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Locally Convex Spaces with the Extension Property

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There are several researches[1)] on a normed space N with the extension property: each continuous linear function f on a subspace of any normed space with values in N has a linear extension f' on the whole space such that $\|f\|=\|f'\|$. Among those, the following result has been obtained by L. Nachbin [4]; a normed space has the extension property if and only if the collection of all its spheres has the binary intersection property. The collection $\mathfrak{U}$ of all spheres is said to have the binary intersection property if every subcollection of $\mathfrak{U}$, any two elements of which intersect, has a nonvoid intersection.

In this paper, we shall give a convenient definition of extension property of locally convex topological vector spaces. Of course the property must be a generalization of the usual extension property. Theorem 1 gives a necessary and sufficient condition in order that a locally convex space should have the extension property in our sense which corresponds with Nachbin's result. Theorem 2 gives a characterization of a locally convex topological vector space having the extension property.

A vector space E is said to be a *topological vector space* if E is a Hausdorff space in which the vector operations, summation and scalar multiplication, are continuous for the topology. Moreover, if the neighbourhood system in a topological vector space consists of convex sets, then E is said to be a *locally convex topological vector space* or a *locally convex space*. In this paper, the neighbourhood system in a locally convex space is always assumed, without loss of generality, to consist of symmetric, convex closed sets. We shall denote by R the real space $(-\infty, +\infty)$ and $\{x; P(x)\}$ the set of all the elements with the property $P(x)$.

1. Definition of the extension property

Let E be a locally convex space and $\mathfrak{S}$ be a subbase, consisting of convex, symmetric and closed sets, for a neighbourhood system of the zero point of E. If, for two elements W and W' of $\mathfrak{S}$, we can find positive numbers λ and μ such that $\lambda W\subset W'$[2)] and $\mu W'\subset W$, then we

1) For example, L. Nachbin [4], D. B. Goodner [2] and J. L. Kelly [3].

2) If A and B are subsets of E and λ is a real number, then $A+B$ and λA denote respectively the set $\{a+b;\ a\in A,\ b\in B\}$ and $\{\lambda a;\ a\in A\}$.

shall say that W and W' are *equivalent* to each other and shall denote it by $W \sim W'$. Evidently the relation $\sim$ satisfies the equivalence law, by which $\mathfrak{S}$ can be divided into classes, the set of the representative elements of which, denoted by $\mathfrak{B}$, will be called a *fundamental base* in E. Since the definition depends on a neighbourhood system, a sub-base and representative elements, there are many fundamental bases coming into considerations in one locally convex space. For example, let $\mathfrak{B}_1$ be the class of two sets $\{(x, y); -1 \leqq x \leqq 1, y \in R\}$ and $\{(x, y); -1 \leqq y \leqq 1, x \in R\}$ and let $\mathfrak{B}_2$ be the class of one set $\{(x, y); x^2+y^2 \leqq 1\}$. Both $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are the fundamental bases in the two-dimensional Euclidian space R^2. If $B \in \mathfrak{B}$, $x \in E$ and $\mu \in R$ then $\mu B + x$ is called a *$\mathfrak{B}$-figure* and two $\mathfrak{B}$-figures denoted by $\mu_1 B + x_1$ and $\mu_2 B + x_2$ ($x_1, x_2 \in E$, $\mu_1, \mu_2 \in R$) are said to be *similar* to each other. Evidently above $\mathfrak{B}$-figures are similar to B.

The following definition is a generalization of the binary intersection property by L. Nachbin [4]. A fundamental base $\mathfrak{B}$ in E is said to have the *binary intersection property* if an arbitrary collection of $\mathfrak{B}$-figures whose every two similar $\mathfrak{B}$-figures intersect has a nonvoid intersection. For example, the fundamental base $\mathfrak{B}_1$ in R^2 defined above has the binary intersection property and $\mathfrak{B}_2$ has not.

