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An Application of Spectral Properties of Non-support Operators to a Theorem of S. Karlin

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Recently, S. Karlin has shown the existence of infinite eigenvalues for integral operators with extended totally positive kernels or more general kernels [1]. Each of these kernels must satisfy some conditions of differentiability. In the following note, we shall show that, applying the spectral properties of non-support operators obtained by the author [3], the above Karlin's results can be extended to the case of integral operators without differentiability conditions. For example, as one of these kernels, we can take a kernel $K(x, s)$ of Schmidt type on $(a, b)\times(a, b)$ which is totally positive, continuous (not necessarily symmetric) and satisfies

$$K\begin{pmatrix} x_1, \cdots, x_n \\ x_1, \cdots, x_n \end{pmatrix} = \det\Big(K(x_i, x_j)\Big)_{i,j=1}^{n} > 0, \quad n=1, 2 \cdots,$$
$$a < x_1 < \cdots < x_n < b .$$

The author wishes to express her hearty thanks to Prof. S. Karlin for his valuable informations and remarks given her during his stay in Japan.

§1. Terminologies and notations. We suppose that S be a measure space where measure μ is finite and that $L_2(S)$ be a L_2-space over S ordered by the positive cone consisting of almost everywhere non-negative functions. The positive cone is clearly closed, proper, generating, normal and minihedral. A linear operator T in $L_2(S)$ is *positive* if T leaves the positive cone invariant. Further, if T is positive and, for arbitrary non-zero elements f, g in the positive cone, there exists a positive integer n_0 such that

$$\int_S T^n f(x) g(x) d\mu(x) > 0 \qquad \text{for} \quad n \geqq n_0 ,$$

then T is called a *non-support operator.* The *spectral radius* $r(T)$ of a linear operator T is defined by $r(T) = \lim \| T^n \|^{1/n}$. A μ^2-measurable function $K(x, s)$ on $S\times S$ is said to be of *Schmidt type* if the integral of $|K(x, s)|^2$ over $S\times S$ is finite. It is well known that the integral operator with a kernel $K(x, s)$ of Schmidt type, i. e.

$$Tf(x)=\int_S K(x, s)f(s)d\mu(s)$$

is completely continuous and the iterated operator T^n is associated with the iterated kernel $K^{(n)}(x, s)$, defined by

$$K^{(n)}(x, s)=\int_S \cdots \int_S K(x, s_{n-1}) \cdots K(s_1, s)\ d\mu(s_1) \cdots d\mu(s_{n-1})$$

which is also of Schmidt type

In the following, we shall make use of terminologies and notations after S. Karlin's paper [1].

$$\varDelta_n=\{\overline{x}=(x_1, \cdots, x_n)\ ;\ a<x_1<\cdots<x_n<b\}$$

denote the open simplex in a n-dimensional Euclidean space E^n. A function

$$K_{[n]}(\overline{x}, \overline{s})=\det\left(K(x_i, s_j)\right)_{i,j=1}^{n}, \quad \begin{array}{l}\overline{x}=(x_1, \cdots x_n)\\ \overline{s}=(s_1, \cdots s_n)\end{array}$$

is defined on $\varDelta_n\times\varDelta_n$ and is called the *compound kernel* of order n induced by a kernel $K(x, s)$. We see easily by the calculation of determinants that if $K(x, s)$ is of Schmidt type and μ is Lebesgue measure then $K_{[n]}(\overline{x}, \overline{s})$ is also of Schmidt type. Therefore, the integral operator $T_{[n]}$ with the compound kernel $K_{[n]}(\overline{x}, \overline{s})$ defined on $L_2(\varDelta_n)$ is also completely continuous with respect to the product measure μ^n. For functions $f_1, \cdots, f_n$ in $L^2(\varDelta)$, the exterior product $f_1\wedge\cdots\wedge f_n$ is defined by

$$f_1\wedge\cdots\wedge f_n(\overline{x})=\det\left(f_i(x_j)\right)_{i,j=1}^{n}.$$

Hereafter, we shall assume that a kernel $K(x, s)$ is always of Schmidt type and almost everywhere non-negative.

