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On the second Betti number of a compact Sasakian space

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Recently S. I. Goldberg [**4**] proved the following

THEOREM A. *If a compact, simply connected, regular* $2m+1$ *dimensional Sasakian space has positive sectional curvature and its scalar curvature is constant, then it is isometric with a sphere* S^{2m+1} *with the natural structure.*

On the other hand the odd dimensional Betti number $b_{2p+1}(M, \boldsymbol{R})$, $1 \leqq 2p+1 \leqq m$, of a compact Sasakian space M is even[1)] and for the even dimensional Betti number of M the following theorem is known [**4**].

THEOREM B. *If a compact, regular* $2m+1$ *dimensional Sasakian space* M *has positive sectional curvature, then* $b_2(M, \boldsymbol{R})=0$.

The assumption "*regular*" in the theorems is essential, because the fibration of Boothby-Wang is used in their proofs.

In this paper we shall prove the following theorem without the assumption "*regular*".

THEOREM C. *If any sectional curvature* $\rho(X, Y)$ *of a complete* $2m+1$ $(\geqq 5)$ *dimensional, Sasakian space* M *satisfies*

$$\rho(X, Y) > \frac{1}{2m},$$

then we have $b_2(M, \boldsymbol{R})=0$.

REMARK. The metric of our Sasakian space is not normalized in the sence that the maximum sectional curvature is 1, though it has been normalized in a certain sence.

As to the notations we follow S. Tachibana [**5**] and give definitions, preliminary facts and formulas in §**1** and §**2**. In §**3**∼§**5** we shall prove Theorem C by the method of Berger [**2**] and Bishop-Goldberg [**3**].

§1. Sectional curvature. Harmonic tensor.

Consider an n dimensional Riemannian space M with local coordinate systems $\{x^\lambda\}$[2)] and denote the Riemannian metric, the curvature tensor by $g_{\lambda\mu}$, $R_{\lambda\mu\nu}{}^{\omega}$ respectively. At a point P of M the sectional

1) In the case when $p=0$, see S. Tachibana [**5**]. For $p>0$, see §**3** in this paper.
2) $\lambda, \mu, \cdots = 1, \cdots, n$.

curvature $\rho(X, Y)$ of 2-plane spanned by linearly independent vectors $X=v^\lambda\partial_\lambda$[3] and $Y=u^\lambda\partial_\lambda$ is given by

$$\rho(X, Y)=-\frac{R_{\lambda\mu\nu\omega}v^\lambda u^\mu v^\nu u^\omega}{(g_{\lambda\nu}g_{\mu\omega}-g_{\lambda\omega}g_{\mu\nu})v^\lambda u^\mu v^\nu u^\omega} .$$

If any sectional curvature of any point of M is positive, then we call M to be *of positive sectional curvature.*

As the Ricci tensor is defined by $R_{\mu\nu}=R_{\lambda\mu\nu}{}^\lambda$, if we take an orthonormal basis $X_{(i)}=v_{(i)}{}^\lambda\partial_\lambda$, $(i=1, \cdots, n)$, at a point P, we have

$$R_{\mu\nu}=-\sum_i R_{\alpha\mu\nu\beta}v_{(i)}{}^\alpha v_{(i)}{}^\beta$$

and hence we have

$$R_{\mu\nu}v_{(1)}{}^\mu v_{(1)}{}^\nu=\sum_{i=2}^n \rho(X_{(1)}, X_{(i)}) .$$

Let $u_{\lambda\mu}$ be a harmonic tensor, then we have

$$\nabla^\alpha\nabla_\alpha u_{\lambda\mu}-R_\lambda{}^\alpha u_{\alpha\mu}-R_\mu{}^\alpha u_{\lambda\alpha}-R_{\lambda\mu}{}^{\rho\sigma}u_{\rho\sigma}=0 .$$

Thus the following well known fact is obtained by taking account of the Laplacian of $u_{\lambda\mu}u^{\lambda\mu}$.

