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On Spectral Properties of Some Positive Operators

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§1. Introduction. In a finite-dimensional Euclidean space with natural order by the positive cone $K=\{x\,;\,x=\{x_j\},\ x_j\geqq 0\ j=1,\cdots,n\}$, the following two theorems concerning spectral properties for non-negative matrices are well known:

THEOREM A[1] *A non-negative matrix T is indecomposable, i.e. it can not be reduced to the form* $\begin{pmatrix} T_{1,1} & T_{1,2} \\ 0 & T_{2,2} \end{pmatrix}$ *with square submatrices $T_{1,1}$, $T_{2,2}$ by applying one and the same permutation to the rows and colums, if and only if T satisfies the following condition* (a):

(a) *Each of the proper spaces of T and T^* corresponding to $r(T)$*[2] *is a one-dimensional subspace passing through a positive element*[3].

THEOREM B[4] *A non-negative matrix T is primitive, i.e. there exists a natural number m such that $T^m=(t_{ij}^{(m)})$, $t_{ij}^{(m)}>0$ $(i,j=1,\cdots,n)$, if and only if T satisfies the above condition* (a) *and the following condition* (b):

(b) *The set of the proper values of T on the circle $|\lambda|=r(T)$, consists only of $r(T)$.*

M. G. Krein and M. A. Rutman generalized the concept of primitive matrices to the case where a Banach space ordered by a closed proper positive cone K having non empty interior and obtained the result corresponding to Theorem B for completely continuous operators in that space [5: Theorem 6.3]. This was partly extended to more general cases by many authors. Among them, S. Karlin [4] and H. Schaefer [8] investigated the extention to the case that an operator, not necessarily completely continuous, has the resolvent with $r(T)$ as its pole. Further, in case of the positive cone with empty interior H. Schaefer [8: Theorem 2] obtained a sufficient condition for an operator to have the properties corresponding to (a). Only recently, F. Niiro generalized the notion of indecomposability to the case of a space l_p with natural order and obtained the results corresponding to Theorem A and B [7: Theorem 1 and 5]. The aim of this note is to

1) See, for example, F.R. Gantmacher [3].

2) $r(T)$ denotes the spectral radius of T. In this case, $r(T)$ is the maximum modulus of all proper values of T and it is also a proper value.

3) An element v is positive if $v=\{v_j\}$, $v_j>0$ for all $j=1,\cdots,n$.

4) See G. Frobenius [2].

obtain a generalization of the results of Theorem A and B.

Let E be a partially ordered real Banach space with a closed and proper positive cone K and T be a positive operator in E. Also the complexification of E and the extension of T to E will be denoted respectively by the same letters E and T, but the cone K is always included in the real Banach space E. Following S. Karlin and H. Schaefer we shall make use of an operator whose resolvent $R(\lambda, T)$ has the point $\lambda=r(T)$ as its pole (it is well known that such operators have the number $r(T)$ as its proper value). Further, we introduce the three concepts of semi-non-support, non-support and strict non-support operators replacing the rôle of positive elements in Theorem A and B by that of non-support points[5]. The first concept is a generalization of the notion of indecomposable non-negative matrices and the second and third ones are generalizations of the notion of primitive non-negative matrices. Let (A) and (B) be the following two conditions for T corresponding to (a) and (b) respectively.

(A) The proper space corresponding to the proper value $r(T)$ of T is a one-dimensional subspace passing through a non-support point of K. The proper space corresponding to $r(T)$ of T^* is also a one-dimensional subspace of E^* passing through a strictly positive functional.[6]

(B) The spectrum of T on the circle $|\lambda|=r(T)$ consists only of $r(T)$.

In this paper our main results are as follows:

Let K be total and T be a positive operator whose resolvent $R(\lambda, T)$ has the point $\lambda=r(T)$ as its pole.

I[7]. *Then, holds the following logical relation:*

$$T \textit{ is a semi-non-support operator} \Leftrightarrow \begin{cases} r(T)>0 & ^{8)} \\ T \textit{ satisfies } (\mathrm{A}). \end{cases}$$

II[9]. *Further, let K be normal and minihedral, and let the dual cone K^* be also normal. Suppose an operator T satisfies the condition that the spectrum of T on the circle $|\lambda|=r(T)$ consists only of poles and each proper space corresponding to proper values on the circle $|\lambda|=r(T)$ is finite-dimensional. Then, holds the following logical relation:*

$$T \textit{ is a non-support operator} \Leftrightarrow \begin{cases} r(T)>0 \\ T \textit{ satisfies } (\mathrm{A}) \textit{ and } (\mathrm{B}). \end{cases}$$

These results include a generalization of the results of the theorem 12 and 13 in S. Karlin's paper [4]. By analyzing the property (A) and the result I, we can obtain the result of the theorem 2, 3° in H. Schaefer [8] without any supplementary conditions a), b) and c) there.

