Almost Periodic Solutions of a System of Ordinary Differential Equations with Periodic Coefficients

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I. Introduction

Usually the study of periodic solutions of ordinary differential equations is reduced to the study of a finite number of relations between certain parameters (or arbitrary constants). Those relations may be derived from the condition that solutions be periodic functions of the independent variable. In this paper we shall study a very special case in which such relations between parameters can be obtained even for almost periodic solutions.

II. Assumption and Main Theorem

§1. Preliminaries: Let a system of ordinary differential equations of the form

$$\frac{dx_j}{dt} = \lambda_j x_j + \delta_j x_{j-1} + f_j(t, x) \qquad (j=1, 2, \cdots, n) \tag{1.1}$$

be given, where λ_j and δ_j are constants and f_j are power series of x with coefficients periodic in t of period 1. We shall assume that δ_j is equal to 1 or 0 and that if $\delta_j=1$ then $\lambda_j=\lambda_{j-1}$. On the other hand, the power series f_j are assumed to contain only those terms of degrees not less than 2. We shall write f_j as follows:

$$f_j(t, x) = \sum_{|\mathfrak{p}| \leqq 2} f_{j\mathfrak{p}}(t) x^{\mathfrak{p}}, \tag{1.2}$$

where $\mathfrak{p}$ is a system of nonnegative integers $p_1, \cdots, p_n$, and

$$\begin{cases} x^{\mathfrak{p}} = x_1^{p_1} \cdots x_n^{p_n}, \\ |\mathfrak{p}| = p_1 + \cdots + p_n. \end{cases} \tag{1.3}$$

Our assumptions say that the coefficients $f_{j\mathfrak{p}}(t)$ are periodic of period 1.

Let us assume that

$$\begin{cases} \Re\lambda_j = 0, \quad \lambda_j \not\equiv 0 \pmod{2\pi i} & (j=1, 2, \cdots, r), \\ \Re\lambda_j \neq 0 \quad \text{or} \quad \lambda_j \equiv 0 \pmod{2\pi i} & (j \neq 1, 2, \cdots, r). \end{cases} \tag{1.4}$$

Then consider a system of ordinary differential equations of the following form:

$$\frac{dy_j}{dt}=\begin{cases}\lambda_j y_j+\delta_j y_{j-1}+f_j(t, y)+A_j(C)e^{\lambda_j t} & (j=1, \cdots, r), \\ \lambda_j y_j+\delta_j y_{j-1}+f_j(t, y) & (j\neq 1, \cdots, r),\end{cases} \tag{1.5}$$

where A_j are power series of the parameters $C_1, \cdots, C_r$ with constant coefficients and $A_j(0)=0$. We shall write A_j as follows:

$$A_j(C)=\sum_{|\mathfrak{p}|\geqq 1} A_{j\mathfrak{p}}C^{\mathfrak{p}}, \tag{1.6}$$

where $\mathfrak{p}$ is a system of nonnegative integers $p_1, \cdots, p_r$, and

$$C^{\mathfrak{p}}=C_1^{p_1}\cdots C_r^{p_r}. \tag{1.7}$$

Now we shall determine the power series A_j in such a way that the system (1.5) admits a solution of the following form:

$$y_j=U_j(t, u)=\sum_{|\mathfrak{p}|\geqq 1} U_{j\mathfrak{p}}(t)u^{\mathfrak{p}} \qquad (j=1, \cdots, n), \tag{1.8}$$

where $U_{j\mathfrak{p}}$ are periodic in t of period 1 and

$$u_j=C_j e^{\lambda_j t} \qquad (j=1, 2, \cdots, r), \tag{1.9}$$

$$u^{\mathfrak{p}}=u_1^{p_1}\cdots u_r^{p_r}. \tag{1.10}$$

To do this, substituting (1.8) in (1.5), we derive a system of linear ordinary differential equations

$$\frac{dU_{j\mathfrak{p}}}{dt}=\begin{cases}\lambda_{j\mathfrak{p}}U_{j\mathfrak{p}}(t)+\delta_j U_{j-1\mathfrak{p}}(t)+H_{j\mathfrak{p}}(t)+A_{j\mathfrak{p}}e^{\lambda_{j\mathfrak{p}}t} & (j=1, \cdots, r), \\ \lambda_{j\mathfrak{p}}U_{j\mathfrak{p}}(t)+\delta_j U_{j-1\mathfrak{p}}(t)+H_{j\mathfrak{p}}(t) & (j\neq 1, \cdots, r),\end{cases} \tag{1.11}$$

where

$$\lambda_{j\mathfrak{p}}=\lambda_j-\sum_{k=1}^{r} p_k\lambda_k, \tag{1.12}$$

and

$$f_j(t, U)=\sum_{|\mathfrak{p}|\geqq 2} H_{j\mathfrak{p}}(t)u^{\mathfrak{p}}. \tag{1.13}$$