In a compact orientable Riemannian space M,

$$\int_M F(u)dV\leqq 0$$

holds good for any harmonic 2-form $u=(1/2)u_{\lambda\mu}dx^\lambda\wedge dx^\mu$, where dV means the volume element of M and $F(u)$ is defined by

$$F(u)=R_{\lambda\mu}u^{\lambda\alpha}u^\mu{}_\alpha+(1/2)R_{\lambda\mu\rho\sigma}u^{\lambda\mu}u^{\rho\sigma} . \tag{1.1}$$

§ 2. Sasakian space.

An n-dimensional Sasakian space M is a Riemannian space which admits a unit Killing vector field $Z=\eta^\lambda\partial_\lambda$ satisfying

$$\nabla_\lambda\nabla_\mu\eta_\nu=\eta_\mu g_{\lambda\nu}-\eta_\nu g_{\lambda\mu} .$$

In this case n is necessarily odd $(n=2m+1)$ and M is orientable. In the following we shall mean by M an n $(=2m+1)$ dimensional Sasakian space.

Now if we define a tensor field $\varphi_{\lambda\mu}$ by

$$\varphi_{\lambda\mu}=\nabla_\lambda\eta_\mu , \qquad \varphi_\mu{}^\nu=g^{\nu\alpha}\varphi_{\mu\alpha} ,$$

then the following formulas valid.

$$\varphi_\alpha{}^\lambda\varphi_\mu{}^\alpha=-\delta_\mu^\lambda+\eta_\mu\eta^\lambda , \qquad \varphi_\alpha{}^\lambda\eta^\alpha=0 ,$$

$$\varphi_{\mu\nu}=-\varphi_{\nu\mu} .$$

For the curvature tensor of M the following identities hold good:

3) $\partial_\lambda=\partial/\partial x^\lambda$.

4) Cf. S. Tachibana [5].

(2.1) $$R_{\lambda\mu\nu}{}^{\varepsilon}\eta_{\varepsilon}=\eta_{\lambda}g_{\mu\nu}-\eta_{\mu}g_{\lambda\nu}$$

(2.2) $$R_{\lambda\mu\rho\sigma}\varphi_{\nu}{}^{\rho}\varphi_{\omega}{}^{\sigma}=R_{\lambda\mu\nu\omega}+g_{\lambda\nu}g_{\mu\omega}-g_{\lambda\omega}g_{\mu\nu}-\varphi_{\lambda\nu}\varphi_{\mu\omega}+\varphi_{\lambda\omega}\varphi_{\mu\nu}$$

(2.3) $$R_{\alpha\beta\rho\sigma}\varphi_{\lambda}{}^{\alpha}\varphi_{\mu}{}^{\beta}\varphi_{\nu}{}^{\rho}\varphi_{\omega}{}^{\sigma}=R_{\lambda\mu\nu\omega}+\eta_{\lambda}\eta_{\nu}g_{\mu\omega}+\eta_{\mu}\eta_{\omega}g_{\lambda\nu}-\eta_{\mu}\eta_{\nu}g_{\lambda\omega}-\eta_{\lambda}\eta_{\omega}g_{\mu\nu}\,.$$

By virtue of (2.1) it follows that $\rho(Z, X)=1$ for any vector X, linearly independent to Z.

Let P be any point of M, then we can take an orthonormal basis $X_{(i)}=v_{(i)}{}^{\lambda}\partial_{\lambda}$, $X_{(i^*)}=v_{(i^*)}{}^{\lambda}\partial_{\lambda}$, $X_{(n)}=Z=\eta^{\lambda}\partial_{\lambda}$ of $T_p(M)$, the tangent space at P, such that

$$g_{\lambda\mu}\eta^{\lambda}v_{(i)}{}^{\mu}=0\,,\qquad v_{(i^*)}{}^{\lambda}=\varphi_{\alpha}{}^{\lambda}v_{(i)}{}^{\alpha}\,,\qquad (i=1,\cdots,m,\ i^*=m+i)\,.$$