Let L be a locally convex space, L' a proper subspace and l' a continuous linear operator from L' into E. Let $\mathfrak{B}$ be a fundamental base in E and $\mathfrak{W}_L$ be a neighbourhood system in L. From the continuity of l', there exists, to each $B \in \mathfrak{B}$, a neighbourhood $T(B)$ in L such that

$$l'(T(B) \cap L') \subset B . \tag{1.1}$$

The relation between B and $T(B)$ gives us a mapping from $\mathfrak{B}$ to $\mathfrak{W}_L$. Let $\mathfrak{T}(\mathfrak{B}, l')$ be the class of all such mappings. Fixing one mapping T of $\mathfrak{T}(\mathfrak{B}, l')$, the linear operator l' is said to have a *T-continuous extension* l'' if l'' is a linear extension of l', where l'' is a linear operator on L'' with $L'' \supset L'$ satisfies $l''=l'$ on L', and furthermore $l''(T(B) \cap L'') \subset B$ for every B of $\mathfrak{B}$. The T-continuous extension of l' is obviously continuous. A fundamental base $\mathfrak{B}$ in E is said to have the *extension property* if every continuous linear operator l', defined on any subspace L' of any locally convex space L, always has a T-continuous extension on L for every mapping T in $\mathfrak{T}(\mathfrak{B}, l')$. If the fundamental base with the extension property exists in E then E is said to have the *extension property*. If N is a normed space then the following proposition (a) and (b) are equivalent to each other;

(a) N has the extension property in a usual sense mentioned at the begining of this paper,

(b) the fundamental base consisting of only the unit sphere in N

has the extension property.
Therefore, it is easy to see that if a normed space N has the extension property in a usual sense then N has the extension property in our sense. On the other hand, we can find the normed space which has the extension property in our sense and has not the extension property in usual sense. For example, n-dimensional Euclidian space R^n has the extension property in our sense but not the extension property in a usual sense. Let us remark here that the extension property of E is a topologically invariant property.

Now, we shall state our main theorem.

Theorem 1. *A locally convex space E has the extension property if and only if the space E has the fundamental base with the binary intersection property.*

2. Proof of sufficiency

Let $\mathfrak{B}$ be the fundamental base with the binary intersection property. We shall prove that the fundamental base $\mathfrak{B}$ has the extension property. Let l' be a continuous linear operator defined on a subspace L' of a locally convex space L and T a mapping from $\mathfrak{B}$ into a neighbourhood system $\mathfrak{W}_L$ in L such that, for every element B of $\mathfrak{B}$, the image $T(B)$ of B satisfies (1.1). In this section, our purpose is to show that l' has a T-continuous extension on L, which will be established in two steps.

The first step. We shall show l' has a T-continuous extension on $\{L', s_0\}=\{s'+\lambda s_0;\ s'\in L',\ \lambda\in R\}$, where $s_0\in L$ and $s_0\notin L'$. Let $\mathfrak{U}(l')$ be the set of all $\mathfrak{B}$-figures such that $\mu B-l'(s')$ where $s_0+s'\in\mu T(B)$, $B\in\mathfrak{B}$. For each $B\in\mathfrak{B}$, every two $\mathfrak{B}$-figures $\mu_1 B-l'(s_1')$ and $\mu_2 B-l'(s_2')$ where $s_0+s_1'\in\mu_1 T(B)$ and $s_0+s_2'\in\mu_2 T(B)$ have at least a common point, since we have $s_1'-s_2'=(s_1'+s_0)-(s_2'+s_0)\in(|\mu_1|+|\mu_2|)T(B)$ and

$$l'(s_1')-l'(s_2')=l'(s_1'-s_2')\in(|\mu_1|+|\mu_2|)B=\mu_1 B-\mu_2 B\ .$$

$\mathfrak{B}$ having the binary intersection property, all the $\mathfrak{B}$-figures in $\mathfrak{U}(l')$ have a point in common. Denote this common point by β and define an operator l on $\{L', s_0\}$ by the following equality,

$$l(s'+\lambda s_0)=l'(s')+\lambda\beta\ .$$

By this definition, we see immediately that the operator l is a linear and T-continuous extension on $\{L', s_0\}$.