5) This definition will be given in section 2.
6) This definition will be given in section 2.
7) In this paper it corresponds theorem 2 in section 4.
8) $X \Rightarrow Y$ or $Y \Leftarrow X$ denotes "X implies Y" and $X \Leftrightarrow Y$ denotes "$X \Rightarrow Y$ and $X \Leftarrow Y$".
9) In this paper it corresponds theorem 5 in section 5.

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§2. Notations and terminologies. We shall denote the complex number field and the real number field by C and R respectively. Let E be a Banach space over R, partially ordered by a proper closed cone K and E^* be the topological dual of E. A cone is *positive* if the order of the space is given by that cone. A cone K is *proper* if $K\cap(-K)=\{0\}$. A cone K is *normal* if there exists a positive number δ such that $\|x+y\|\geqq\delta$ whenever $\|x\|=\|y\|=1$, $x, y\in K$. A cone K is *minihedral* if there exists $\sup\{x, y\}$ whenever $x, y\in K$. A cone K is *total* if $(K-K)^a=E$[10]. A linear functional f is called *positive* (resp. *strictly positive*) if $f\in E^*$ and $f(x)\geqq 0$ for $x\in K$ (resp. $f(x)>0$ for nonzero $x\in K$). *Dual cone* K^* is the set of all positive linear functionals. $x\in K$ is a *non-support* point of K if $f(x)>0$ whenever $f\in K^*$, $f\neq 0$.

We shall denote by $\mathfrak{L}(E)$ the set of bounded linear operators mapping the space E into itself. An operator $T\in\mathfrak{L}(E)$ is called *positive* if $TK\subset K$ and *strongly positive*[11] with respect to K if T is positive and for each nonzero $x\in K$ there exists a natural number $n=n(x)$ such that $T^n x$ is an interior point in K. The *resolvent* of T, denoted by $R(\lambda, T)$ or $R(\lambda)$, is a bounded linear operator $(\lambda I-T)^{-1}$ whenever it exists for λ. The set of λ where $R(\lambda, T)$ exists is denoted by $\rho(T)$. The complement of $\rho(T)$ in C is the *spectrum* of T, denoted by $\mathfrak{S}(T)$. If there exists a nonzero $x\in E$, which satisfies $Tx=\lambda x$, λ is called a *proper value* and x a *proper vector* corresponding to λ. The proper space corresponding to λ is the set $\{x\,;\,Tx=\lambda x\}$. The *spectral radius* of T, denoted by $r(T)$ or simply r, is the maximum modulus of the elements of $\mathfrak{S}(T)$, i.e.,

$$r=\max_{\lambda\in\mathfrak{S}(T)}|\lambda|=\lim_{n\to\infty}\sqrt[n]{\|T^n\|}.$$

§3. Fundamental definitions and their direct consequences. In this section we shall define semi-non-support, non-support and strict non-support operators with respect to the positive cone K in E and discuss their simple properties[12].

DEFINITION 1. *An operator T is a semi-non-support operator in E with respect to K if T is positive and for each nonzero $x\in K$ and for each nonzero $f\in K^*$ there exists a natural number $n=n(x,f)$ such that $f(T^n x)>0$.*

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10) X^a is the closure of X and X^b is the boundary of X.

11) The terminology by Krein-Rutman [5], S. Karlin called it quasi-strictly positive [4].

12) For these definitions, see propositions 2, 3 at the end of this paper.