We shall call such a basis an *adapted* basis. Components of tensors $g_{\lambda\mu}$, $\varphi_{\lambda\mu}$, η^{λ} with respect to an adapted basis $X_{(\lambda)}$ at the point P have following forms:

$$g_{\lambda\mu}=\delta_{\lambda\mu}\,,\qquad \eta^{\lambda}=\eta_{\lambda}=(0,\cdots,0,1)$$

$$\varphi_{\lambda\mu}=\varphi_{\lambda}{}^{\mu}=\begin{cases}1, & \text{if } \lambda=i,\quad \mu=i^*\,,\\ -1, & \text{if } \lambda=i^*,\quad \mu=i\,,\\ 0, & \text{otherwise}\,.\end{cases}$$

If we put $\rho(\lambda,\mu)=\rho(X_{(\lambda)}, X_{(\mu)})$ for an adapted basis $X_{(\lambda)}$, then we can get from (2.1) and (2.2) the following relations:

(2.4) $$\rho(i,j)=\rho(i^*,j^*)\,,\qquad \rho(i,j^*)=\rho(i^*,j)$$

$$R_{jii^*j^*}=\rho(i,j)-1$$

$$R_{ji^*j^*i}=\rho(i,j^*)-1$$

(2.5) $$R_{jj^*ii^*}=-\rho(i,j)-\rho(i,j^*)+2\,,$$

where $i,j=1,\cdots,m$, $i^*=m+i$, $j^*=m+j$ and $i\neq j$.

We shall call a skew-symmetric tensor $u_{\lambda\mu}$ to be *pure* if it satisfies

$$\eta^{\lambda}u_{\lambda\mu}=0\,,\qquad \varphi_{\lambda}{}^{\alpha}\varphi_{\mu}{}^{\beta}u_{\alpha\beta}=-u_{\lambda\mu}$$

and to be *hybrid* if it satisfies

$$\eta^{\lambda}u_{\lambda\mu}=0\,,\qquad \varphi_{\lambda}{}^{\alpha}\varphi_{\mu}{}^{\beta}u_{\alpha\beta}=u_{\lambda\mu}\,.$$

§ 3. Proof of Theorem C.

The following two theorems are known [**5**].

THEOREM D. *In an n $(=2m+1)$ dimensional compact Sasakian space, any harmonic p-form $u: u_{\lambda_1\cdots\lambda_p}$, $(1\leqq p\leqq m)$, satisfies*

(3.1) $$\eta^{\alpha}u_{\alpha\lambda_2\cdots\lambda_p}=0\,.$$

THEOREM E. *If u is a harmonic p-form, $(1\leqq p\leqq m)$, in an n $(=2m+1)$ dimensional compact Sasakian space, then the p-form Φu defined by*

$$\Phi u:\ (\Phi u)_{\lambda_1\cdots\lambda_p}=\sum_{i=1}^{p}\varphi_{\lambda_i}{}^{\alpha}u_{\lambda_1\cdots\alpha\cdots\lambda_p}$$

is harmonic.

Operating $\nabla^{\lambda_2}=g^{\lambda_2\alpha}\nabla_\alpha$ to (3.1) we get

$$\varphi^{\rho\sigma}u_{\rho\sigma\lambda_3\cdots\lambda_p}=0 \qquad (2\leqq p\leqq m)\,. \tag{3.2}$$

REMARK. It is known that the first Betti number $b_1(M, \boldsymbol{R})$ of an $2m+1$ dimensional compact Sasakian space M is even [**5**, Theorem 4.3]. More generally we can prove that $b_{2p+1}(M, \boldsymbol{R})$ *is even*, if $1\leqq 2p+1\leqq m$.[5] In fact, let $\mathfrak{H}^q(M)$ be the vector space of harmonic q-form over M $(1\leqq q\leqq m)$, and define $\Psi: u\rightarrow\Psi u$ by

$$(\Psi u)_{\lambda_1\cdots\lambda_q}=\varphi_{\lambda_1}{}^{\alpha_1}\cdots\varphi_{\lambda_q}{}^{\alpha_q}u_{\alpha_1\cdots\alpha_q}\,.$$

Then, by virtue of Theorem D and E, Ψ is a linear transformation of $\mathfrak{H}^q(M)$. As we have $\Psi^2=-I$ (I means the identity transformation of $\mathfrak{H}^q(M)$) for $q=2p+1$, so $\mathfrak{H}^{2p+1}(M)$ admits a complex structure and hence $\dim\mathfrak{H}^{2p+1}(M)$ is even.