The second step. We shall show l' has a T-continuous extension on L. Here, we shall make use of the transfinite method. Let l_α be a linear operator which is a T-continuous extension of l' on L_α where L_α is a subspace of L containing L'. We define the order $\prec$ in the set $\mathfrak{L}$[3] of all such operators l_α, $\alpha \in A$, by setting $l_\alpha \prec l_\beta$ if and only if

(i) $L_\alpha \subset L_\beta$,

(ii) l_β is a T-continuous extension of l_α[4].

Under this ordering every totally ordered set $\{l_\alpha; \alpha \in A' \subset A\}$ in $\mathfrak{L}$ has an upper bound determined by the linear operator l_0' on $\bigcup_{\alpha \in A'} L_\alpha$ defined by setting $l_0'(s) = l_\alpha(s)$ when $s \in L_\alpha$. Then, by Zorn's Lemma[5], $\mathfrak{L}$ has a maximal element l_0. Let L_0 be the domain of l_0. We shall prove that l_0 is a T-continuous extension on L. If $L \neq L_0$, then by what is obtained in the first step[6] we can define a T-continuous extension l_1 on $\{L_0, s_0\}$, $s_0 \in L$ and $s_0 \notin L_0$. The linear operator l_1 belongs to $\mathfrak{L}$ and satisfies $l_0 \prec l_1$ and $l_0 \neq l_1$. This contradicts to the fact that l_0 is maximal. Thus, the sufficiency is proved.

3. Proof of Necessity. To prove the necessity, let us begin with the following Lemma 1, 2 and 3.

Lemma 1. *Let $\mathfrak{B}$ be a fundamental base in E. Every two $\mathfrak{B}$-figures similar to B, $\mu_1 B + x_1$ and $\mu_2 B + x_2$, have a common point if and only if the element $x_1 - x_2$ belongs to $(|\mu_1| + |\mu_2|)B$.*

The proof is easy and omitted here.

Lemma 2. *Let $\mathfrak{B}$ be a fundamental base in E and ρ_B be the semi-norm defined by putting $\rho_B(x) = \inf \{|\lambda|; x \in \lambda B\}$. Suppose for every element B of $\mathfrak{B}$ there exists a non-negative function r_B with the following properties*

(2.1) $r_B(x) + r_B(y) \geqq \rho_B(x-y)$ *when* $x, y \in E$,

(2.2) $|r_B(x) - r_B(y)| \leqq \rho_B(x-y)$ *when* $x, y \in E$,

(2.3) $r_B(\lambda x + (1-\lambda)y) \leqq \lambda r_B(x) + (1-\lambda) r_B(y)$ *when* $x, y \in E$ *and* $0 \leqq \lambda \leqq 1$.

Suppose further that the class of all such functions $\{r_B; B \in \mathfrak{B}\}$ has the property such that

(2.4) *for every element x in E, there is a function r_B in the class satisfying $r_B(x) > 0$.*

Then, choosing an abstract element ξ (of course not contained in E), we can define a semi-norm $\bar{\rho}_B$ on $\{E, \xi\} = \bar{E}$, such that

(2.5) $\bar{\rho}_B(x - \xi) = r_B(x)$ *when* $x \in E$,

3) $\mathfrak{L}$ is nonvoid.

4) Since l_α is T-continuous extension of l', $l_\alpha(T(B) \cap L_\alpha) \subset B$, the mapping T belongs to $\mathfrak{T}(\mathfrak{B}, l_\alpha)$.

5) See J. W. Tukey [5, p. 7].