DEFINITION 2. *An operator T is a non-support operator in E with respect to K if T is positive and for each nonzero $x \in K$ and for each nonzero $f \in K^*$ there exists a natural number $n_0 = n(x, f)$ such that $f(T^n x) > 0$ whenever $n \geqq n_0$.*

DEFINITION 3. *An operator T is a strict non-support operator in E with respect to K if T is positive and for each nonzero $x \in K$, there exists a natural number $n_0 = n(x)$ such that $T^n x$ is a non-support point of K whenever $n \geqq n_0$.*

It is obvious that a strict non-support operator is a non-support one and a non-support operator is a semi-non-support one. If the positive cone has the non empty interior, the notion of strict non-support operator coincides with that of strongly positive one. In a n-dimensional Euclidean space with natural order, it is easily seen that the four notions of non-support operators, strict non-support operators, strongly positive operators and primitive matrices coincide. Also, when n is not 1, the notion of semi-non-support operators coincides with that of indecomposable non-negative matrices. The proof is the following: Since $T = \begin{pmatrix} T_{1,1} & T_{1,2} \\ 0 & T_{2,2} \end{pmatrix}$ ($T_{1,1}$, $T_{2,2}$ are square matrices)is not a semi-non-support operator, it is obvious that semi-non-support operators are indecomposable non-negative matrices. Conversely, a indecomposable non-negative matrix T is a semi-non-support operator. Because, if T is not a semi-non-support operator, then there exist two elements e_{i_0}, e_{k_0}, $i_0 \neq k_0$, such that $e_{i_0} = \{x_j\}$, $x_j = \begin{cases} 1 & (j = i_0) \\ 0 & (j \neq i_0) \end{cases}$, $e_{k_0} = \{y_j\}$, $y_j = \begin{cases} 1 & (j = k_0) \\ 0 & (j \neq k_0) \end{cases}$ and $\langle T^m e_{i_0} | e_{k_0} \rangle$[13] $= 0$ for all m. Put $J = \{j \,;\, \langle T^m e_{i_0} | e_j \rangle > 0$ for some $m\}$. If $J = \phi$, then the subspace generated by $\{e_{i_0}\}$ is invariant under T. If $J \neq \phi$, then $\phi \neq J \subsetneqq \{1, \cdots, n\}$ and T leaves the subspace generated by $\{e_j \,;\, j \in J\}$ invariant. These contradict the indecomposability of T. Similarly, in the space l_p $(1 < p < \infty)$ with natural order the notions of semi-non-support operators and indecomposable positive operators in the sense of F. Niiro [7] coincide.

LEMMA 1. *Let E be a finite-dimensional linear space, and the positive cone K be minihedral and have at least one interior point. Then the notions of strict non-support operators and non-support operators coincide.*

PROOF. If the cone with non empty interior is minihedral then there exists a base $\{e_1, \cdots, e_n\}$ of E such that $K = \{\sum_{j=1}^{n} c_j e_j \,;\, c_j \geqq 0\}$[14]. Therefore, the space E is topologically order isomorphic to the n-dimensional Euclidean space with natural order. Hence lemma 1 is proved.

13) $<x|y>$ denotes the inner product of $x = \{x_j\}$ and $y = \{y_j\}$, i. e. $<x|y> = \sum_{j=1}^{n} x_j y_j$.

14) See, for example, Sz. Nagy [6].

REMARK. If the positive cone with non empty interior is not minihedral then even in a finite dimensional linear space lemma 1 does not hold. For example, let E be a 3-dimensional Euclidean space ordered by the Lorenz cone K, i. e. $K=\{(x_1, x_2, x_3)\ ;\ x_1 \geqq \sqrt{x_2^2+x_3^2}\}$ and T be the operator represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

where θ/π is irrational. Then T is a non-support operator but not a strict non-support operator.

It is obviously seen that the condition for T to be a semi-non-support operator is equivalent to the following condition: there exists a number $\lambda > r(T)$ such that $TR(\lambda)x$ is a non-support point of K for each nonzero $x \in K$. Therefore, a quasi-interior operator[15] is a semi-non-support operator. If the positive cone K has a non empty interior then the notion of semi-non-support operator coincides with that of quasi-interior one. Also, if E is a space l_p $(1 < p < \infty)$ with natural order, the above two notions coincide.

LEMMA 2. *If there exists a semi-non-support operator T in E with respect to K then the positive cone K is total in E.*

PROOF. Let E_0 be the smallest closed linear subspace including K. If $E \neq E_0$ then there exists a nonzero continuous linear functional f_0 such that $f_0(x)=0$ whenever $x \in K$. Since $f_0 \in K^*$ and T is a semi-non-support operator, for the linear function f_0 and for each nonzero $x \in K$ there exists a natural number $n=n(x, f_0)$ such that $f_0(T^n x) > 0$. But $T^n x \in K$ and $f_0(T^n x)=0$, which is a contradiction. Thus we have $E_0 = E$.

