Natural Science Report, Ochanomizu University, Vol. 10, No. 2, 1959
お茶の水女子大学 自然科学報告 第10巻 第2号

On Infinitesimal Holomorphically Projective Transformations in Certain Almost-Hermitian Spaces

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Recently we have defined an almost-Hermitian space which is a generalization of a Kählerian space and called it a K-space [4][1]. A K-space is characterized by the fact that the associated tensor field φ_{ji} is a Killing tensor. An example of the non-Kählerian K-space is given by a six-dimensional sphere S^6 by virtue of the structure defined by Fukami, T. and S. Ishihara [1].

On the other hand, we have discussed in detail infinitesimal holomorphically projective transformations in Kählerian spaces [5]. In an almost-complex space, such a transformation has been defined in the case when the affine connection under consideration is a φ-connection. But in a K-space such transformations are defined naturally in terms of the Riemannian connection which is not necessarily a φ-connection.

In this paper we shall generalize some results in [5] to K-spaces and, in the last section, see a fact that on the S^6 as a K-space an analytic infinitesimal holomorphically projective transformation is necessarily an isometry.

§ 1. **K-spaces.** In an n dimensional space, an almost-complex structure is defined by assigning to the space a tensor field $\varphi_i^{\ h}$ such that $\varphi_i^{\ r}\varphi_r^{\ j}=-\delta_i^j$.[2] Then an almost-complex space, i. e. a space with an almost-complex structure, is necessarily of even dimension and orientable.

The tensor $N_{ji}^{\ \ h}$ defined by

$$N_{ji}^{\ \ h}=\varphi_j^{\ r}(\partial_r\varphi_i^{\ h}-\partial_i\varphi_r^{\ h})-\varphi_i^{\ r}(\partial_r\varphi_j^{\ h}-\partial_j\varphi_r^{\ h}), \qquad \partial_j=\partial/\partial x^j,$$

is called the Nijenhuis' tensor of the almost-complex structure $\varphi_i^{\ h}$.

1) The number in brackets refers to the Bibliography at the end of the paper.

2) As to the notations we follow Tachibana, S. [4]. We shall express any quantities in terms of their components with respect to natural reperes $\partial/\partial x^i$, where x^i denotes local coordinates. Indices run over $1, 2, \cdots, n=2m$.

A vector field v^i is called almost-analytic or analytic [4] if it satisfies

$$\underset{v}{\pounds}\varphi_j{}^i = v^r\partial_r\varphi_j{}^i - \varphi_j{}^r\partial_r v^i + \varphi_r{}^i\partial_j v^r = 0 ,$$

where $\underset{v}{\pounds}$ denotes the operator of Lie derivation with respect to v^i.

If a Riemannian metric (positive definite) tensor g_{ji} satisfies $g_{rs}\varphi_j{}^r\varphi_i{}^s = g_{ji}$, then it is called that the pair $(g_{ji}, \varphi_i{}^k)$ assigns an almost-Hermitian structure to the space. A Kählerian structure $(g_{ji}, \varphi_i{}^h)$ is an almost-Hermitian one such that $\nabla_j\varphi_i{}^h = 0$. In this case the tensor field $\varphi_{ji} = \varphi_j{}^r g_{ri}$ is harmonic and at the same time a Killing tensor. An almost-Kählerian structure is an almost-Hermitian one such that φ_{ji} is harmonic. Hence an almost-Kählerian structure is a generalization of a Kählerian one.

An almost-Hermitian space is called a K-space [4] if φ_{ji} is a Killing tensor, i. e. it satisfies the equation

$$\nabla_j\varphi_{ih} + \nabla_i\varphi_{jh} = 0 , \tag{1.1}$$

where ∇_j denotes the operator of the Riemannian covariant derivation. A K-space is another generalization of a Kählerian space.

In this paper we shall be only concerned about K-spaces and assume that $n = 2m > 2$.