6) The result in the first step is applied to arbitrary element of $\mathfrak{L}$.

(2.6) $\bar{B}\cap E=B$ *where* $\bar{B}=\{\bar{x};\ \bar{\rho}_B(\bar{x})\leqq 1,\ x\in\bar{E}\}$

and

(2.7) $\{\bar{E}, \bar{\xi}\}$ *is a locally convex topological vector space with the topology induced by the class* $\{\bar{\rho}_B;\ B\in\mathfrak{B}\}$ *and has* E *as a topological subspace of* $\{\bar{E}, \bar{\xi}\}$.

Proof. Define $\bar{\rho}_B$ by setting

$$\bar{\rho}_B(x+\lambda\xi)=\begin{cases}|\lambda|r_B\left(-\dfrac{x}{\lambda}\right) & \text{when} \quad \lambda\neq 0\ ,\\ \rho_B(x) & \text{when} \quad \lambda=0\ ,\end{cases}$$

where $x\in E$, $\lambda\in R$. We can prove that $\bar{\rho}_B$ is a semi-norm by the same methods used in the proof of Nachbin's Lemma 1 [4]. Thus the properties (2.5) and (2.6) immediately follow. From the property (2.4) we observe that the topology of $\bar{E}$ defined by $\{\bar{\rho}_B;\ B\in\mathfrak{B}\}$ satisfies the separation axiom, whence we have (2.7).

Lemma 3. *Let* ρ *be a semi-norm on* E *and* λ *be the function on* E *such that* $\lambda(x)+\lambda(y)\geqq\rho(x-y)$ *for every* $x, y\in E$. *Then there exists a function* r *on* E *with the properties (2.1), (2.2) and (2.3) and* $r(x)\leqq\lambda(x)$ *for every* $x\in E$.

As the proof is easily obtained following after that of Nachbin's Lemma 2 [4], it is omitted here.

We assume that $\mathfrak{B}$ has the extension property, that is, every continuous linear operator l defined on any subspace of any locally convex space always has a T-continuous extension for every mapping T of $\mathfrak{T}(\mathfrak{B}, l)$. We have to show that this fundamental base has the binary intersection property. Let $\mathfrak{U}$ be a collection of $\mathfrak{B}$-figures whose any two similar $\mathfrak{B}$-figures of $\mathfrak{U}$ have a common point. Let $\mathfrak{U}_B$ be the collection of $\mathfrak{B}$-figures similar to B in $\mathfrak{U}$. If $\mathfrak{U}_B$ is empty then let $\mathfrak{U}_B=\{B\}$[7]. Let A_B be the subset $\{x;\ \mu B+x\in\mathfrak{U}_B\}$ and ξ_B be a fixed element in A_B. We shall define a non-negative function λ_B on E as follows,

$$\lambda_B(x)=\begin{cases}\inf\{|\lambda|;\ \lambda B+x\in\mathfrak{U}_B\} & \text{when} \quad x\in A_B\ ,\\ \lambda_B(\xi_B)+\rho_B(x-\xi_B) & \text{when} \quad x\in E \text{ and } x\notin A_B\ ,\end{cases}$$

where ρ_B is the semi-norm defined by $\rho_B(x)=\inf\{|\lambda|;\ x\in\lambda B\}$. Now, we shall show that the function λ_B has the property (2.1), that is, for every $x, y\in E$, holds

$$x-y\in(\lambda_B(x)+\lambda_B(y))B$$

7) $\{B\}$ denotes the class of one element B.

