_{\lambda\alpha}\varphi_\nu{}^\alpha ,$$

$$S_{\mu\nu} = -(1/2)\varphi^{\alpha\beta}R_{\alpha\beta\mu\nu} ,$$

$$2\nabla_\alpha S_\lambda{}^\alpha = \varphi_\lambda{}^\alpha \nabla_\alpha R .$$

As the differential form $S = (1/2)S_{\lambda\mu}dx^\lambda \wedge dx^\mu$ is closed,[5] it follows that

$$\varphi_\lambda{}^\alpha \nabla_\alpha S_{\mu\nu} = -\nabla_\mu R_{\nu\lambda} + \nabla_\nu R_{\mu\lambda} .$$

It is also known that if R is constant then the 2-form S is harmonic.[5]

Now we shall consider a tensor $K_{\lambda\mu\nu}{}^\kappa$ defined by

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3) M. Tani, [**5**].

4) $\partial_\lambda = \partial/\partial x^\lambda$, where $\{x^\lambda\}$ denotes real local coordinates.

5) K. Yano, [**7**] p. 72, S. Tachibana, [**3**].

$$K_{\lambda\mu\nu}{}^{\kappa}=R_{\lambda\mu\nu}{}^{\kappa}+\frac{1}{n+4}(R_{\lambda\nu}\delta_{\mu}^{\kappa}-R_{\mu\nu}\delta_{\lambda}^{\kappa}+g_{\lambda\nu}R_{\mu}^{\kappa}-g_{\mu\nu}R_{\lambda}^{\kappa}$$
$$+S_{\lambda\nu}\varphi_{\mu}{}^{\kappa}-S_{\mu\nu}\varphi_{\lambda}{}^{\kappa}+\varphi_{\lambda\nu}S_{\mu}{}^{\kappa}-\varphi_{\mu\nu}S_{\lambda}{}^{\kappa}$$
$$+2S_{\lambda\mu}\varphi_{\nu}{}^{\kappa}+2\varphi_{\lambda\mu}S_{\nu}{}^{\kappa})$$
$$-\frac{R}{(n+2)(n+4)}(g_{\lambda\mu}\delta_{\mu}^{\kappa}-g_{\mu\nu}\delta_{\lambda}^{\kappa}+\varphi_{\lambda\nu}\varphi_{\mu}{}^{\kappa}-\varphi_{\mu\nu}\varphi_{\lambda}{}^{\kappa}+2\varphi_{\lambda\mu}\varphi_{\nu}{}^{\kappa}),$$

which is constructed formally from $C_{\lambda\mu\nu}{}^{\kappa}$ by taking account of the formal resemblance between $W_{\lambda\mu\nu}{}^{\kappa}$ and $P_{\lambda\mu\nu}{}^{\kappa}$. Then we can prove that the tensor $K_{\lambda\mu\nu\omega}=g_{\kappa\omega}K_{\lambda\mu\nu}{}^{\kappa}$ has components of the tensor given by S. Bochner with respect to complex local coordinates. Hence we call this tensor the *Bochner curvature tensor.*

REMARK. If we put

$$L_{\lambda\mu}=R_{\lambda\mu}-\frac{R}{2(n+2)}g_{\lambda\mu}, \qquad M_{\lambda\mu}=\varphi_{\lambda}{}^{\alpha}L_{\alpha\mu}=S_{\lambda\mu}-\frac{R}{2(n+2)}\varphi_{\lambda\mu},$$

$K_{\lambda\mu\nu}{}^{\kappa}$ has the following form:

$$K_{\lambda\mu\nu}{}^{\kappa}=R_{\lambda\mu\nu}{}^{\kappa}+\frac{1}{n+4}(L_{\lambda\mu}\delta_{\mu}^{\kappa}-L_{\mu\nu}\delta_{\lambda}^{\kappa}+g_{\lambda\nu}L_{\mu}{}^{\kappa}-g_{\mu\nu}L_{\lambda}{}^{\kappa}$$
$$+M_{\lambda\nu}\varphi_{\mu}{}^{\kappa}-M_{\mu\nu}\varphi_{\lambda}{}^{\kappa}+\varphi_{\lambda\nu}M_{\mu}{}^{\kappa}-\varphi_{\mu\nu}M_{\lambda}{}^{\kappa}+2M_{\lambda\mu}\varphi_{\nu}{}^{\kappa}+2\varphi_{\lambda\mu}M_{\nu}{}^{\kappa}).$$

The following identities are obtained by the straightforward computations.

$$K_{\lambda\mu\nu}{}^{\kappa}=-K_{\mu\lambda\nu}{}^{\kappa}, \qquad K_{\lambda\mu\nu\omega}=-K_{\lambda\mu\omega\nu},$$
$$K_{\lambda\mu\nu}{}^{\kappa}+K_{\mu\nu\lambda}{}^{\kappa}+K_{\nu\lambda\mu}{}^{\kappa}=0,$$
$$K_{\alpha\mu\nu}{}^{\alpha}=0, \qquad K_{\lambda\mu\alpha}{}^{\alpha}=0,$$
$$K_{\lambda\mu\nu}{}^{\alpha}\varphi_{\alpha}{}^{\kappa}=K_{\lambda\mu\alpha}{}^{\kappa}\varphi_{\nu}{}^{\alpha}, \qquad K_{\alpha\mu\nu}{}^{\kappa}\varphi_{\lambda}{}^{\alpha}=-K_{\lambda\alpha\nu}{}^{\kappa}\varphi_{\mu}{}^{\alpha},$$
$$K_{\lambda\mu\alpha}{}^{\beta}\varphi_{\beta}{}^{\alpha}=0, \qquad K_{\alpha\mu\nu}{}^{\beta}\varphi_{\beta}{}^{\alpha}=0.$$ [6)]