LEMMA 3. *Let P be a bounded positive projection and T leave PE invariant. If x_0 is a non-support point of K then Px_0 is a non-support point of PK in PE. If T is a semi-non-support operator in E with respect to K then the restriction of T to PE is a semi-non-support one in PE with respect to PK. If T is a non-support operator in E with respect to K then the restriction of T to PE is a non-support one in PE with respect to PK. If T is a strict non-support operator in E with respect to K then the restriction of T to PE is a strict non-support one in PE with respect to PK.*

PROOF. For each linear functional f_1 in $(PK)^*$, f_1P is a linear functional in K^* whence all the assertions of lemma 3 obviously follow.

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15) H. Schaefer defined a quasi-interior operator as follows [7]: An element u is quasi-interior if $u \in K$ and $\{y\ ;\ 0 \leqq y \leqq u\}$ is total in E, and T is quasi-interior to K if there exists a number $\lambda > r$ such that $TR(\lambda)x$ is quasi-interior to K for each nonzero $x \in K$.

§ 4. **Properties of semi-non-support operators.** Throughout the sections 4 and 5 we shall assume that an operator is positive and its resolvent $R(\lambda, T)$ has the point $\lambda=r$ as its pole. In this section, we shall analyze the condition (A)[16] and, as a generalization of the result of Theorem A, show that the necessary and sufficient condition for an operator T to be a semi-non-support operator is (A) and $r(T)>0$.

Let the point λ_0 $(\in C)$ be a pole of the resolvent $R(\lambda, T)$ with the expansion $\sum_{n=-k}^{\infty} A_n(\lambda-\lambda_0)^n$ at $\lambda=\lambda_0$. Then the following properties are well known:

(4.1) $$A_{-k}=\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)^k R(\lambda) \text{ and } R(\lambda)=\sum_{n=0}^{\infty}\frac{T^n}{\lambda^{n+1}} \text{ for } |\lambda|>r(T).$$

(4.2) $$A_{-l}=(T-\lambda_0 I)^{l-1}P=P(T-\lambda_0 I)^{l-1}$$

where $l=1, \cdots, k$ and $P=\frac{1}{2\pi i}\int_\gamma R(\lambda)d\lambda$ is a bounded projection (γ is a positive oriented sufficiently small circle enclosing λ_0).

(4.3) $$A_{-k}E\subset\{x\,;\ Tx=\lambda_0 x\}\subset PE.$$

(4.4) If K is total then there exist a nonzero $x\in K$ and a nonzero $f\in K^*$ such that $Tx=rx$ and $T^*f=rf$ respectively[17].

Now we shall present theorem 1.

THEOREM 1. *Let K be total and T be a positive operator whose resolvent $R(\lambda, T)$ has the point $\lambda=r(T)$ as its pole. If every proper vector corresponding to $r(T)$ lying in K is a non-support point of K then $R(\lambda, T)$ has a pole of order 1 at $\lambda=r(T)$ and further the proper spaces corresponding to proper value $r(T)$ of T and T^* are one-dimensional.*

PROOF. Let $R(\lambda, T)$ have a pole of order k at $\lambda=r(T)=r$, i.e. $R(\lambda, T)=\sum_{n=-k}^{\infty}\Gamma_n(\lambda-r)^n$, $\Gamma_{-k}\neq 0$. Then $R(\lambda, T^*)$ also have a pole of order k at $\lambda=r$. By (4.1), Γ_{-k} is positive and by (4.3) and (4.4) we have

(4.5) $$\{0\}\neq\Gamma_{-k}K\subset\{x\,;\ Tx=rx,\ x\in K\}\subset PE\cap K$$

and

$$\{0\}\neq\{f\,;\ T^*f=rf,\ f\in K^*\}\subset P^*E\cap K^*.$$

From these facts and the assumption, there exist $x_0\in K$, $x_0\neq 0$ and $f_0\in K^*$, $f_0\neq 0$ such that $\Gamma_{-k}x_0$ is a non-support point of K and $T^*f_0=rf_0$. Therefore using (4.2), (4.5) we have

16) See the introduction.