§ 2. **Preliminaries.** Let us consider a K-space whose structure is given by $(g_{ji}, \varphi_i{}^j)$. We denote the Riemannian curvature tensor by $R_{kji}{}^h$ and put

$$R_{ji} = R_{rji}{}^r , \qquad R_{kjih} = R_{kji}{}^r g_{rh} , \qquad R^*_{ji} = (1/2)\varphi^{rs}R_{rtsi}\varphi_j{}^s ,$$

then the following identities are known [4]

$$R_{rs}\varphi_j{}^r\varphi_i{}^s = R_{ji} , \tag{2.1}$$

$$R^*_{rs}\varphi_j{}^r\varphi_i{}^s = R^*_{ji} , \qquad R^*_{ji} = R^*_{ij} . \tag{2.2}$$

The Ricci's identity for a tensor $a_i{}^h$ is given by

$$\nabla_k\nabla_j a_i{}^h - \nabla_j\nabla_k a_i{}^h = R_{kjr}{}^h a_i{}^r - R_{kji}{}^r a_r{}^h .$$

If we put

$$t_{ji}{}^h \equiv \underset{v}{\pounds}\left\{{}_j{}^h{}_i\right\} = \nabla_j\nabla_i v^h + R_{rji}{}^h v^r \tag{2.3}$$

for a vector, the following identities are valid [7]

$$\underset{v}{\pounds}\nabla_j\varphi_i{}^h - \nabla_j\underset{v}{\pounds}\varphi_i{}^h = t_{jr}{}^h\varphi_i{}^r - t_{ji}{}^r\varphi_r{}^h , \tag{2.4}$$

$$\underset{v}{\pounds}R_{kji}{}^h = \nabla_k t_{ji}{}^h - \nabla_j t_{ki}{}^h . \tag{2.5}$$

For a vector field v^i we shall define $N(v)_i$ by

$$N(v)_i=(1/4)\, N_{tri}\nabla^t v^r\,,$$

where we have put $N_{tri}=N_{tr}{}^h g_{hi}$ and $\nabla^t=g^{ts}\nabla_s$. Since our space is a K-space, it obtains easily that

$$N(v)_i=\varphi_i{}^r(\nabla_r\varphi_{ts})\,\nabla^t v^s\,. \tag{2.6}$$

In terms of these quantities, the following theorem is known [4].

Theorem. *In a compact K-space, a necessary and sufficient condition for a vector field v^i to be analytic is that*

$$\nabla^r\nabla_r v^i+R_r{}^i v^r=0\,, \tag{2.7}$$

$$(R_{ir}-R^*_{ir})\, v^r+2N(v)_i=0\,. \tag{2.8}$$

A vector field v^i is called an infinitestimal holomorphically projective transformation or, for simplicity, *HP*-transformation, if it satisfies

$$t_{ji}{}^h\equiv \underset{v}{\pounds}\{{}^{\,h}_{j\,i}\}=\rho_j\delta_i^h+\rho_i\delta_j^h-\tilde\rho_j\varphi_i{}^h-\tilde\rho_i\varphi_j{}^h\,, \tag{2.9}$$

where ρ_i is a vector and $\tilde\rho_i=\varphi_i{}^r\rho_r$. We shall call ρ_i in (2.9) the associated vector of the *HP*-transformation.

Contracting (2.9) with respect to i and h, we get

$$\rho_i=\{1/(n+2)\}\,\nabla_i\nabla_r v^r\,, \tag{2.10}$$

which shows that ρ_i is gradient.

In particular, if $\rho_i=0$, then the *HP*-transformation is an infinitesimal affine transformation.

§ 3. Holomorphically planar curves.

In a K-space, let us consider a curve $x^i=x^i(t)$ which satisfies the following differential equation

$$\frac{d^2x^h}{dt^2}+\{{}^{\,h}_{j\,i}\}\frac{dx^j}{dt}\frac{dx^i}{dt}=\alpha\frac{dx^h}{dt}+\beta\varphi_j{}^h\frac{dx^j}{dt}\,, \tag{3.1}$$

where α and β are certain functions of the parameter t.

We shall call such a curve a holomorphically planar curve. The curve is characterized by the fact that the field of planes spanned by $\dot x^i=dx^i/dt$ and $\varphi_j{}^i\dot x^j$ are parallel along the curve. If we take the arc length s as the parameter, (3.1) becomes the following form

$$\frac{d^2x^h}{ds^2}+\{{}^{\,h}_{j\,i}\}\frac{dx^j}{ds}\frac{dx^i}{ds}=\gamma(s)\,\varphi_j{}^h\frac{dx^j}{ds}\,.$$

Hence we have another characterization that the first normal of the curve is $\pm\varphi_j{}^i(dx^j/ds)$.