First of all, we put

$$U_{j\mathfrak{e}_k}=\delta_{jk} \qquad (j=1, \cdots, n;\ k=1, \cdots, r) \tag{1.14}$$

and

$$A_{j\mathfrak{e}_k}=\begin{cases}-\delta_j & (k=j-1), \\ 0 & (k\neq j-1),\end{cases} \tag{1.15}$$

where δ_{jk} is the Kronecker's delta and

$$\mathfrak{e}_k=(\delta_{1k}, \cdots, \delta_{rk}).$$

Then for $|\mathfrak{p}|\geqq 2$ we put

$$A_{j\mathfrak{p}}=\begin{cases}-\int_0^1\{\delta_j U_{j-1\mathfrak{p}}(t)+H_{j\mathfrak{p}}(t)\}e^{-\lambda_{j\mathfrak{p}}t}dt & (\lambda_{j\mathfrak{p}}\equiv 0 \mod 2\pi i), \\ 0 & (\lambda_{j\mathfrak{p}}\not\equiv 0 \mod 2\pi i),\end{cases} \tag{1.16}$$

$$(1.17)\qquad U_{j\mathfrak{p}}(t)=\begin{cases}\displaystyle\int_0^1 s\{\delta_j U_{j-1\mathfrak{p}}(t+s)+H_{j\mathfrak{p}}(t+s)+A_{j\mathfrak{p}}e^{\lambda_{j\mathfrak{p}}(t+s)}\}e^{-\lambda_{j\mathfrak{p}}s}ds & (\lambda_{j\mathfrak{p}}\equiv 0 \mod 2\pi i)\,,\\ \displaystyle E_{j\mathfrak{p}}\int_0^1\{\delta_j U_{j-1\mathfrak{p}}(t+s)+H_{j\mathfrak{p}}(t+s)\}e^{-\lambda_{j\mathfrak{p}}s}ds & (\lambda_{j\mathfrak{p}}\not\equiv 0 \mod 2\pi i)\,,\end{cases}$$

where

$$(1.18)\qquad E_{j\mathfrak{p}}=\frac{e^{\lambda_{j\mathfrak{p}}}}{1-e^{\lambda_{j\mathfrak{p}}}}\qquad (\lambda_{j\mathfrak{p}}\not\equiv 0 \mod 2\pi i)\,.$$

It is easily seen that the series A_j and U_j thus determined satisfy the relations (1.5) formally.

§ 2. **Main theorem:** So far we have determined the formal power series A_j and U_j in such a way that they satisfy the relations (1.5). If they are convergent, then the almost periodic solutions of the systmem (1.1) will be given by

$$(2.1)\qquad \begin{cases} x_j=U_j(t,u) & (j=1,\cdots,n)\,,\\ 0=A_j(C) & (j=1,\cdots,r)\,.\end{cases}$$

The convergence of the series A_j and U_j can be proved under certain conditions on λ_j. Namely, we shall prove the following

THEOREM: *Suppose that, except for a finite number of* $\mathfrak{p}$, *the quantities* $\lambda_{j\mathfrak{p}}$ *satisfy the following conditions:*

$$(\text{i})\qquad \lambda_{j\mathfrak{p}}\not\equiv 0 \qquad (\text{mod } 2\pi i)\,;$$

$$(\text{ii})\qquad |E_{j\mathfrak{p}}|\leqq K|\mathfrak{p}|^{v_0}\,,$$

when K *and* v_0 *are positive constants independent of* $(j,\mathfrak{p})$. *Then the quantities* A_j *are polynomials of* C *and the series* U_j *are convergent.*

§ 3. **Remarks:** Consider a system of ordinary differential equations of the form

$$(3.1)\qquad \frac{dx}{dt}=Ax+f(t,x)\,,$$

where x is an n-dimensional vector, A is an n by n constant matrix, and f is an n-dimensional vector whose components are similar to the functions given by (1.2). By a linear transformation of the unknown vector x, the system (3.1) can be reduced to a system of the form (1.1). The same is true for the case where the matrix A is periodic of period 1 with respect to t.