§ 2. Non-support operators in $L_2(S)$. In this section, we shall show some conditions for kernels in order that integral operators with the kernels be non-support operators.

PROPOSITION 1. *Let a kernel $K(x, s)$ on $S\times S$ satisfy the following condition* (a):

(a) *For every positive number ε, there exists an integer $n_0\geqq 1$ such that*

$$\mu^2\{(x, s)\ ;\ K^{(n)}(x, s)=0\}<\varepsilon^2 \quad \textit{for} \quad n\geqq n_0\,. \tag{2.1}$$

Then the integral operator T with the kernel $K(x, s)$ is a non-support operator and the spectral radius $r(T)$ is positive.

PROOF. We shall prove that T is a non-support operator. Let $f\neq 0$ and $g\neq 0$ be almost everywhere non-negative functions in $L_2(S)$. Then measures of the supports S_f, S_g, i.e.

$$S_f=\{s\,;\,f(s)>0\},\quad S_g=\{s\,;\,g(s)>0\}\,,$$

are positive. By the condition (a), for a positive number $\varepsilon^2=\mu^2(S_g\times S_f)$, there exists an integer $n_0(\varepsilon)\geqq 1$ satisfying (2.1). Put $A_n=\{(x,s)\,;\,K^{(n)}(x,s)=0\}$ for $n\geqq n_0(\varepsilon)$. Then, we have $\mu^2(A_n)<\mu^2(S_g\times S_f)$ which implies that the set $(S_g\times S_f)\ominus A_n$[1] is of measure positive. Therefore, we have

$$\langle T^n f\mid g\rangle=\int_{S\times S}K^{(n)}(x,s)f(s)g(x)\;d\mu^2(x,s)$$

$$=\int_{(S_g\times S_f)\ominus A_n}K^{(n)}(x,s)f(s)g(x)\;d\mu^2(x,s)>0$$

for $n\geqq n_0$. Thus, T is a non-support operator.

Next, we shall show $r(T)>0$. We can assume that $\mu(S)=1$ without loss of generality. By the condition (a), there exists an integer $n\geqq 1$ satisfying

$$\mu^2\{(x,s)\,;\,K^{(n)}(x,s)=0\}<(1/3)^2\,.$$

Hence, there exists a positive number δ such that

$$\mu^2\{(x,s)\,;\,K^{(n)}(x,s)<\delta\}<(1/3)^2\,.$$

This implies

$$\mu[x\,;\,\mu\{s'\,;\,K^{(n)}(x,s')<\delta\}\geqq(1/3)]<1/3\,.$$

That is,

$$\mu[x\,;\,\mu\{s'\,;\,K^{(n)}(x,s')<\delta\}<(1/3)]>2/3\,. \tag{2.2}$$

Let X be the set of the left hand side of (2.2). Then, we have $\mu(X)>2/3$ and

$$X\subset[x\,;\,\mu\{s'\,;\,K^{(n)}(x,s')<\delta,\ s'\in X\}<1/3]$$

$$=[x\,;\,\mu\{s'\,;\,K^{(n)}(x,s')\geqq\delta,\ s'\in X\}>\mu(X)-1/3]\,.$$

If $x\in X$, then

$$T^n\chi_X(x)=\int_S K^{(n)}(x,s)\chi_X(s)\;d\mu(s)$$

$$=\int_X K^{(n)}(x,s)\;d\mu(s)$$

$$>(\mu(X)-1/3)\delta>\delta/3>0$$

where χ_X denotes the characteristic function of X. Therefore, $T^n\chi_X\geqq(\delta/3)\chi_X$ which implies $r(T)\geqq(\delta/3)^{1/n}>0$. This completes the proof of

1) $\ominus$ is the set theoretical notation, i.e. $X\ominus Y=\{z\,;\,z\in X$ and $z\notin Y\}$.

proposition 1.