the proof of which is established in three cases (i), (ii) and (iii). (i) both x and y are contained in A_B. By the definition of λ_B, for every $\varepsilon>0$, there exist numbers λ and μ such that $\lambda_B(x)+\varepsilon>\lambda\geqq\lambda_B(x)$, $\lambda_B(y)+\varepsilon>\mu\geqq\lambda_B(y)$ and $\lambda B+x$, $\mu B+y\in\mathfrak{U}_B$. Since every two elements of $\mathfrak{U}_B$ intersect, we have $x-y\in(\lambda+\mu)B\subset(\lambda_B(x)+\lambda_B(y)+2\varepsilon)B$ by Lemma 1. But as B is closed, we have $x-y\in(\lambda_B(x)+\lambda_B(y))B$. (ii) both x and y are not contained in A_B. By the definition of λ_B, $\lambda_B(z)\geqq\rho_B(z-\xi_B)$ when $z\in E$ and $z\notin A_B$. Hence $\xi_B-z\in\lambda_B(z)B$. Therefore $\lambda_B(x)B+x$ and $\lambda_B(y)B+y$ have the common point ξ_B. By Lemma 1, we have $x-y\in(\lambda_B(x)+\lambda_B(y))B$. (iii) x is contained in A_B and y is not. Since $x-y=(x-\xi_B)+(\xi_B-y)$ and x, $\xi_B\in A_B$ and by (i), $x-y$ belongs to $(\lambda_B(x)+\lambda_B(\xi_B))B+\rho_B(\xi_B-y)B$. Using the convexity of B, we have $x-y\in(\lambda_B(x)+(\lambda_B(\xi_B)+\rho_B(\xi_B-y))B$ $=(\lambda_B(x)+\lambda_B(y))B$. Thus we have proved that λ_B has always the property (2.1).

Next, let us consider the collection $\mathfrak{U}_0$ of $\mathfrak{B}$-figures represented by $\lambda_B(x)B+x$ where $B\in\mathfrak{B}$, $x\in E$. If the collection $\mathfrak{U}_0$ has a nonvoid intersection then $\mathfrak{U}$ has also. So we have only to show that $\mathfrak{U}_0$ has a nonvoid intersection. By Lemma 3, there exists a non-negative function r_B with the properties (2.1)—(2.3) satisfying $\lambda_B\geqq r_B$ for each $B\in\mathfrak{B}$. If, on one hand, $\{r_B;\ B\in\mathfrak{B}\}$ has not the property (2.4), that is, there is an element x_0 of E such that $r_B(x_0)=0$ for all r_B, then by x_0-x $\in\rho_B(x_0-x)B\subset(r_B(x_0)+r_B(x))B=r_B(x)B$, x_0 is easily seen to be a common point of $r_B(x)B+x$ for all $x\in E$ and $B\in\mathfrak{B}$. Since $r_B(x)B+x\subset\lambda_B(x)B+x$ holds for all $x\in E$ and $B\in\mathfrak{B}$, $\mathfrak{U}_0$ has a nonvoid intersection. On the other hand, if $\{r_B;\ B\in\mathfrak{B}\}$ has the property (2.4) then we shall be able to make use of Lemma 2. That is, if ζ is an abstract element, $\zeta\notin E$, then we can consider a locally convex space $\{E,\zeta\}$ with the topology induced by the class of the semi-norms having the properties (2.5) and (2.6) and has E as a topological subspace of $\{E,\zeta\}$. Now we shall make use of the hypothesis that $\mathfrak{B}$ has the extension property. Let e be the identity mapping on E and T the mapping on $\mathfrak{B}$ to the neighbourhood system in $\{E,\zeta\}$ defined by $T(B)=\overline{B}$. The mapping e is a continuous linear operator on a subspace E of a locally convex space $\{E,\zeta\}$ and, by the property (2.6) T is contained in $\mathfrak{T}(\mathfrak{B},e)$. Then we have a T-continuous extension $\bar{e}$ on $\{E,\zeta\}$. Hence, $\bar{e}(\bar{x})\in B$ when $\bar{x}\in T(B)=\overline{B}$, and $\bar{e}(-x+\zeta)\in\bar{\rho}_B(-x+\zeta)B$. If $x\in E$ then $\bar{\rho}_B(-x+\zeta)=r_B(x)$ by the property (2.5) and $\bar{e}(-x+\zeta)=\bar{e}(\zeta)-x$. Therefore, $\bar{e}(\zeta)$ belongs to all the $\mathfrak{B}$-figures $r_B(x)B+x$ for $x\in E$ and $B\in\mathfrak{B}$. Since $\lambda_B(x)B+x$ contains $r_B(x)B+x$, all the $\mathfrak{B}$-figures in $\mathfrak{U}_0$ contain the element $\bar{e}(\zeta)$ of E. Thus the proof of the theorem is completed.