We now ask for the condition that an infinitesimal point transformation $x^i\rightarrow x^i+\varepsilon v^i$ transforms any holomorphically planar curve into such a curve, ε being an infinitesimal constant. It is easily seen that a necessary and sufficient condition for a vector field v^i to be such a transformation is that the equations

$$\dot{x}^j \underset{v}{\pounds} \varphi_j{}^i = a\dot{x}^i + b\varphi_j{}^i\dot{x}^j , \tag{3.2}$$

$$\dot{x}^j\dot{x}^i t_{ji}{}^h = p\dot{x}^h + q\varphi_j{}^h\dot{x}^j \tag{3.3}$$

are valid for any direction $\dot{x}^i$, where a, b, p and q are some functions of x^i and $\dot{x}^i$.

Now let v^i be such a transformation, then from (3.2) it follows

$$\underset{v}{\pounds} \varphi_j{}^i = 0 , \tag{3.4}$$

by virtue of Lemma 1 in [5, Appendix I].

Next from (3.3) and Lemma 3 in [5, Appendix I], we have

$$t_{ji}{}^h = \rho_j\delta_i^h + \rho_i\delta_j^h + \sigma_j\varphi_i{}^h + \sigma_i\varphi_j{}^h , \tag{3.5}$$

where ρ_i and σ_i are vectors.

On the other hand, if we substitute (3.4) into (2.4), then we get

$$\underset{v}{\pounds} \nabla_j\varphi_i{}^h = t_{jr}{}^h\varphi_i{}^r - t_{ji}{}^r\varphi_r{}^h ,$$

from which we find that $g^{ji}t_{ji}{}^h = 0$.

Substituting (3.5) into the last equation, we obtain $\sigma_j = -\tilde{\rho}_j$. Thus we have

$$t_{ji}{}^h = \rho_j\delta_i^h + \rho_j\delta_i^h - \tilde{\rho}_j\varphi_i{}^h - \tilde{\rho}_i\varphi_j{}^h .$$

Consequently the vector v^i is analytic and at the same time an *HP*-transformation. Since the converse is obvious, we obtain the following

Theorem 1. *In a K-space, in order that an infinitesimal transformation carries any holomorphically planar curve into such a curve, it is necessary and sufficient that it is an analytic HP-transformation.*

Consider an *HP*-transformation v^i, then from (2.9) we have

$$\nabla_j\nabla_i v^h + R_{rji}{}^h v^r = \rho_j\delta_i^h + \rho_i\delta_j^h - \tilde{\rho}_j\varphi_i{}^h - \tilde{\rho}_i\varphi_j{}^h .$$

Transvecting this with g^{ji}, we find that

$$\nabla^r\nabla_r v^h + R_r{}^h v^r = 0 ,$$

from which and the well known theorem [7] we have $\int_M (R_{ji}v^jv^i)d\sigma \geqq 0$ for an *HP*-transformation provided that the *K*-space M under consideration is compact, where $d\sigma$ means the volume element of M. The equality holds when and only when v^i is parallel.

§ 4. The associated vector of an analytic *HP*-transformation.

Let v^i be an analytic *HP*-transformation, then it holds that

$$\underset{v}{\pounds}\, \varphi_j{}^i = 0\,, \tag{4.1}$$

$$\underset{v}{\pounds}\, \{{}_j{}^h{}_i\} = \rho_j\delta_i^h + \rho_i\delta_j^h - \tilde{\rho}_j\varphi_i{}^h - \tilde{\rho}_i\varphi_j{}^h\,. \tag{4.2}$$