REMARK 1. Using the following lemma, we can replace the condition (a) in proposition 1 by the apparently weaker condition (a′)

(a′) *For every positive number* ε, *there exists an integer* $n_0 \geqq 1$ *such that*

$$\mu^2\{(x, s)\,;\, K^{(n_0)}(x, s)=0\}<\varepsilon^2\,. \tag{2.1′}$$

LEMMA 1. *The conditions* (a) *and* (a′) *for a kernel* $K(x, s)$ *are equivalent to each other.*

PROOF. We shall prove that (a′) implies (a). We may suppose $\mu(S)=1$. To each ε, $0<\varepsilon<1/2$, we shall consider three kinds of sets of positive integers,

$$N(\varepsilon)=[n\,;\,\mu^2\{(x, s)\,;\,K^{(n)}(x, s)=0\}<\varepsilon^2]\,,$$

$$M_x(\varepsilon)=[n\,;\,\mu[x\,;\,\mu\{s'\,;\,K^{(n)}(x, s')>0\}\geqq 1-\varepsilon]\geqq 1-\varepsilon]\,,$$

$$M_s(\varepsilon)=[n\,;\,\mu[s\,;\,\mu\{x'\,;\,K^{(n)}(x', s)>0\}\geqq 1-\varepsilon]\geqq 1-\varepsilon]\,.$$

Then, we have

$$N(\varepsilon)\neq\phi\,, \tag{2.3}$$

$$N(\varepsilon)\subset M_x(\varepsilon)\cap M_s(\varepsilon) \quad \text{and} \quad M_x(\varepsilon)\cup M_s(\varepsilon)\subset N(\sqrt{2\varepsilon})\,, \tag{2.4}$$

$$n,\, m\in M_x(\varepsilon)\cap M_s(\varepsilon) \quad \text{implies} \quad n+m\in M_x(\varepsilon)\cap M_s(\varepsilon)\,. \tag{2.5}$$

Moreover, we shall prove that

(2.6) there exist two integers $k_0(\varepsilon)$ and $n_0(\varepsilon)$ satisfying

$$k_0\in M_x(\varepsilon)\cap M_s(\varepsilon) \quad \text{and} \quad n_0+r\in M_x(\varepsilon)\cap M_s(\varepsilon),\ r=1, \cdots, k_0.$$

Let k_0 be any element of $N(\varepsilon)$. Then, there exists a positive number $\delta(k_0, \varepsilon)$ satisfying $\varepsilon>\delta>0$ and

$$\mu[s\,;\,\mu\{x'\,;\,K^{(r)}(x', s)>0\}>\delta]\geqq 1-\varepsilon \quad \text{for} \quad r=1, \cdots, k_0\,. \tag{2.7}$$

Because, by (a′)

$$\mu[s\,;\,\mu\{x'\,;\,K^{(r)}(x', s)>0\}>0]=1 \quad \text{for every integer} \quad r\geqq 1\,.$$

Let n_0 be an integer in $M_x(\delta)\cap M_s(\delta)$. Putting

$$X_{n_0}=[x\,;\,\mu\{s'\,;\,K^{(n_0)}(x, s')>0\}\geqq 1-\delta]$$

and

$$S_r=[s\,;\,\mu\{x'\,;\,K^{(r)}(x', s)>0\}>\delta]\,,$$

we have,

$$\mu(X_{n_0})\geqq 1-\delta \text{ and by (2.7) } \mu(S_r)\geqq 1-\varepsilon \text{ for } r=1, \cdots, k_0$$

and, for $(x, s)\in X_{n_0}\times S_r$,

$$K^{(n_0+r)}(x, s)=\int_S K^{(n_0)}(x, s')K^{(r)}(s', s)d\mu(s')>0$$

from which follow

$$\mu[x\,;\,\mu\{s'\,;\,K^{(n_0+r)}(x, s')>0\}\geqq 1-\varepsilon]\geqq 1-\delta>1-\varepsilon\,,$$

and

$$\mu[s\,;\,\mu\{x'\,;\,K^{(n_0+r)}(x', s)>0\}\geqq 1-\delta]\geqq 1-\varepsilon\,.$$

This implies $n_0+r\in M_x(\varepsilon)\cap M_s(\varepsilon)$, $r=1, \cdots, k_0$, which completes the proof of (2.6).