According to the process in the proof of Theorem 1, the following corollary is immediately obtained.

Corollary 1. *A fundamental base $\mathfrak{B}$ has the extension property if and only if the fundamental base $\mathfrak{B}$ has binary intersection property.*

Specially, let E be a normed space and $\mathfrak{B}$ the fundamental base consisting of the unit sphere. In this case Corollary 1 is equivalent to the result of L. Nachbin, cited at the beginning of this paper, containing Hahn-Banach's extension theorem[8] as a special case.

4. Characterization of a locally convex space with the extension property. To each index $\alpha \in A$ let E_α be a locally convex space. The Cartesian product ΠE_α is said to be a *topological product* of E_α, if its topology is defined by taking, as neighbourhood system in ΠE_α, the collection of all sets $U = \Pi U_\alpha$ where each U_α is a neighbourhood in E_α and where $U_\alpha = E_\alpha$ except for a finite set of indices α. The space ΠE_α is known to be locally convex.

Lemma 4. *If E_α has the extension property then the topological product ΠE_α has the extension property.*

Proof. To each index $\alpha \in A$ there exists by Theorem 1 a fundamental base $\mathfrak{B}_\alpha$ in E_α with the binary intersection property. Let $\mathfrak{B}$ be the collection of neighbourhoods B in ΠE_α such that $B = \Pi B_\alpha$ where $B_\alpha = E_\alpha$ except for an arbitrary index α' in A and where $B_{\alpha'}$ is an element of $\mathfrak{B}_{\alpha'}$. It is easy to see that $\mathfrak{B}$ is a fundamental base in ΠE_α with the binary intersection property. From Theorem 1 ΠE_α has the extension property.

By Lemma 4, it follows immediately that if a locally convex space E is equivalent as a topological vector space to the topological product of normed spaces with the extension property then E has also the property. Is the converse true? The question is answered affirmatively. That is, if E has the extension property then E is topologically equivalent to the product space of normed spaces with the extension property. In order to prove this, we shall first prove some lemmas.

A locally convex space E is said to be *complete*, if every Cauchy's directed family in E converges to an element of E[9].

Lemma 5. *A locally convex space E with the extension property is complete.*

Proof. Let $\widetilde{E}$ be a complete locally convex space containing E as a dense subspace and let $\{x_\alpha\}$ be a Cauchy's directed family in E. In general, the existence of completion $\widetilde{E}$ of E is well known[10]. The

8) S. Banach [1].

9) See J. W. Tukey [5].

10) See J. W. Tukey [5, pp. 65–70].

family $\{x_\alpha\}$ converges to $\tilde{x}$ of $\widetilde{E}$. By the hypothesis, the identity mapping e on E has a continuous extension $\bar{e}$. Therefore, $\bar{e}(x_\alpha)$ converges to $\bar{e}(\tilde{x})$. Since $\bar{e}(x_\alpha)=e(x_\alpha)=x_\alpha$ and $\bar{e}(\tilde{x})\in E$, by the uniquenses of a limit, we obtain that $\tilde{x}=\bar{e}(\tilde{x})$ is contained in E. Thus Cauchy's directed family $\{x_\alpha\}$ converges to $\bar{e}(\tilde{x})$ of E.