In the following we shall always mean by M an n $(=2m+1)$ dimensional compact Sasakian space and assume $n\geqq 5$.

Theorem C follows from the following three lemmas.

LEMMA 1. *Any harmonic 2-form* $u=(1/2)u_{\lambda\mu}dx^\lambda\wedge dx^\mu$ *in* M *is written in the following form*

$$u_{\lambda\mu}=\zeta_{\lambda\mu}+\xi_{\lambda\mu}\,,$$

where $\zeta_{\lambda\mu}$ *is pure harmonic and* $\xi_{\lambda\mu}$ *is hybrid harmonic.*

LEMMA 2. *If* M *has positive sectional curvature, then there does not exist a pure harmonic 2-form other than zero.*

LEMMA 3. *If any sectional curvature* $\rho(X, Y)$ *of* M *satisfies*

$$\rho(X, Y)>\frac{1}{2m}\,,$$

then there does not exist a hybrid harmonic 2-form other than zero.

PROOF OF LEMMA 1. For a harmonic tensor $u_{\lambda\mu}$.

$$(\Phi^2u)_{\lambda\mu}=-2u_{\lambda\mu}+2\varphi_\lambda{}^\alpha\varphi_\mu{}^\beta u_{\alpha\beta}$$

is harmonic, by virtue of Theorem D and E, and hence so is $\varphi_\lambda{}^\alpha\varphi_\mu{}^\beta u_{\alpha\beta}$. Thus putting

$$\zeta_{\lambda\mu}=(1/2)(u_{\lambda\mu}-\varphi_\lambda{}^\alpha\varphi_\mu{}^\beta u_{\alpha\beta})\,,\qquad \xi_{\lambda\mu}=(1/2)(u_{\lambda\mu}+\varphi_\lambda{}^\alpha\varphi_\mu{}^\beta u_{\alpha\beta})\,,$$

we can get Lemma 1.

§ 4. Proof of Lemma 2.

Let $\zeta_{\lambda\mu}$ be pure harmonic in M with positive sectional curvature. Taking account of (2.2) and the purity of $\zeta_{\lambda\mu}$, we can get

5) This is due to H. Wakakuwa.

$$F(\zeta)=R_{\lambda\mu}\zeta^{\lambda\alpha}\zeta^{\mu}{}_{\alpha}+(1/2)R_{\lambda\mu\nu\omega}\zeta^{\lambda\mu}\zeta^{\nu\omega}$$
$$=R_{\lambda\mu}\zeta^{\lambda\alpha}\zeta^{\mu}{}_{\alpha}-\zeta_{\lambda\mu}\zeta^{\lambda\mu}\,,$$

where $F(\cdot)$ is defined by (1.1).

Now assume that $\zeta_{\lambda\mu}$ is not a zero tensor at a point P. Let $X_{(\lambda)}$ be an orthonormal basis of $T_p(M)$, then in terms of components with respect to this basis $F(\zeta)$ becomes the following form:

$$F(\zeta)=\sum_{\lambda,\mu,\alpha}R_{\lambda\mu}\zeta_{\lambda\alpha}\zeta_{\mu\alpha}-\sum_{\lambda,\mu}\zeta_{\lambda\mu}\zeta_{\lambda\mu}\,.$$