Now, since, for n_0 and k_0 in (2.6), every integer $n>n_0(\varepsilon)$ can be written in the form $n=n_0+jk_0+r$ where $r=1, \cdots, k_0$, $j=0, 1, \cdots$ then, by (2.4), (2.5) and (2.6), n belongs to $N(\sqrt{2\varepsilon})$. This shows that to each ε, $0<\varepsilon<1/2$, there exists an integer $n_0\geqq 1$ satisfying $\{n\,;\,n\geqq n_0\}\subset N(\sqrt{2\varepsilon})$ which accomplishes the proof of lemma 1.

THEOREM 1. *If the integral operator T is a non-support operator and the spectral radius $r(T)$ is positive, then T has the following spectral properties* (i)–(iii)[2] :

(i) *$\gamma_0=r(T)$ is a simple eigenvalue possessing an eigenfunction $\varphi_0(x)$ which is positive almost everywhere; the eigenmanifold of γ_0 is one-dimensional and γ_0 is a simple pole of the resolvent $R(\lambda, T)=(\lambda I-T)^{-1}$.*

(ii) *If λ is an eigenvalue of T, $\lambda\neq\gamma_0$, then $|\lambda|<\gamma_0$.*

(iii) *$(T/\gamma_0)^n$ converges to the one-dimensional projection operator on the eigenmanifold $\mathcal{M}(\gamma_0)=\{\varphi\,;\,(T-\gamma_0 I)\varphi=0\}$. Moreover, $L_2(S)$ decomposes into the direct sum $\mathcal{M}(\gamma_0)+\mathcal{U}(\gamma_0)$ where $\mathcal{U}(\gamma_0)$ is a complementary invariant subspace on which T has spectral radius less than γ_0.*

PROOF. In $L_2(S)$, the positive cone is closed, proper, generating, normal and minihedral. Since T is completely continuous and $\gamma_0=r(T)$ is positive, γ_0 is a pole of the resolvent $R(\lambda, T)$. Using theorem 5 and the corollary of theorem 2 in the author's paper [3], we have (i) and (ii). The latter part of (iii) is clear by (i), (ii). Therefore, denoting by P the projection operator on $\mathcal{M}(\gamma_0)$ and putting $T'=T(I-P)$, we have

$$r(T'/\gamma_0)<1 \tag{2.8}$$

and

$$(T/\gamma_0)^n=((TP+T')/\gamma_0)^n=(P+T'/\gamma_0)^n$$
$$=P+(T'/\gamma_0)^n\,.$$

By (2.8), $(T'/\gamma_0)^n$ uniformly converges to 0. Therefore, $(T/\gamma_0)^n$ uniformly converges to P. Thus we complete the proof of theorem 1.

COROLLARY 1. *The integral operator T with a kernel $K(x, s)$ satis-*

2) These spectral properties (i)-(iii) are identical with (ii)-(iv) of theorem 2 in Karlin's paper [1] except for an insertion of 'almost everywhere'.

fying condition (a′) *has the spectral properties* (i)-(iii).

This is clear by proposition 1 and remark 1.

REMARK 2. Some spectral properties of the integral operator with a kernel satisfying condition (a′) is shown in M. G. Krein and M. A. Rutman's paper [2; (β') in p. 274]. But we obtain corollary 1 as one special case of a general theorem for non-support operator obtained recently by the author [3].

§ 3. **Totally non-support kernels.** Throughout this section, we shall suppose Δ be an open interval (a, b), μ be Lebesque measure and T be an integral operators in $L_2(\Delta, \mu)$ associated with a kernel $K(x, s)$ which is almost everywhere totally positive, i. e., for every integer $n \geqq 1$, the compound kernel $K_{[n]}(\bar{x}, \bar{s})$ of order n induced by $K(x, s)$ is almost everywhere non-negative on $\Delta_n \times \Delta_n$. Our purpose in this section is to extend Karlin's theorems 3, 4 and 5 to more general kernels.

At first, we shall define totally non-support kernels as follows:

DEFINITION. *If, for every integer* $n \geqq 1$, *the integral operator* $T_{[n]}$ *associated with a kernel* $K_{[n]}(\bar{x}, \bar{s})$ *is a non-support operator in* $L_2(\Delta_n)$ *and the spectral radius* $r(T_{[n]})$ *is positive, then the kernel* $K(x, s)$ *is called a totally non-support kernel on* $\Delta \times \Delta$.