Next we introduce a tensor $K_{\lambda\mu\nu}$ given by

$$K_{\lambda\mu\nu}=\nabla_{\lambda}R_{\mu\nu}-\nabla_{\mu}R_{\lambda\nu}$$
$$+\frac{1}{2(n+2)}(g_{\lambda\nu}\delta_{\mu}^{\varepsilon}-g_{\mu\nu}\delta_{\lambda}^{\varepsilon}+\varphi_{\lambda\nu}\varphi_{\mu}{}^{\varepsilon}-\varphi_{\mu\nu}\varphi_{\lambda}{}^{\varepsilon}+2\varphi_{\lambda\mu}\varphi_{\nu}{}^{\varepsilon})\nabla_{\varepsilon}R,$$

then we can get the following identity:

$$\nabla_{\alpha}K_{\lambda\mu\nu}{}^{\alpha}=\frac{n}{n+4}K_{\lambda\mu\nu}.$$

Now we consider a tensor $U_{\lambda\mu\nu}{}^{\kappa}$ given by

6) The holomorphically projective curvature tensor satisfies these all identities except the second one. See, S. Tachibana and S. Ishihara, [4].

$$U_{\lambda\mu\nu}{}^{\kappa}=R_{\lambda\mu\nu}{}^{\kappa}+\frac{R}{n(n+2)}(g_{\lambda\nu}\delta_{\mu}^{\kappa}-g_{\mu\nu}\delta_{\lambda}^{\kappa}+\varphi_{\lambda\nu}\varphi_{\mu}{}^{\kappa}-\varphi_{\mu\nu}\varphi_{\lambda}{}^{\kappa}+2\varphi_{\lambda\mu}\varphi_{\nu}{}^{\kappa}),$$

then we can obtain the following

THEOREM 1. *The Bochner curvature tensor coincides with* $U_{\lambda\mu\nu}{}^{\kappa}$ *of a Kählerian space* K^n *if and only if* K^n *is an Einstein space.*

REMARK. The tensor of a Riemannian space defined by

$$Z_{\lambda\mu\nu}{}^{\kappa}=R_{\lambda\mu\nu}{}^{\kappa}+\frac{R}{n(n-1)}(g_{\lambda\nu}\delta_{\mu}^{\kappa}-g_{\mu\nu}\delta_{\lambda}^{\kappa})$$

is called the concircular curvature tensor and is invariant under any concircular correspondence.[7] $U_{\lambda\mu\nu}{}^{\kappa}$ corresponds to $Z_{\lambda\mu\nu}{}^{\kappa}$.

A Kählerian space is called a space of constant holomorphic sectional curvature if its $U_{\lambda\mu\nu}{}^{\kappa}$ vanishes identically. As a corollary of Theorem 1 we have

THEOREM 2.[8] *The Bochner curvature tensor of a space of constant holomorphic sectional curvature vanishes identically.*

The following theorem is known.

THEOREM.[9] *If a compact Kählerian space* K^{2m} *with vanishing Bochner curvature tensor has positive definite Ricci form, then we have*

$$b_{2l}=1\,,\qquad b_{2l+1}=0\,,\qquad 0\leqq 2l,\ 2l+1\leqq (m/2)+2\,,$$

where b_i *denotes the i-th Betti number of* K^{2m}.

Under the assumption stated in this theorem, if the scalar curvature R is constant then the differential 2-form S is harmonic. Hence there exists a scalar function ρ such that $S_{\lambda\mu}=\rho\varphi_{\lambda\mu}$ by virtue of the theorem stated above. Thus our K^{2m} is an Einstein space and hence a space of constant holomorphic sectional curvature, because of Theorem 1. On the other hand, we know the following theorem.

THEOREM.[10] *If a compact Kählerian space of constant scalar curvature has positive sectional curvature, then it is a complex projective space with the natural metric.*

Thus we have the following

THEOREM 3. *If a compact Kählerian space with vanishing Bochner curvature tensor of constant scalar curvature has positive definite Ricci form, then it is a complex projective space with the natural metric.*

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7) K. Yano, [**6**].
8) S. Bochner, [**2**].
9) S. Bochner, [**2**], K. Yano and S. Bochner, [**8**] p. 164.
10) R. L. Bishop and S. I. Goldberg, [**1**].

Bibliography

[1] R. L. Bishop and S. I. Goldberg, Rigidity of positively curved Kaehler manifolds, Proc. Natl. Acad. Sc. U. S. A., 54 (1965) 1037-1041.

[2] S. Bochner, Curvature and Betti numbers, II, Ann. of Math., 50 (1949) 77-93.

[3] S. Tachibana, On automorphisms of certain compact almost Hermitian space, Tôhoku Math. J., 13 (1961) 179-185.

[4] S. Tachibana and S. Ishihara, On infinitesimal holomorphically projective transformations in Kählerian manifolds, Tôhoku Math. J., 12 (1960) 77-101.

[5] M. Tani, On a compact conformally flat space with positve Ricci curvature, to appear.

[6] K. Yano, Concircular geometry, I, II, III., Proc. Imp. Acad., 16 (1940) 195-200, 354-360, 442-448.

[7] K. Yano, Differential geometry on complex and almost complex spaces, Pergamon Press, 1965.

[8] K. Yano and S. Bochner, Curvature and Betti numbers, Ann. of Math. Stud., 32, 1953.