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A Remark on Linear Connections with some Properties

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§ 1. Let B and G be an n-dimensional differentiable manifold[1] and a general linear group $GL(n, R)$ respectively. By $P(B, \pi, G)$ we shall denote the principal fibre bundle consisting of all n-frames over B, where P is the bundle space and π is the natural projection. An element $x \in P$ is represented by a form (u, X_i),[2] where u means a point of B and X_i an n-frame at u. Let F be an n-dimensional vector space and ξ_i its fixed base. Then $x=(u, X_i)$ is considered as a linear mapping such that

$$x: F \rightarrow T_u(B) = \text{the tangent vector space at } u,$$

$$x(\xi_i) = X_i = X_i^a \frac{\partial}{\partial u^a}. \quad {}^{3)}$$

A linear connection in P or on B is given by a field $\mathfrak{H}$ of horizontal planes or equivalently a connection form ω.[4] Let A_i^k be a base of the Lie algebra $\mathfrak{G}$ of G such that $A_i^k \cdot \xi_j = \delta_i^k \xi_j$. Then a connection form ω is given by the following equation:

$$\omega = \omega_k^i A_i^k,$$

$$\omega_k^i = Y_a^i (dX_k^a + \Gamma_b{}^a{}_c X_k^c du^b),$$

where Y is the inverse matrix of (X_i^a) and Γ[5] is coefficients of the connection form ω.

The field $\mathfrak{H}$ is spanned by n basic vectors

$$B(\xi_i) = X_i^a \left(\frac{\partial}{\partial u^a} - \Gamma_a{}^b{}_c X_j^c \frac{\partial}{\partial X_j^b} \right).$$

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1) By differentiability we shall always mean that class C^∞. As to the notations, we follow K. Nomizu [6].

2) The indicies $i, j, k, \ldots$ run over $1, 2, \ldots, n$.

3) The indicies $a, b, c, \ldots$ run over $1, 2, \ldots, n$ and by these we denote components with respect to a natural repere $\partial/\partial u^a$, where u^a is a local coordinate.

4) Cf. K. Nomizu [6]. By $\mathfrak{H}$ we shall always mean the field of horizontal planes of a connection form ω. Hence we shall say that "connection ω" or equivalently "connection $\mathfrak{H}$".

5) For simplicity we shall denote φ resp. Γ instead of $\varphi_a{}^b$ resp. $\Gamma_b{}^a{}_c$.

A one to one mapping $f: P \rightarrow P$ is called an automorphism of P, if it satisfies the following two conditions.

(i) There exists an automorphism σ of G such that

$$f(x \cdot a) = f(x) \cdot \sigma(a), \qquad x \in P, a \in G.$$

(ii) $$\pi \circ f(x) = \pi(x).$$

Let f be an automorphism of P and σ the corresponding one of G. For a given connection $\mathfrak{H}$ we shall define a new linear connection $\bar{\mathfrak{H}}$ by $\bar{\mathfrak{H}} = f^{-1}\mathfrak{H}$.[6] The connection form $\bar{\omega}$ corresponding to $\bar{\mathfrak{H}}$ satisfies $\sigma \circ \bar{\omega} = f^* \omega$, in which f^* means the transposed operator of f.[7]

Let φ be a regular tensor field of type (1,1) on B and $\varphi_a{}^b$ its components. Then the matrix $(\varphi_a{}^b)$ has the inverse, say $(\psi_a{}^b)$. If we put, for an n-frame X_i,

$$X_i' = X'_i{}^a \frac{\partial}{\partial u^a}, \quad \text{where} \quad X'_i{}^a = \varphi_b{}^a X_i{}^b,$$

the mapping

$$f_\varphi : x = (u, X_i) \rightarrow x' = (u, X_i')$$

is an automorphism of P. We call $\Phi_\varphi \equiv f_\varphi^{-1}$ the associated map of φ. Consider a connection $\mathfrak{H}$ and introduce a new connection $\bar{\mathfrak{H}}$ by f_φ in the sence of the preceeding argument. Then we have $\Phi_\varphi \mathfrak{H} = \bar{\mathfrak{H}}$. The coefficients $\bar{\Gamma}$ of the linear connection $\bar{\mathfrak{H}}$ are given by

$$\bar{\Gamma}_b{}^a{}_c = \Gamma_b{}^a{}_c + \psi_e{}^a \nabla_b \varphi_c{}^e,^{8)}$$

where Γ denotes the coefficients of $\mathfrak{H}$ and

$$\nabla_b \varphi_c{}^e = \partial_b \varphi_c{}^e + \varphi_c{}^a \Gamma_b{}^e{}_a - \varphi_a{}^e \Gamma_b{}^a{}_c, \quad \partial_c \equiv \partial / \partial u^c.$$

A linear connection $\mathfrak{H}$ is called a φ-connection if the tensor field φ is covariantly constant with respect to $\mathfrak{H}$. Thus we have easily

Theorem 1. *In order that a linear connection $\mathfrak{H}$ is a φ-connection, it is necessary and sufficient that the field $\mathfrak{H}$ of horizontal planes is invariant under the associated map of φ.*