17) We shall remark that if K is closed, proper and total, then K^* is closed, proper and w^*-total.

$$\begin{aligned} 0<f_0(\Gamma_{-k}x_0)&=f_0(P(T-rI)^{k-1}x_0)\\ &=(T^*-rI)^{k-1}P^*f_0(x_0)\\ &=(T^*-rI)^{k-1}f_0(x_0)\,. \end{aligned}$$

Hence, k must equal to 1. That is, r is a pole of order 1 of $R(\lambda, T)$. Therefore $\Gamma_{-1}=P$ is a bounded positive projection. By (4.3) and (4.5) we have

$$\{x\,;\ Tx=rx\}=PE$$

and

$$\{0\}\neq\Gamma_{-1}K=\{x\,;\ Tx=rx,\ x\in K\}=PE\cap K=PK\,.$$

From the assumption and lemma 3, every nonzero element of PK is a non-support point of PK. Since PE is a closed subspace and $PK=PE\cap K$, PK is a closed proper cone in Banach space PE. Here, we shall use the following result by E. Bishop and R. R. Phelps [1 : Theorem 1] ; if K is a closed convex subset of a Banach space E, then the support points of K are dense in the boundary of K. Therefore, the boundary of PK must be $\{0\}$. Since PK is closed and proper in PE, PE is one dimensional. Let $PE=\{ax_0\,;\ a\in R\}$ where x_0 is a non-support point of K. Then, putting $f_0(x)=a$ for $Px=ax_0$, we have $f_0\in K^*$, $f_0\neq 0$ and $Px=f_0(x)x_0$ whence $P^*f=f(x_0)f_0$. Since the point $\lambda=r$ is also a pole with order 1 of the resolvent $R(\lambda, T^*)$, P^*E is the proper space corresponding to r, of T^*. Thus the proper space is one-dimensional. This completes the proof of theorem 1.

Let (A_1) and (A_2) be the following conditions for T:

(A_1) Every proper vector corresponding to the proper value $r(T)$ lying in K is a non-support point of K and every proper vector corresponding to $r(T)$ lying in K^* is strictly positive.

(A_2) The proper space corresponding to the proper value $r(T)$ is a one-dimensional subspace of E passing through a non-support point K and there exists a strictly positive proper functional corresponding to $r(T)$.

From theorem 1, we immediately see that $(A_1)\Rightarrow(A_2)$ and $(A_2)\Rightarrow(A)$. Therefore we obtain the following corollary :

COROLLARY 1. *Let K be total. Then the three conditions* (A), (A_1) *and* (A_2) *are equivalent to each other.*

In the proof of theorem 1, the totality of K is used only to prove the existence of nonzero proper vectors in K and K^*. Therefore corollary 2 follows.

COROLLARY 2. *If T satisfies the condition* (A), *then its resolvent $R(\lambda, T)$ has a pole of order* 1.

Here we shall obtain theorem 2 as a generalization of Theorem A.

THEOREM 2. *Let T have the resolvent $R(\lambda, T)$ with the point $\lambda=r$ as its pole. Then, holds the following logical relation :*

$$T \textit{ is a semi-non-support operator} \Leftrightarrow \begin{cases} r(T)>0 \\ T \textit{ satisfies } (A)\,. \end{cases}$$

PROOF. $\Rightarrow$: Let T be a semi-non-support operator and x (resp. f) a proper vector corresponding to r in K (resp. K^*). Then, for $x \in K$ and for every nonzero $f \in K^*$ (resp. for every nonzero $x \in K$ and for $f \in K^*$), there exists a natural number $n = n(x, f)$ for which $0 < f(T^n x) = r^n f(x)$. That is, $r > 0$ and $f(x)$ is always positive for every nonzero $f \in K^*$ (resp. $x \in K$). Hence, x is a non-support point of K (resp. f is strictly positive on K). Then T satisfies the conditions $r > 0$ and (A_1). By lemma 2 and corollary 1 of theorem 1, T satisfies the condition (A).

$\Leftarrow$: Let T satisfy the condition (A) and $r(T) > 0$. Then we can assume $r(T)$ to be 1. By corollary 2 of theorem 1, the point 1 is a pole of order 1 of the resolvent $R(\lambda, T)$. Let $\sum_{n=-1}^{\infty} \Gamma_n (\lambda - 1)^n$, $\Gamma_{-1} \neq 0$ be the expansion of $R(\lambda, T)$ at $\lambda = 1$. Then, by (4.1), $P = \Gamma_{-1}$ is a positive projection onto the proper space corresponding to 1 and