If a kernel $K(x, s)$ is almost everywhere strictly totally positive[3)], then this is a totally non-support one by proposition 1. If a kernel $K(x, s)$ is a totally non-support kernel, then this is almost everywhere totally positive. Further, we have

PROPOSITION 2. *If* $K(x, s)$ *is totally positive and continuous on* $\Delta \times \Delta$ *and, for each integer* $n \geqq 1$, $K_{[n]}(\bar{x}, \bar{x})$ *is positive on* Δ_n, *then* $K(x, s)$ *is a totally non-support kernel.*

PROOF. Let n be an arbitrary fixed positive integer. We can choose a sequence $\{A_{n,j}\,;\, j=1, \cdots\}$ of compact convex subsets of Δ_n satisfying

$$A_{n,j} \subset A_{n,j+1} \quad \text{and} \quad \Delta_n = \bigcup_{j=1}^{\infty} A_{n,j}\,. \tag{3.1}$$

(For example, $A_{n,j} = \{(x_1, \cdots, x_n)\,;\, x_i - x_{i-1} \geqq (b-a)/(n+1+j),\ i=1, \cdots, n+1\}$ where $x_0 = a$ and $x_{n+1} = b$). Since $K_{[n]}(\bar{x}, \bar{s})$ is continuous on $A_{n,j} \times A_{n,j}$, $K_{[n]}(\bar{x}, \bar{x})$ is positive on $A_{n,j}$ and $A_{n,j}$ is compact, there exists a positive number $\delta_{n,j}$ such that $K_{[n]}(\bar{x}, \bar{s}) > 0$ whenever $\operatorname{dist}(\bar{x}, \bar{s}) < \delta_{n,j}$. Therefore, if $k > (\operatorname{diam} A_{n,j})/\delta_{n,j}$, then

$$K_{[n]}(\bar{x}, \bar{\tau}_{k-1}),\ K_{[n]}(\bar{\tau}_{k-1}, \bar{\tau}_{k-2}),\ \cdots,\ K_{[n]}(\bar{\tau}_1, \bar{s}) > 0$$

where $\bar{x}, \bar{s} \in A_{nj}$, $\bar{\tau}_i = (i/k)\bar{x} + (1-i/k)\bar{s}$, $i=1, \cdots, k-1$. Since μ is Lebes-

3) If, for every integer $n \geqq 1$, $K_{[n]}(\bar{x}, \bar{s})$ is almost everywhere positive on $\Delta_n \times \Delta_n$, then $K(x, s)$ is called almost everywhere strictly totally positive.

que measure we have, for every integer $k>(\mathrm{diam}\, A_{n,j})/\delta_{n,j}$,

$$K_{[n]}{}^{(k)}(\bar{x}, \bar{s})=\int_{\Delta_n}\cdots\int_{\Delta_n} K_{[n]}(\bar{x}, \bar{t}_{k-1})\cdots K_{[n]}(\bar{t}_1, \bar{s})\, d\mu^n(\bar{t}_1)\cdots d\mu^n(\bar{t}_{k-1})>0 \tag{3.2}$$

whenever $\bar{x}, \bar{s}\in A_{n,j}$. By σ-additivity of measure μ and (3.1), the kernel $K_{[n]}(\bar{x}, \bar{s})$ satisfies condition (a). Then, by proposition 1, $T_{[n]}$ is a non-support operator and $r(T_{[n]})$ is positive. This completes the proof of proposition 2.

From remark 1, follows immediately

PROPOSITION 3. *If $K(x, s)$ is almost everywhere totally positive on $\Delta\times\Delta$ and, for each integer $n\geqq 1$ and positive number ε, there exists an integer $k_0=k_0(n, \varepsilon)$ such that*

$$\mu^{2n}\{(\bar{x}, \bar{s})\ ;\ K_{[n]}{}^{(k_0)}(\bar{x}, \bar{s})=0\}<\varepsilon^2$$

then $K(x, s)$ is a totally non-support kernel.

The following theorem corresponds to theorem 3, 4, 5 in Karlin's paper [1].