If every characteristic value of the matrix A is purely imaginary, and if the conditions (i) and (ii) of our theorem are satisfied for *every* $\mathfrak{p}$, then the general solution of the system (1.1) is almost periodic. This is essentially due to C. L. Siegel [2] and one of the corollaries of our theorem.

In one of our previous papers [1] we used the similar ideas for the construction of periodic solutions. In that case the proof of the convergence of the series A_j and U_j was not complicated. However, in the present case, because of the small divisors, such a proof is very complicated. Ours is essentially based on the ideas invented by C. L. Siegel [2].

The case when A is a real matrix is not covered by our theorem. In fact, if λ is a characteristic value of A and purely imaginary, then $-\lambda$ is also a characteristic value of A. Therefore, the conditions of our theorem are not satisfied.

III. Proof of Main Theorem

§ 4. **Part I:** The following facts can be derived from our assumptions:

(i) $$\lambda_1 p_1 + \lambda_2 p_2 + \cdots + \lambda_r p_r \not\equiv 0 \qquad (\text{mod } 2\pi i)$$

for every $\mathfrak{p}$;

(ii) *there exists a positive number* L *such that we have*

$$U_{j\mathfrak{p}}(t) = E_{j\mathfrak{p}} \int_0^1 \{\delta_j U_{j-1\mathfrak{p}}(t+s) + H_{j\mathfrak{p}}(t+s)\} e^{-\lambda_{j\mathfrak{p}} s} ds \tag{4.1}$$

for $|\mathfrak{p}| \geqq L$.

Let us put

$$H_{\mathfrak{p}} = \max_j \max_t | H_{j\mathfrak{p}}(t) | , \tag{4.2}$$

$$U_{\mathfrak{p}} = \max_j \max_t | U_{j\mathfrak{p}}(t) | , \tag{4.3}$$

and

$$M_{\mathfrak{p}} = \begin{cases} 1 & (|\mathfrak{p}| = 1), \\ nU_{\mathfrak{p}} & (|\mathfrak{p}| > 1). \end{cases} \tag{4.4}$$

Then for $\Re\lambda_j \neq 0$ let us put

$$\sigma_{j\mathfrak{p}} = | E_{j\mathfrak{p}} | \int_0^1 | e^{-\lambda_{j\mathfrak{p}} s} | ds \tag{4.5}$$

and

$$\sigma = \sup_{(j,\mathfrak{p})} \{1, \sigma_{j\mathfrak{p}}\} . \tag{4.6}$$

On the other hand, we put

$$E_{j\mathfrak{p}} = 1 \qquad (\lambda_{j\mathfrak{p}} \equiv 0 \quad \text{mod } 2\pi i) , \tag{4.7}$$

and

$$E_{\mathfrak{p}} = \max_{\Re\lambda_j = 0} \{1, | E_{j\mathfrak{p}} |\} . \tag{4.8}$$

Since (4.1) implies

$$|U_{j\mathfrak{p}}(t)|\leqq j\sigma^j E_{\mathfrak{p}}^j H_{\mathfrak{p}} \qquad (j=1,\cdots,n\,;\ |\mathfrak{p}|\geqq L), \tag{4.9}$$

we have

$$M_{\mathfrak{p}}\leqq n^2\sigma^n E_{\mathfrak{p}}^n H_{\mathfrak{p}} \qquad (|\mathfrak{p}|\geqq L). \tag{4.10}$$

§ **5. Part II:** Let

$$|f_j(t,x)|\leqq N \tag{5.1}$$

for

$$|x_k|<\delta, \quad -\infty<t<+\infty, \tag{5.2}$$

where N and δ are positive constants. Then we have

$$f_j(t,U)\ll N\sum_{m=2}^{\infty}\Big(\sum_{|\mathfrak{p}|\leqq 1} M_{\mathfrak{p}}u^{\mathfrak{p}}\Big)^m\delta^{-m}, \tag{5.3}$$

where $\ll$ means that the left-hand members are majorized by the right-hand member as power series of u. Therefore

$$H_{\mathfrak{p}}\leqq N\sum_{\substack{\mathfrak{p}_1+\cdots+\mathfrak{p}_\nu=\mathfrak{p}\\ \nu>1}} M_{\mathfrak{p}_1}\cdots M_{\mathfrak{p}_\nu}\delta^{-\nu}. \tag{5.4}$$