$$P = \lim_{\xi \downarrow 1} (\xi - 1) R(\xi) . \tag{4.6}$$

Since strictly positive proper vector f_0 corresponding to 1 of T^* exists, $f_0(Px) = P^* f_0(x) = f_0(x) > 0$ for every nonzero $x \in K$. Therefore $Px \neq 0$ and $Px \in K$ which implies Px is a non-support point for every nonzero $x \in K$. Thus, if x is a nonzero element in K and f is a nonzero functional in K^*, then $f(Px) > 0$. By (4.6), for $f(Px)/(2\|x\|\,\|f\|\,\|T\|) > 0$, there exists $\xi > 1$ such that

$$\|(\xi - 1) R(\xi) - P\| < \frac{f(Px)}{2\|x\|\,\|f\|\,\|T\|} ,$$

whence

$$|f\{(\xi - 1) T R(\xi) x - Px\}| = |f\{(\xi - 1) T R(\xi) x - TPx\}| < \frac{f(Px)}{2} .$$

From $TR(\xi) = \sum_{n=1}^{\infty} \frac{T^n}{\xi^n}$, by (4.1),

$$0 < \frac{f(Px)}{2} < (\xi - 1) \sum_{n=1}^{\infty} \frac{f(T^n x)}{\xi^n} .$$

Thus, there exists a natural number n such that

$$f(T^n x) > 0$$

which completes the proof of theorem 2.

REMARK 1. Using the corollary 1 of theorem 1, we see that this theorem includes a generalization of the result of Theorem 13 in S. Karlin [4] to the case where the positive cone has not necessarily a non empty interior.[18)]

REMARK 2. F. Niiro obtained a result corresponding to this

18) T is a semi-non-support operator if and only if, for each nonzero $x \in K$ and each nonzero $f \in K^*$, there exists a natural number n such that $f\left(\frac{T + \cdots + T^n}{n} x\right) > 0$.

theorem in the space l_p $(1<p<\infty)$ with natural order [7: Theorem 1].

Combining corollary 2 of theorem 1 and theorem 2, we have immediately the following corollary.

COROLLARY. *If T is a semi-non-suppor operator then T satisfies the conditions $r>0$, r is a pole of order* 1 *and* (A).

REMARK. This corollary corresponds to Theorem 2 in H. Schaefer [8]. It is seen from the corollary that the supplementary conditions a), b) and c) for Theorem 2, 3° in H. Schaefer are unnecessary.

§ 5. **Properties of non-support operators.** In the previous section we had the result that an operator T is a semi-non-support operator if and only if the operator T has the properties $r>0$ and (A) corresponding to the properties (a) in Theorem A. In this section we shall consider a generalization of the result of Theorem B. It is immediately seen that a semi-non-support operator does not necessarily satisfy the condition (B)[19] from the simple example $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$[20].

First, we shall obtain a necessary condition which follows from the condition $r>0$, (A) and (B).

THEOREM 3. *Let T be a positive operator whose resolvent $R(\lambda, T)$ has the point $\lambda=r(T)$ as its pole. Then, holds the following logical relation:*

$$\left.\begin{array}{r} r(T)>0 \\ T \text{ satisfies (A) and (B)} \end{array}\right\} \Rightarrow T \text{ is a non-support operator.}$$

PROOF. We can suppose r to be 1. In the same way as the proof of "$\Leftarrow$" of theorem 2, the leading coefficient $P=\Gamma_{-1}$ of the expansion of the resolvent at $\lambda=1$ is a bounded positive projection onto the proper space and Px is a non-support point for every nonzero $x\in K$. Put

$$T_1\equiv PT \quad \text{and} \quad T_2\equiv(I-P)T\,.$$

Then

$$T_1=PT=TP=P,\ T_1T_2=T_2T_1=0 \quad \text{and} \quad T^n=P+T_2^n\,.$$

By the condition (B), we have

$$r(T_2)=\lim_{n\to\infty}\sqrt[n]{\|T_2^n\|}<1\,.$$

Therefore $\|T_2^n\|\to 0$ $(n\to\infty)$ whence $f(T_2^n x)\to 0$ $(n\to\infty)$ whenever $x\in E$ and $f\in E^*$. Let x be a nonzero element in K and f a nonzero element in K^*. Then $f(Px)>0$ from which follows that there exists a natural number $n_0=n(x,f)$ such that $|f(T_2^n x)|<\dfrac{f(Px)}{2}$ whenever $n\geqq n_0$ whence

$$f(T^n x)=f(Px)+f(T_2^n x)>0\,.$$

19) See the introduction in this paper.

20) See H. Schaefer [7: p. 1018].