THEOREM 2. *Let $K(x, s)$ be a totally non-support kernel. Then the integral operator T has the following spectral properties:*

(i) *T possesses a countable set $\{\lambda_j\}$ of simple positive eigeuvalues $\lambda_0>\lambda_1>\cdots$ decreasing to zero. There exists no other nonzero spectrum.*

(ii) *Let $\varphi_0(x), \varphi_1(x), \cdots$ denote the corresponding eigenfunctions (each uniquely determined in $L_2(\Delta)$ except for a multiplicative factor). Then*

$$\varepsilon_n(\varphi_0\wedge\cdots\wedge\varphi_{n-1})(\bar{x})>0$$

almost everywhere on Δ_n, $n=1, \cdots$ where ε_n is appropriately $+1$ or -1.

PROOF. Since $K(x, s)$ is a totally non-support kernel, by theorem 1 we have that, for each integer $n\geqq 1$, the integral operator $T_{[n]}$ associated with the kernel $K_{[n]}(\bar{x}, \bar{s})$ has spectral properties (i)-(iii) in theorem 1 replacing T, γ_0 and φ_0 by $T_{[n]}$, γ_{n-1} and φ_{n-1} respectively. Using this fact, we can complete the proof following after S. Karlin's proof of theorem 3 in his paper [1].

COROLLARY 2. *Let a kernel $K(x, s)$ be almost everywhere totally positive on $\Delta\times\Delta$ and, for each integer $n\geqq 1$ and positive number ε, there exists an integer $k_0=k_0(n, \varepsilon)$ such that*

$$\mu^{2n}\{(\bar{x}, \bar{s})\ ;\ K_{[n]}{}^{(k_0)}(\bar{x}, \bar{s})=0\}<\varepsilon^2\ .$$

Then the integral operator T has the spectral properties (i) *and* (ii) *in theorem 2.*

This corollary is clear by proposition 3.

As a more special case, we obtain the following corollary.

COROLLARY 3. *Let $K(x, s)$ be bounded continuous totally positive kernel and, for every integer $n\geqq 1$, $K_{[n]}(\bar{x}, \bar{x})>0$ on Δ_n. Then the integral operator T has the spectral properties* (i) *in theorem 2 and* (ii$'$) *:*

(ii′) *Let* $\varphi_0(x)$, $\varphi_1(x)$, $\cdots$ *denote the corresponding eigenfunctions (each uniquely determined in* $L_2(\varDelta)$ *except for a multiplicative factor). Then each* $\varphi_j(x)$ *is almost everywhere equal to a continuous function* $\psi_j(x)$ *and*

$$\varepsilon_n(\psi_0\wedge\cdots\wedge\psi_{n-1})(\bar{x})>0$$

for every $\bar{x}\in\varDelta_n$, $n=1,\cdots$ *where* ε_n *is appropreately* $+1$ *or* -1.

PROOF. The integral operator T has the properties (i) and (ii) in theorem 2 by proposition 2. From the continuity and boundedness of $K(x, s)$, we have obviously that $T\varphi_j$, $j=0, 1, \cdots$ are continuous. Then each $\varphi_j(x)$ is almost everywhere equal to the continuous function $\psi_j(x)=T\varphi_j(x)/\lambda_j$ which is also an eigenfunction. It is shown that

$$\psi_0\wedge\cdots\wedge\psi_{n-1}(\bar{x})=\int_{\varDelta_n}K_{[n]}(\bar{x}, \bar{s})\frac{\psi_0\wedge\cdots\wedge\psi_{n-1}}{\lambda_0\cdots\lambda_{n-1}}(\bar{s})\; d\mu^n(\bar{s})$$

on $\varDelta_n$ which along with $K_{[n]}(\bar{x}, \bar{x})>0$ on $\varDelta_n$ proves $\varepsilon_n(\psi_0\wedge\cdots\wedge\psi_{n-1})>0$ on $\varDelta_n$.

Remark. In this corollary 3, we can replace $\varDelta_n$ by $\tilde{\varDelta}_n=\{\bar{x}=(x_1,\cdots, x_n)\,;\, a\leqq x_1<\cdots<x_n\leqq b\}$.

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