If $\sum_{\lambda}\zeta_{\lambda1}^2=a^2\neq 0$, then we can take an orthonormal vectors $Y_{(1)}, Y_{(2)}, \cdots, Y_{(n)}$ such that

$$Y_{(1)}=\sum_{\lambda}\frac{\zeta_{\lambda1}}{a}X_{(\lambda)}\,,\qquad Y_{(2)}=Z$$

and we have

$$\sum_{\lambda,\mu}R_{\lambda\mu}\zeta_{\lambda1}\zeta_{\mu1}=a^2\rho(Y_{(1)}, Y_{(2)})+a^2\sum_{i=3}^{n}\rho(Y_{(1)}, Y_{(i)})$$
$$>a^2\rho(Y_{(1)}, Y_{(2)})=a^2=\sum_{\lambda}\zeta_{\lambda1}^2\,.$$

Thus if $\zeta_{\lambda\mu}$ is not a zero tensor at P, then we have $F(\zeta)>0$ which contradicts to the argument in § **1**.

§ 5. Proof of Lemma 3.

Let $\xi_{\lambda\mu}$ be a hybrid harmonic tensor in M with sectional curvature $\geqq\delta>\dfrac{1}{2m}$.

Assume that $\xi_{\lambda\mu}$ is not a zero tensor at a point P and in the following we consider all quantities at the point P. Now we define $a_{\lambda}{}^{\mu}$ by $a_{\lambda}{}^{\mu}=\varphi_{\lambda}{}^{\alpha}\xi_{\alpha}{}^{\mu}$, then we have $\varphi_{\lambda}{}^{\alpha}a_{\alpha}{}^{\mu}=a_{\lambda}{}^{\alpha}\varphi_{\alpha}{}^{\mu}$. Hence if $X=v^{\lambda}\partial_{\lambda}$ is a proper vector of the matrix $(a_{\lambda}{}^{\mu})$, so is $\varphi X=\varphi_{\alpha}{}^{\lambda}v^{\alpha}\partial_{\lambda}$. As $Z=\eta^{\lambda}\partial_{\lambda}$ is a proper vector of $(a_{\lambda}{}^{\mu})$, we can get an adapted basis $X_{(i)}$, $X_{(i^*)}=\varphi X_{(i)}$, $X_{(n)}=Z$, where $X_{(i)}$, $(i=1, \cdots, m)$, are proper vectors of $(a_{\lambda}{}^{\mu})$. With respect to such an adapted basis, components of $\xi_{\lambda\mu}$ are all zero except $\xi_i=\xi_{ii^*}=-\xi_{i^*i}$. From (3.2) it follows that

$$\sum_{i=1}^{m}\xi_i=0\,.$$

Taking account of this equation, (2.4), (2.5) and

$$R_{ii}=R_{i^*i^*}=\sum_{\substack{j=1\\ j\neq i}}^{m}\rho(i, j)+\sum_{j=1}^{m}\rho(i, j^*)+1\,,$$

$F(\xi)$ becomes the following form:

$$
\begin{aligned}
F(\xi) &= \sum_{\lambda,\mu,\alpha} R_{\lambda\mu}\xi_{\lambda\alpha}\xi_{\mu\alpha} + (1/2)\sum_{\lambda,\mu,\nu,\omega} R_{\lambda\mu\nu\omega}\xi_{\lambda\mu}\xi_{\nu\omega} \\
&= \sum_i (R_{ii} + R_{i^*i^*})\xi_i^2 + 2\sum_{i,j} R_{jj^*ii^*}\xi_j\xi_i \\
&= 2\sum_i \{\sum_{j\neq i}\rho(i, j) + \sum_j \rho(i, j^*) + 1\}\xi_i^2 \\
&\quad -2\sum_i \{\sum_{j\neq i}(\rho(i, j) + \rho(i, j^*) - 2)\xi_j\xi_i + \rho(i, i^*)\xi_i^2\} \\
&= \sum_i\sum_{j\neq i}\{\rho(i, j) + \rho(i, j^*)\}(\xi_i - \xi_j)^2 - 2\sum_i \xi_i^2 \\
&\geqq 2\delta\sum_i\sum_{j\neq i}(\xi_i^2 - 2\xi_j\xi_i + \xi_j^2) - 2\sum_i \xi_i^2 \\
&= 2(2m\delta - 1)\sum_i \xi_i^2 > 0 .
\end{aligned}
$$

Thus the lemma is proved.

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