That is, T is a non-support operator which completes the proof of theorem 3.

By theorem 2 and theorem 3, we obtain the following corollary.

COROLLARY. *Let T satisfy the condition* (B). *Then, for T*

$$\begin{array}{lcl} \left.\begin{array}{l} r(T)>0 \\ (\mathrm{A}) \end{array}\right\} & \Rightarrow & \text{to be a non-support operator} \\ & \nwarrow & \Downarrow \\ & & \text{to be a semi-non-support operator.} \end{array}$$

In general, the condition that T be a non-support operator does not imply the condition (B)[21]. The following theorem is a generalization of the result obtained by S. Karlin [4: Theorem 12] to the case where the positive cone K has not necessarily non empty interior. Let (C) be the following condition for T:

(C) The spectrum of T on the circle $|\lambda|=r(T)$ consists only of poles and each proper space corresponding to proper values on the circle $|\lambda|=r(T)$ is finite-dimensional.

THEOREM 4. *Let the positive cone K be normal and minihedral and the dual cone K^* be normal. Suppose an operator T satisfy the condition* (C). *Then, holds the following logical relation:*

$$T \text{ is a non-support operator} \Rightarrow T \text{ satisfies (B).}$$

PROOF. Since T is a semi-non-support operator, the spectral radius r is positive by theorem 2. Therefore we may suppose $r=1$ without loss of generality. Since a pole is an isolated point, there are at most finite poles $1=\lambda_1, \cdots, \lambda_\nu$ on the circle $|\lambda|=1$. Let E_1 be the subspace spanned by all the proper vectors corresponding to $\lambda_1, \cdots, \lambda_\nu$.

First step: We shall show that there exists a positive projection P onto E_1. By corollary of theorem 2, the point $\lambda=1$ is a pole of order 1. It is known[22] that if K and K^* are normal then the order of a pole on the circle $|\lambda|=1$ does not exceed the order of the pole at $\lambda=1$. Therefore, $\lambda_2, \cdots, \lambda_\nu$ are also poles of order 1. Put

$$P_j=\frac{1}{2\pi i}\int_{\gamma_j} R(\lambda)d\lambda \quad \text{for each } j=1, \cdots, \nu$$

where γ_j is a sufficiently small circle enclosing λ_j. Then P_j is a bounded projection onto the proper space corresponding to λ_j by (4.2) and (4.3). It is easy to see that

$$P_iP_j=P_jP_i=0 \quad \text{for } i\neq j \tag{5.1}$$

and

$$P_jT=TP_j=\lambda_jP_j\,. \tag{5.2}$$

Let

$$P=\sum_{j=1}^{\nu} P_j\,.$$

21) See remark 1 of theorem 4.

22) See, for example proposition 1 in H. Schaefer [8].

Then P is a bounded projection onto E_1. Furthermore, since $P=\lim_{k\to\infty} T^{n_k}$ [23], P is positive.

Second step: Let $K_1=PK$ and T_1 be the restriction of T to $E_1=PE$. We shall prove that T_1 is a strict non-support operator in E_1 with respect to K_1. Since $PT=TP$, T_1 leaves E_1 invariant, by lemma 3 T_1 is a non-support operator with respect to K_1 in E_1. By lemma 2, K_1 is total in E_1. Since E_1 is finite-dimensional K_1 has non empty interior in E_1. The positivity of P implies that $K_1=K\cap E_1$ and K_1 is closed, proper and minihedral cone. In fact, for each $x, y\in K$, $P(\sup\{x, y\})$ is the $\sup\{x, y\}$ in E_1. Therefore, by lemma 1, T_1 is a strict non-support operator. Since K_1 has non empty interior, T_1 is a strongly positive operator in E_1. Using $P=\lim_{k\to\infty} T^{n_k}$, strong positivity of T_1 and the fact that if $T^{n_0}x_0$ is an interior point of K_1 then $\{T^{n_0+j}x_0\,;\ j=1, 2, \cdots\}$ lies at a positive distance from K_1^b [24], we have $K_1^b=\{0\}$. Therefore $E_1=PE$ is one-dimensional. Hence $P=P_1$ and $P_j=0$ $(j=2, \cdots, \nu)$. That is, T has only one spectrum $\lambda=1$ on the circle $|\lambda|=1$. This completes the proof of theorem 4.