Hence

$$M_{\mathfrak{p}}\leqq n^2\sigma^n E_{\mathfrak{p}}^n N\sum_{\substack{\mathfrak{p}_1+\cdots+\mathfrak{p}_\nu=\mathfrak{p}\\ \nu>1}} M_{\mathfrak{p}_1}\cdots M_{\mathfrak{p}_\nu}\delta^{-\nu} \qquad (|\mathfrak{p}|\geqq L). \tag{5.5}$$

§ **6. Part III:** Let us define a function Φ of a single variable v by the following equation:

$$\Phi=v+n^2\sigma^n N\sum_{m=2}^{\infty}\Phi^m\delta^{-m}, \tag{6.1}$$

and put

$$\Psi(u)=\Phi(u_1+\cdots+u_r). \tag{6.2}$$

Let

$$\Psi(u)=u_1+\cdots+u_r+\sum_{|\mathfrak{p}|\leqq 2}\tau_{\mathfrak{p}}u^{\mathfrak{p}}. \tag{6.3}$$

Then

$$\tau_{\mathfrak{p}}=n^2\sigma^n N\sum_{\substack{\mathfrak{p}_1+\cdots+\mathfrak{p}_\nu=\mathfrak{p}\\ \nu>1}}\tau_{\mathfrak{p}_1}\cdots\tau_{\mathfrak{p}_\nu}\delta^{-\nu}. \tag{6.4}$$

Now we put

$$\sigma_{\mathfrak{p}}=\begin{cases} 1 & (|\mathfrak{p}|=1),\\ E_{\mathfrak{p}}^n \max\limits_{\substack{\mathfrak{p}_1+\cdots+\mathfrak{p}_\nu=\mathfrak{p}\\ \nu>1}} \sigma_{\mathfrak{p}_1}\cdots\sigma_{\mathfrak{p}_\nu} & (|\mathfrak{p}|>1).\end{cases} \tag{6.5}$$

Then if we choose a sufficiently large positive number a in a suitable way, we have the following estimates:

(6.6) $$M_{\mathfrak{p}} \leqq (an^2\sigma^n N)^{|\mathfrak{p}|-1}\sigma_{\mathfrak{p}}\tau_{\mathfrak{p}} \qquad (|\mathfrak{p}| \geqq 1).$$

(6.6) can be proved by induction.

Hereafter we shall aim at the proof of the existence of a sufficiently large positive constant b such that we have

(6.7) $$\sigma_{\mathfrak{p}} \leqq b^{|\mathfrak{p}|-1}|\mathfrak{p}|^{-2v_0nL} \qquad (|\mathfrak{p}| \geqq 1).$$

It is evident that (6.6) and (6.7) imply the convergence of the series U_j.

§ 7. **Part IV:** First of all we shall prove the following:

Let us put

(7.1) $$\alpha_{\mathfrak{p}} = b^{|\mathfrak{p}|-1}|\mathfrak{p}|^{-2v_0nL}.$$

Then if

(7.2) $$b > 2^{2v_0nL},$$

we have the inequality

(7.3) $$\alpha_{\mathfrak{p}_1}\alpha_{\mathfrak{p}_2} < \alpha_{\mathfrak{p}_1+\mathfrak{p}_2}.$$

In fact

$$\frac{\alpha_{\mathfrak{p}_1}\alpha_{\mathfrak{p}_2}}{\alpha_{\mathfrak{p}_1+\mathfrak{p}_2}} = \left\{\frac{1}{|\mathfrak{p}_1|} + \frac{1}{|\mathfrak{p}_2|}\right\}^{2v_0nL} b^{-1} \leqq 2^{2v_0nL}b^{-1} < 1.$$