§ 2. Let φ be a tensor field such that

(1) $$\varphi_{c_1}{}^a \varphi_{c_2}{}^{c_1} \cdots \varphi_b{}^{c_{p-1}} = \varepsilon \delta_b{}^a, \quad \varepsilon = \pm 1,$$

or by matrix notation, $\varphi^p = \varepsilon I$, where I denotes a unit matrix of order n. Making use of the associated map Φ_φ, we shall introduce linear connections in the following manner:

6) For a map f, we shall denote by the same letter f the differential map of f.
7) A. C. Allamigeon [1].
8) I. C. Gasparini [3].

$$\overset{0}{\mathfrak{H}}=\mathfrak{H}\,,\quad \overset{\alpha}{\mathfrak{H}}=\Phi_{\varphi}\overset{\alpha-1}{\mathfrak{H}}\,,\qquad \alpha=1,2,\cdots,p-1\,.$$

Denoting by $\overset{\alpha}{\omega}$ the corresponding connection form, we shall define ω_{φ} by

$$\omega_{\varphi}=\frac{1}{p}\sum_{\alpha=0}^{p-1}\overset{\alpha}{\omega}$$

and it call the r-mean connection with respect to φ.

Theorem 2. *Let φ be a tensor field of type* (1,1) *such that $\varphi^{p}=\varepsilon I$. Then, for any linear connection ω, the r-mean connection ω_{φ} with respect to φ is a φ-connection.*

In order to prove the last theorem, we shall prepar a well known lemma.[9] Let ω_{α} the linear connections whose field of horizontal planes is $\mathfrak{H}_{\alpha}$. We shall mean by

$$\frac{1}{p}\,(\mathfrak{H}_1\dot{+}\mathfrak{H}_2\dot{+}\cdots\dot{+}\mathfrak{H}_p) \tag{2}$$

the vector space which consists of vectors such that

$$X=\frac{1}{p}\sum_{\alpha=1}^{p}X_{\alpha},\ \text{where}\ \pi(X_{\alpha})=\pi(X_{\beta})\ \text{and}\ X_{\alpha}\in\mathfrak{H}_{\alpha}\ \text{for all}\ \alpha,\beta.$$

Lemma. *Let ω_{α}, $\alpha=1,\cdots,p$, be linear connections whose field of horizontal planes is $\mathfrak{H}_{\alpha}$. Then the linear connection $\overset{*}{\omega}=\frac{1}{p}\sum_{\alpha=1}^{p}\omega_{\alpha}$ has* (2) *as the field of horizontal planes.*

Proof of theorem 2. By virtue of the lemma the field of horizontal planes of ω_{φ} is given by

$$\overset{*}{\mathfrak{H}}=\frac{1}{p}\,(\overset{0}{\mathfrak{H}}\dot{+}\cdots\dot{+}\overset{p-1}{\mathfrak{H}})\,.$$

Hence

$$\Phi_{\varphi}\overset{*}{\mathfrak{H}}=\frac{1}{p}\,(\Phi_{\varphi}\overset{0}{\mathfrak{H}}\dot{+}\cdots\dot{+}\Phi_{\varphi}\overset{p-1}{\mathfrak{H}})=\overset{*}{\mathfrak{H}}\,,$$

from which and theorem 1 we see that ω_{φ} is a φ-connection. *q. e .d*

Next we shall give the coefficients $\overset{*}{\Gamma}$ of ω_{φ} explicitly. Let $\overset{\alpha}{\varphi}{}_{b}{}^{a}$ be components of a matrix φ^{α}, then inverse of φ^{α} is given by $\psi^{\alpha}=\varepsilon\varphi^{p-\alpha}$ and it holds that

$$\overset{*}{\Gamma}{}_{b}{}^{a}{}_{c}=\Gamma_{b}{}^{a}{}_{c}+\frac{\varepsilon}{p}\sum_{\alpha=0}^{p-1}\overset{p-\alpha}{\varphi}{}_{e}{}^{a}\nabla_{b}\overset{\alpha}{\varphi}{}_{c}{}^{e}\,.$$

9) K. Nomizu [6].

Let ω be an arbitrary φ-connection, θ another linear connection. Then there exists a tensor field T of type (1,2) such that $\theta=\omega+T$.

A necessary and sufficient condition in order that θ is also a φ-connection is that

$$T_{bc}{}^{a}=T_{bq}{}^{p}\varphi_{c}{}^{q}\psi_{p}{}^{a}\,, \tag{3}$$

or in the matrix notation, $T_b=\varphi\cdot T_b\cdot\psi$.

If T satisfies (3), we have

$$\varphi^{\alpha}\cdot T_b\cdot\psi^{\alpha}=T_b\,, \qquad \alpha=1,\cdots,p\,.$$

Consequently

$$T_b=S_b+\varphi\cdot S_b\cdot\psi+\cdots+\varphi^{p-1}\cdot S\cdot\psi^{p-1}, \tag{4}$$

where $pS_b=T_b$.