Substituting (4.2) into (2.5), we get

$$\underset{v}{\pounds}\, R_{kji}{}^h = \delta_j^k\nabla_k\rho_i - \delta_k^h\nabla_j\rho_i - \varphi_j{}^h\nabla_k\tilde{\rho}_i + \varphi_k{}^h\nabla_j\tilde{\rho}_i - \varphi_i{}^h(\nabla_k\tilde{\rho}_j - \nabla_j\tilde{\rho}_k)$$
$$- \tilde{\rho}_j\nabla_k\varphi_i{}^h + \tilde{\rho}_k\nabla_j\varphi_i{}^h - 2\tilde{\rho}_i\nabla_k\varphi_j{}^h\,.$$

If we contract this equation with respect to h and k and take account of (1.1), $\nabla_j\rho_i = \nabla_i\rho_j$, $\varphi_r{}^r = 0$ and $\nabla_r\varphi_i{}^r = 0$, then we have

$$\underset{v}{\pounds}\, R_{ji} = -n\nabla_j\rho_i - 2\varphi_j{}^r\varphi_i{}^t\nabla_r\rho_t\,. \tag{4.3}$$

Operating $\underset{v}{\pounds}$ to (2.1) and making use of (4.1) and (4.3), then we find

$$\underset{v}{\pounds}\, R_{ji} = -n\varphi_j{}^r\varphi_i{}^t\nabla_r\rho_t - 2\nabla_j\rho_i\,.$$

Comparing (4.3) with the last equation, we obtain

$$\nabla_j\rho_i = \varphi_j{}^r\varphi_i{}^t\nabla_r\rho_t\,. \tag{4.4}$$

Hence by virtue of (1.1) and (4.4) it holds that

$$\nabla_j\tilde{\rho}_i + \nabla_i\tilde{\rho}_j = (\nabla_j\varphi_i{}^t + \nabla_i\varphi_j{}^t)\rho_t + \varphi_i{}^t(\nabla_j\rho_t - \varphi_j{}^r\varphi_t{}^s\nabla_r\rho_s) = 0\,,$$

which shows that $\tilde{\rho}^i$ is a Killing vector. Thus we have

Theorem 2. *If ρ_i is the associated vector of an analytic HP-transformation, then $\tilde{\rho}^i$ is a Killing vector i.e. an infinitesimal isometry.*

From (4.3) and (4.4) we have

$$\underset{v}{\pounds}\, R_{ji} = -(n+2)\,\nabla_j\rho_i\,. \tag{4.5}$$

§ 5. An Einstein *K*-space.

In this section we shall always consider an Einstein *K*-space with non-vanishing scalar curvature R. Let v^i be an analytic *HP*-transformation, then the equation (4.5) holds. Since we have $R_{ji} = (R/n)\,g_{ji}$ and $\underset{v}{\pounds}\, g_{ji} = \nabla_j v_i + \nabla_i v_j$, it holds that

$$\nabla_j\{v_i - (1/2k)\rho_i\} + \nabla_i\{v_j - (1/2k)\rho_j\} = 0\,,$$

where we have put $k = -R/n(n+2)$. Therefore if we define p_i by $v_i = p_i + (1/2k)\rho_i$, then p^i is a Killing vector. Now if we put $q_i = (1/2k)\tilde{\rho}_i$, then q^i is also a Killing vector, because of Theorem 2. Thus we can obtain easily the following

Theorem 3. *In an Einstein K-space with $R \neq 0$, an analytic HP-transformation v^i is decomposed into the form*

$$v^i = p^i + \varphi_r{}^i q^r , \tag{5.1}$$

where p^i and q^i are both Killing vectors and $\varphi_i{}^r q_r$ is gradient. The decomposition stated above is unique.

From (5.1) we get

$$\underset{v}{\mathfrak{L}}\{{}_j{}^h{}_i\} = \underset{p}{\mathfrak{L}}\{{}_j{}^h{}_i\} - \underset{\tilde{q}}{\mathfrak{L}}\{{}_j{}^h{}_i\} = (1/2k)\underset{\rho}{\mathfrak{L}}\{{}_j{}^h{}_i\} .$$

Substituting the last equation into (4.2), we find

$$\nabla_j \nabla_i \rho^h + R_{rji}{}^h \rho^r = 2k(\rho_j \delta_i^h + \rho_i \delta_j^h - \tilde{\rho}_j \varphi_i{}^h - \tilde{\rho}_i \varphi_j{}^h) , \tag{5.2}$$

from which we obtain

Theorem 4. *In an Einstein K-space with $R \neq 0$, the associated vector of an analytic HP-transformation is an HP-transformation.*