Now according to the definition (6.5) of $\sigma_{\mathfrak{p}}$, we shall write

(7.4) $$\sigma_{\mathfrak{p}} = (E_{\mathfrak{p}_0}E_{\mathfrak{p}_1}\cdots E_{\mathfrak{p}_s})^n \Delta_1\Delta_2\cdots\Delta_s\Delta_{s+1},$$

where

$$\mathfrak{p}_0 = \mathfrak{p} = \mathfrak{p}_{11} + \cdots + \mathfrak{p}_{1r_1},$$

$$\mathfrak{p}_j = \mathfrak{p}_{j1} = \mathfrak{p}_{j+11} + \cdots + \mathfrak{p}_{j+1r_{j+1}} \qquad (j=1, \cdots, s),$$

$$|\mathfrak{p}_s| > \frac{1}{2},$$

$$|\mathfrak{p}_{s+1\nu}| \leqq \frac{1}{2}|\mathfrak{p}| \qquad (\nu=1, \cdots, r_{s+1}),$$

$$\Delta_j = \sigma_{\mathfrak{p}_{j2}} \cdots \sigma_{\mathfrak{p}_{s+1r_{s+1}}} \qquad (j=1, \cdots, s)$$

and

$$\Delta_{s+1} = \sigma_{\mathfrak{p}_{s+11}} \cdots \sigma_{\mathfrak{p}_{s+1r_{s+1}}}.$$

If every component of a vector $\mathfrak{p}-\mathfrak{q} = (p_1-q_1, \cdots, p_r-q_r)$ is nonnegative, then we write

$$\mathfrak{p} > \mathfrak{q} . \tag{7.5}$$

According to this notation, we have

$$\mathfrak{p}_0 > \mathfrak{p}_1 > \cdots > \mathfrak{p}_s . \tag{7.6}$$

Let us assume that b *is sufficiently large so that we have* (7.2) *and* (6.7) *for* $|\mathfrak{p}| \leqq 2L$. *Then assume that*

$$|\mathfrak{p}| > 2L . \tag{7.7}$$

If we assume that we have (6.6) for $|\mathfrak{p}'| < |\mathfrak{p}|$ then (7.3) implies that we have

$$\varDelta_j \leqq b^{|\mathfrak{p}_{j-1} - \mathfrak{p}_j| - 1} |\mathfrak{p}_{j-1} - \mathfrak{p}_j|^{-2v_0 nL} \qquad (j = 1, \cdots, s) .$$

On the other hand,

$$\varDelta_{s+1} \leqq b^{|\mathfrak{p}_s| - \rho_0} \left\{ \prod_{\nu=1}^{\rho_0} \mathfrak{q}_\nu \right\}^{-2v_0 nL} ,$$

where

$$r_{s+1} = \rho_0, \quad \mathfrak{p}_{s+1\nu} = \mathfrak{q}_\nu \qquad (\nu = 1, \cdots, \rho_0) .$$

Hence

$$\sigma_{\mathfrak{p}} \leqq b^{|\mathfrak{p}| - s - r_0} \left\{ \prod_{j=1}^{s} |\mathfrak{p}_{j-1} - \mathfrak{p}_j| \prod_{\nu=1}^{\rho_0} \mathfrak{q}_\nu \right\}^{-2v_0 nL} (E_{\mathfrak{p}_0} E_{\mathfrak{p}_1} \cdots E_{\mathfrak{p}_s})^n . \tag{7.8}$$

§ 8. **Part V:** We are now going to prove the following estimates:

$$E_{\mathfrak{p}_0} E_{\mathfrak{p}_1} \cdots E_{\mathfrak{p}_s} \leqq E^{s+1} \left\{ |\mathfrak{p}_0| \prod_{j=1}^{s} |\mathfrak{p}_{j-1} - \mathfrak{p}_j| \right\}^{v_0 L} , \tag{8.1}$$

where

$$E = (2^{2v_0+1} L^{v_0} K)^L . \tag{8.2}$$

To do this, we need the following results:

(i) *If* $\mathfrak{p} > \mathfrak{p}'$ *and* $E_{j\mathfrak{p}} = E_{k\mathfrak{p}'}$, *we have* $|\mathfrak{p} - \mathfrak{p}'| < L$;

(ii) *if* $\mathfrak{p} > \mathfrak{p}'$ *and* $E_{j\mathfrak{p}} \neq E_{k\mathfrak{p}'}$, *we have*

$$\min \{ |E_{j\mathfrak{p}}|, |E_{k\mathfrak{p}'}| \} \leqq 2^{v_0+1} K |\mathfrak{p} - \mathfrak{p}'|^{v_0} .$$