Conversely, if ω is a φ-connection and T is a tensor field of type (1,2) with form (4), then the linear connection θ defined by $\theta=\omega+T$ is also a φ-connection. Thus we have

Theorem 3.[10] *Let θ be a linear connection, ω a fixed linead connection and φ a tensor field satisfying $\varphi^p=\varepsilon I$. A necessary and sufficient condition in order that a linear connection θ is a φ-connection is that the tensor field $T=\theta-\omega_\varphi$ takes a form*

$$T_b=S_b+\varphi\cdot S_b\cdot\psi+\cdots+\varphi^{p-1}\cdot S_b\cdot\psi^{p-1}\,,$$

where S is a tensor field of type (1,2).

The general solution of φ-connections is also given in the following way. If θ is a φ-connection, then we have $\theta=\theta_\varphi$. Conversely, if θ is a form $\theta=(\omega+T)_\varphi$, where ω is a fixed linear connection and T is any tensor field of type (1,2), then θ is a φ-connection. Hence[11]

A necessary and sufficient condition in order that a linear connection θ is a φ-connection is that there exists a tensor field T such that $\theta=(\omega+T)_\varphi$, where ω is a fixed linear connection.

In (1), if $p=2$ and $\varepsilon=-1$, then φ is an almost-complex structure.[12] If $p=2$ and $\varepsilon=1$, then φ is a (real) almost-product structure.[13]

In the last place we remark a following fact. If a manifold B admits a tensor field satisfying (1) and satisfies the second axiom of countability, it also admits a positive definite Riemannian metric such

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10) Cf. M. Obata [7].
11) Cf. A. G. Walker [10].
12) A. Frölicher [2], M. Obata [7].
13) G. Legrand [5].

that $g_{ab}=g_{ce}\varphi_a{}^c\varphi_b{}^e$.

In fact, as B admits a positive definite Riemannian metric,[14] say a_{ab}, it is sufficient to define g_{ab} by

$$g_{ab}=\frac{1}{p}\,(a_{ab}+a_{ce}\varphi_a{}^c\varphi_b{}^e+\cdots+a_{ce}\overset{p-1}{\varphi_a}{}^c\overset{p-1}{\varphi_b}{}^e)^{15)}$$

§ 3. In this section we shall give another applications of theorem 1. Consider a manifold B which admits an almost-quarternion structure (ξ,η), that is, tensor fields $\xi_a{}^b$, $\eta_a{}^b$ such that

$$\xi_e{}^a\xi_b{}^e=\eta_e{}^a\eta_b{}^e=-\delta_b^a\,,\quad \xi_e{}^a\eta_b{}^e=-\eta_e{}^a\xi_b{}^e\,.^{16)}$$

A linear connection ω is called a (ξ,η)-connection if ξ and η are both covariantly constant with respect to it. In this case the tensor field ζ defined by $\zeta_a{}^b=\xi_e{}^a\eta_b{}^e$ is also covariantly constant. Then we have

Theorem 4.[17] *Let ω be a linear connection on B which admits an almost-quaternion structure (ξ,η). Then the connection $(\omega_\xi)_\eta$, which is the 2-mean connection with respect to η of the 2-mean connection of ω with respect to ξ, is a (ξ,η)-connection.*

In fact, the field of horizontal planes is given by

$$\mathfrak{H}=\frac{1}{4}\,(\mathfrak{H}\dotplus\Phi_\xi\mathfrak{H}\dotplus\Phi_\eta\mathfrak{H}\dotplus\Phi_\zeta\mathfrak{H}),$$

so the theorem is trivial.

The analogous arguments are applicable to the complete distribution in the sence of A. G. Walker.[18] Let a_ρ, $\rho=1,\cdots,m$, be tensor fields of type (1,1) such that

$$\sum_{\rho=1}^{m}a_\rho=I\,,\quad a_\rho^2=a_\rho\,,\quad a_\rho\cdot a_\sigma=a_\sigma\cdot a_\rho=0,\quad \text{if}\quad \rho\neq\sigma\,. \tag{5}$$

Now we define tensors α_ρ by $\alpha_\rho=2a_\rho-I$, then it follows that

$$\alpha_\rho^2=I\,,\quad \alpha_\rho\cdot\alpha_\sigma=\alpha_\sigma\cdot\alpha_\rho=I-2(a_\rho+a_\sigma)\,. \tag{6}$$

From (6) we see that each α_ρ defines an almost-product structure.[19] Let Φ_ρ be the associated map of α_ρ, ω_ρ the 2-mean connection of ω with respect to α_ρ and $\omega_{\rho\sigma}$ the 2-mean connection of ω_ρ with respect to α_σ etc., then we can easily obtain the following

Theorem 5. *Let a_ρ be tensor fields satisfying* (5). *Then for any linear connection ω, the linear connection $\omega_{1,2,\cdots,(m-1)}$ makes each a_ρ covariantly constant.*

14) K. Nomizu [6].
15) Cf. A. Frölicher [2], M. Obata [7].
16), 17) M. Obata [7].
18) A. G. Walker [10].
19) G. Legrand [5].

Making use of a regular tensor field of type (0,2), we can also define an automorphism of P.[20] If the tensor is a Riemannian metric one, the analogous methods are applicable to the discussion of metrical connections.[21]

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