Now let $x^i = x^i(s)$ be a geodesic such that $\rho_i(dx^i/ds) \neq 0$ at a point on it, where s is the arc length. If we define a function $f(s) = \rho_i(dx^i/ds)$ along the geodesic, then it follows that $f''(s) = 4kf(s)$ by virtue of (5.2). If $R < 0$, it holds that $f(s) = A \exp(2\sqrt{k}\, s) + B \exp(-2\sqrt{k}\, s)$, where A and B are constant. Hence we have

Theorem 5. *In a complete Einstein K-space with $R < 0$, the length of the associated vector of an analytic HP-transformation can not be bounded.*

From (5.2) we have

$$\nabla_j \nabla_i \rho_h + R_{rjih} \rho^r = 2k(\rho_j g_{ih} + \rho_i g_{jh} - \tilde{\rho}_j \varphi_{ih} - \tilde{\rho}_i \varphi_{jh}) . \tag{5.3}$$

Taking the symmetric and alternating parts of (5.3) with respect to i and h respectively, it holds that

$$\nabla_j \nabla_i \rho_h = k(\rho_i g_{jh} + \rho_h g_{ji} - \tilde{\rho}_i \varphi_{jh} - \tilde{\rho}_h \varphi_{ji} + 2\rho_j g_{ih}) , \tag{5.4}$$

$$R_{rjih} \rho^r = k(\rho_i g_{jh} - \rho_h g_{ji} - \tilde{\rho}_i \varphi_{jh} + \tilde{\rho}_h \varphi_{ji} - 2\tilde{\rho}_j \varphi_{ih}) . \tag{5.5}$$

If we transvect (5.4) with g^{ji}, we find

$$\nabla^r \nabla_r \rho_h + (R/n)\rho_h = 0 . \tag{5.6}$$

On the other hand, if we transvect (5.5) with $\varphi^{hi}\varphi_p{}^j$, then it follows that

$$R^*_{pi} \rho^i = (R/n)\rho_p . \tag{5.7}$$

Since ρ_i is gradient, we have $N(\rho)_i = 0$, from which and (5.7) it is seen that the equation (2.8) in Theorem in § 2 is valid for ρ_i. As (5.6) is nothing but (2.7) for ρ_i, we have

Theorem 6. *In a compact Einstein K-space with $R \neq 0$, the associated vector of an analytic HP-transformation is also an analytic HP-transformation.*

The equation (5.5) is also written in the following form.

$$R_{rjih}\rho^r = k(g_{ri}g_{jh} - g_{ji}g_{rh} + \varphi_{ri}\varphi_{jh} - \varphi_{ji}\varphi_{rh} + 2\varphi_{rj}\varphi_{ih})\rho^r .$$

Hence applying Lemma in [5, Appendix II], we find

Theorem 7. *If an Einstein K-space with $R \neq 0$ admits an analytic non-affine HP-transformation, then the restricted homogeneous holonomy group contains the full unitary group $U(n/2)$.*

§ 6. **Spaces of constant curvature.** Consider a K-space of constant curvature with $R \neq 0$. Then the curvature tensor takes the form

$$R_{kjih} = a(g_{ki}g_{jh} - g_{ji}g_{kh}) , \qquad a = -R/n(n-1) .$$

Hence we have

$$R^*_{ji} = -ag_{ji} . \tag{6.1}$$

Now let v^i be an analytic HP-transformation and ρ_i its associated vector, then (5.7) is valid. If we substitute (6.1) into (5.7), then we get $(n-2)R\rho_i/n(n-1) = 0$, from which we have $\rho_i = 0$.

On the other hand, it is known [4] that there does not exist a K-space of negative constant curvature. Thus we have

Theorem 8. *In a K-space of positive constant curvature, an analytic HP-transformation is necessarily affine.*

Corollary. *In a compact K-space of positive constant curvature, an analytic HP-transformation is necessarily an isometry.*

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(*Received Sept. 1, 1959*)