In fact, $E_{j\mathfrak{p}} = E_{k\mathfrak{p}'}$ implies

$$e^{\lambda_{j\mathfrak{p}}} = e^{\lambda_{k\mathfrak{p}'}} .$$

Hence

$$\lambda_j \equiv \lambda_k + \sum_{\nu=q+1}^{r} \lambda_\nu (p_\nu - p'_\nu) \qquad (\mathrm{mod}\ 2\pi i) .$$

This proves the statement (i). On the other hand, the inequality

$$\frac{1}{|E_{j\mathfrak{p}-\mathfrak{p}'+e_k}|}\leqq\frac{1}{|E_{j\mathfrak{p}}|}+\frac{1}{|E_{k\mathfrak{p}'}|}\leqq\frac{2}{\min\{|E_{j\mathfrak{p}}|,|E_{k\mathfrak{p}'}|\}}$$

and the assumption (ii) of our theorem imply the statement (ii).

We shall prove (8.1) by the use of the mathematical induction on s. To do this we consider the following four cases:

Case I: $E_{\mathfrak{p}_0}=\cdots=E_{\mathfrak{p}_s}$;

Case II: $E_{\mathfrak{p}_0}=\cdots=E_{\mathfrak{p}_j}=\min_{\nu=0}^{s} E_{\mathfrak{p}_\nu}$,

$$E_{\mathfrak{p}_{j+1}}>E_{\mathfrak{p}_j},\qquad j<s\,;$$

Case III: $E_{\mathfrak{p}_j}=\cdots=E_{\mathfrak{p}_s}=\min_{\nu=0}^{s} E_{\mathfrak{p}_\nu}$,

$$E_{\mathfrak{p}_{j-1}}>E_{\mathfrak{p}_j},\qquad j>0\,;$$

Case IV: $E_{\mathfrak{p}_j}=\cdots=E_{\mathfrak{p}_k}=\min_{\nu=0}^{s} E_{\mathfrak{p}_\nu}$,

$$E_{\mathfrak{p}_{j-1}}>E_{\mathfrak{p}_j},\qquad E_{\mathfrak{p}_{k+1}}>E_{\mathfrak{p}_k}\,,$$

$$0<j\leqq k<s\,.$$

In Case I, we have $|\mathfrak{p}_0-\mathfrak{p}_s|<L$. Then

$$s\leqq|\mathfrak{p}_0-\mathfrak{p}_s|<L\,.$$

Hence

$$E_{\mathfrak{p}_0}^{s+1}\leqq(K|\mathfrak{p}_0|^{v_0})^L\,.$$

This implies (8.1).

In Case II, we assume

$$E_{\mathfrak{p}_{j+1}}\cdots E_{\mathfrak{p}_s}\leqq E^{s-j}\left\{|\mathfrak{p}_{j+1}|\prod_{\nu=j+2}^{s}|\mathfrak{p}_{\nu-1}-\mathfrak{p}_\nu|\right\}^{v_0L}$$

in order to apply the mathematical induction on s. Since

$$|\mathfrak{p}_0|\geqq|\mathfrak{p}_{j+1}|\,,$$

we have

$$E_{\mathfrak{p}_{j+1}}\cdots E_{\mathfrak{p}_s}\leqq E^{s-j}\left\{|\mathfrak{p}_0|\prod_{\nu=j+2}^{s}|\mathfrak{p}_{\nu-1}-\mathfrak{p}_\nu|\right\}^{v_0L}\,. \tag{8.3}$$

On the other hand, we have

$$j+1\leqq L$$

and

$$E_{\mathfrak{p}_j}\leqq 2^{v_0+1}K|\mathfrak{p}_j-\mathfrak{p}_{j+1}|^{v_0}\,.$$

Therefore we have

$$E_{\mathfrak{p}_0}\cdots E_{\mathfrak{p}_j}\leqq(2^{v_0+1}K)^L|\mathfrak{p}_j-\mathfrak{p}_{j+1}|^{v_0L}\,. \tag{8.4}$$

(8.3) and (8.4) imply (8.1).