Let E be a locally convex space and $\mathfrak{B}$ a fundamental base in E. The following facts are generally known. Fix an arbitrary B of $\mathfrak{B}$. Let ρ_B be the semi-norm defined by setting $\rho_B(x)=\inf\{|\lambda|;\, x\in\lambda B\}$ and let O_B be the set $\{x;\, \rho_B(x)=0\}$. Since O_B is a subspace, we can consider the factor space E/O_B that is the set of cosets of O_B which is the collection of all sets of the form $x+O_B$, where $x\in E$. Let π_B be the natural mapping on E to E/O_B defined by $\pi_B(x)=x+O_B$. The algebraic operations in E/O_B is defined by the equations $\pi_B(x)+\pi_B(y)=\pi_B(x+y)$ and $\lambda\pi_B(x)=\pi_B(\lambda x)$, and the norm $\|\ \|_B$ in E/O_B is defined by putting $\|\pi_B(x)\|_B=\rho_B(x)$. Thus the factor space E/O_B becomes normed space. Let us denote such normed space E/O_B by N_B. Now, consider the topological product ΠN_B of normed spaces N_B for all $B\in\mathfrak{B}$. Then, there exists a linear homeomorphic mapping ψ from E onto the subspace $\psi(E)$ of ΠN_B. The mapping ψ is defined by $\psi(x)=\{\pi_B(x)\}\in\Pi N_B$. In general $\psi(E)$, is not necessarily equal to ΠN_B.

Lemma 6. *If E has the extension property then, for some fundamental base $\mathfrak{B}$ in E, the normed space N_B also has the extension property.*

Proof. Let $\mathfrak{B}$ be the fundamental base in E with the binary intersection property and $\mathfrak{B}_B$ be the class of the unit sphere in N_B. It is easy to see that, by the binary intersection property of $\mathfrak{B}$, $\mathfrak{B}_B$ is a fundamental base in N_B with the binary intersection property. Hence N_B has the extension property by Theorem 1.

Let a locally convex space E have the extension property. By Lemma 6, there exists a fundamental base $\mathfrak{B}$ in E such that, to each $B\in\mathfrak{B}$, N_B has the extension property. By Lemma 5, E is complete and since ψ is a linear homeomorphic mapping, $\psi(E)$ is also complete. Therefore, to prove $\psi(E)=\Pi N_B$ we need only to show that $\psi(E)$ is dense in ΠN_B. Let $\{x_B\}$ be an element of ΠN_B and U be a neighbourhood in ΠN_B. Our purpose is to show that there exists an element x of E such that $\psi(x)\in U+\{x_B\}$. The neighbourhood U is the set ΠU_B where U_{B_j} is a set $\{x'_{B_j};\, \|x'_{B_j}\|_{B_j}\leqq\lambda_j,\ \lambda_j\in R\}$ for $B_j\in\mathfrak{B}$, $j=1, 2, \cdots, n$ and $U_B=N_B$ except for $B=B_j$, $j=1, 2, \cdots, n$. On the other hand, since $\mathfrak{B}$ has the binary intersection property and the collection $\{\lambda_j B_j + y_{B_j};\ \pi_{B_j}(y_{B_j})=x_{B_j},\ \lambda_j\in R,\ j=1, 2, \cdots, n\}$ is the set of $\mathfrak{B}$-figures, any two of which are not similar to each other, all $\mathfrak{B}$-figures $\lambda_j B_j+y_{B_j}$ have a common point x of E. Hence we have $\pi_{B_j}(x)\in\pi_{B_j}(\lambda_j B_j+y_{B_j})=\lambda_j\pi_{B_j}(B_j)$

$+x_{B_j}=U_{B_j}+x_{B_j}$ for all $j=1, 2, \cdots, n$. This shows that $x \in U+\{x_B\}$. Thus we have proved the following

Theorem 2. *A locally convex space with the extension property is equivalent, as a topological vector space, to the topological product of normed spaces with the extension property.*

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