In the first step of the above proof we did not need the condition that K is minihedral and in the second step we have made use of the condition to obtain the fact that T_1 is a strict non-support operator. Therefore, we have the following corollary:

COROLLARY. *Let K and K^* be normal and T satisfy the condition* (C). *Then, holds the following logical relation:*

$$T \textit{ is a strict non-support operator} \Rightarrow T \textit{ satisfies } (\mathrm{B}).$$

Combining theorem 3, its corollary and theorem 4, we get theorem 5 which is a generalization of Theorem B.

THEOREM 5. *Let K be minihedral and normal and let K^* be also normal. Suppose T satisfy the condition* (C). *Then, holds the following logical relation:*

$$T \textit{ is a non-support operator} \Leftrightarrow \begin{cases} r(T)>0 \\ T \textit{ satisfies } (\mathrm{A}) \textit{ and } (\mathrm{B}). \end{cases}$$

REMARK 1. In theorem 5, the assumption that the positive cone be minihedral is indespensable. Indeed, the example noted in the remark of lemma 1 shows that the operator represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

where θ/π is irrational, satisfies the condition (C) and is a non-support

23) This equation is noted in Krein-Rutman [5: p. 288] (replace A by T) where they obtained the equation using only the fact that T has no spectral value except for finite proper values on the circle $|\lambda|=1$ and the above equations (5.1) and (5.2).

24) See Lemma 6.1 in Krein-Rutman [5].

operator. Further, the Lorentz cone K which is identical with K^* is normal. That is, all the assumptions in theorem 5 except for the minihedrality of the positive cone K are satisfied in this example. However, the spectrum of the operator on circle $|\lambda|=1$ is $\{1, e^{\pm i\theta}\}$.

REMARK 2. In the case of l_p with the natural order, theorem 5 has been concluded by F. Niiro [7 : Theorem 5] without supplementary condition (C) as follows:

$$\text{to be a non-support operator} \Leftrightarrow \begin{cases} r(T)>0 \\ (A) \\ (B) \end{cases}.$$

After this manuscript was written, the following three propositions are obtained:

PROPOSITION 1. *If T is a semi-non-support operator and x is a non-support point of K, then Tx is a non-support point of K.*

PROOF. As $T^*f\neq 0$ for every nonzero $f\in K^*$, we have $f(Tx)=T^*f(x)>0$.

PROPOSITION 2. *T is a semi-non-support operator if and only if T is positive and, for each nonzero $x\in K$ and for each nonzero $f\in K^*$, there exists an infinite set of natural numbers m satisfying $f(T^m x)>0$.*

PROOF. Suppose T be a semi-non-support operator. Let $x\in K$, $x\neq 0$ and $f\in K^*$, $f\neq 0$. Then, for every $\xi>r(T)$, $TR(\xi)x$ is a non-support point of K. By proposition 1, $f(T^k R(\xi)x)=\sum_{n=0}^{\infty} f(T^{n+k}x)/\xi^{n+1}>0$. That is, there exists an infinite set of m satisfying $f(T^m x)>0$.

PROPOSITION 3. *T is a strict non-support operator, if and only if T is positive and for each nonzero $x\in K$ there exists a natural number $n=n(x)$ such that $T^n x$ is a non-support point of K.*

PROOF. It follows immediately from proposition 1.

References

[1] E. Bishop and R. R. Phelps, The support functionals of a convex set, Proc. Symp. Pure Math. 7 (1963) 27-35.

[2] G. Frobenius, Über Matrizen aus nicht negativen Elementen, Sitzungsber. Preuss. Akad. Wiss. Berlin, (1912) 456-477.

[3] F. R. Gantmacher, The theory of matrices, Chelsea Pub. Comp. (1959).

[4] S. Karlin, Positive operators, J. Math. Mech. 8 (1959) 907-937.

[5] M. G. Krein and M. A. Rutman, Linear operators leaving invariant a cone in a Banach space, Uspehi Mat. Nauk 3 (23) (1948) 3-95; Amer. Math. Soc. Transl. Ser. I, 10 (1962) 199-325.

[6] Sz. Nagy, Sur les lattis linéaires de dimension finie, Comment. Math. Helv. 17 (1945) 209-213.

[7] F. Niiro, On indecomposable operators in l_p $(1<p<\infty)$ and a problem of H. Schaefer, Sci. Pap. Coll. Gen. Educ. 14 (1964).

[8] H. Schaefer, Some spectral properties of positive linear operators, Pacific J. Math. 10 (1960) 1009-1019.