The Case III can be treated in a similar way. In Case IV, as assume

$$E_{\mathfrak{p}_0}\cdots E_{j-1}E_{\mathfrak{p}_{k+1}}\cdots E_{\mathfrak{p}_s}$$
$$\leqq E^{s-k+j}\left\{|\mathfrak{p}_0|\prod_{\nu=1}^{j-1}|\mathfrak{p}_{\nu-1}-\mathfrak{p}_\nu||\mathfrak{p}_{j-1}-\mathfrak{p}_{k+1}|\prod_{\nu=k+2}^{s}|\mathfrak{p}_{\nu-1}-\mathfrak{p}_\nu|\right\}^{v_0L}.$$

Since we have

$$E_{\mathfrak{p}_j}\leqq 2^{v_0+1}K\min\{|\mathfrak{p}_{j-1}-\mathfrak{p}_j|^{v_0},|\mathfrak{p}_k-\mathfrak{p}_{k+1}|^{v_0}\}$$

and

$$k-j+1\leqq L,$$

we have

$$E_{\mathfrak{p}_j}\cdots E_{\mathfrak{p}_k}\leqq(2^{v_0+1}K)^L[\min\{|\mathfrak{p}_{j-1}-\mathfrak{p}_j|,|\mathfrak{p}_k-\mathfrak{p}_{k+1}|\}]^{v_0L}. \tag{8.5}$$

On the other hand,

$$E_{\mathfrak{p}_0}\cdots E_{\mathfrak{p}_{j-1}}E_{\mathfrak{p}_{k+1}}\cdots E_{\mathfrak{p}_s} \tag{8.6}$$
$$\leqq E^{s-k+j}\left\{|\mathfrak{p}_0|\prod_{\nu=1}^{s}|\mathfrak{p}_{\nu-1}-\mathfrak{p}_\nu|\right\}^{v_0L}\left[\frac{2+L}{\min\{|\mathfrak{p}_{j-1}-\mathfrak{p}_j|,|\mathfrak{p}_k-\mathfrak{p}_{k+1}|\}}\right]^{v_0L}$$

because

$$\frac{|\mathfrak{p}_{j-1}-\mathfrak{p}_{k+1}|}{\prod_{\nu=j}^{k+1}|\mathfrak{p}_{\nu-1}-\mathfrak{p}_\nu|}\leqq\frac{|\mathfrak{p}_{j-1}-\mathfrak{p}_j|+|\mathfrak{p}_j-\mathfrak{p}_k|+|\mathfrak{p}_k-\mathfrak{p}_{k+1}|}{|\mathfrak{p}_{j-1}-\mathfrak{p}_j||\mathfrak{p}_k-\mathfrak{p}_{k+1}|}\leqq\frac{2+L}{\min\{|\mathfrak{p}_{j-1}-\mathfrak{p}_j|,|\mathfrak{p}_k-\mathfrak{p}_{k+1}|\}}.$$

(8.5) and (8.6) imply (8.1).

This completes the proof of (8.1).

§ 9. **Part VI:** From (7.8) and (8.1) we derive the following inequality:

$$\sigma_{\mathfrak{p}}\leqq b^{|\mathfrak{p}|-s-\rho_0}E^{n(s+1)}\left\{\prod_{j=1}^{s}|\mathfrak{p}_{j-1}-\mathfrak{p}_j|\prod_{\nu=1}^{\rho_0}|\mathfrak{q}_\nu|^2\right\}^{-v_0nL}|\mathfrak{p}_0|^{v_0nL}. \tag{9.1}$$

On the other hand, C. L. Siegel [1] proved an inequality

$$\prod_{j=1}^{s}|\mathfrak{p}_{j-1}-\mathfrak{p}_j|\prod_{\nu=1}^{\rho_0}|\mathfrak{q}_\nu|^2\geqq|\mathfrak{p}_0|^3 8^{1-(\rho_0+s)}. \tag{9.2}$$

(9.1) and (9.2) imply

$$\sigma_{\mathfrak{p}}\leqq b^{|\mathfrak{p}|-s-\rho_0}E^{n(s+1)}(8^{\rho_0+s-1})^{v_0nL}|\mathfrak{p}_0|^{-2v_0nL}, \tag{9.3}$$

Therefore if

$$b\geqq 8^{v_0nL}E^n,$$

then we have (6.7).

Thus the proof of our theorem is completed.

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