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Some Results on Abelian Varieties[1)]

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The theory of divisors on an abelian variety over the field of complex numbers has been much developed by making use of the theory of theta functions. But, in the case of an abstract abelian variety, many problems are still left open. In the present paper we shall study some properties of non-degenerate divisors.

First the theorem of Riemann-Roch will be stated as follows: Let X be a positive non-degenerate divisor on an abelian variety A of dimension n. Then the dimension $l(X)$ of the complete linear system $|X|$ is equal to $(X^{(n)})/n!$ and also equal to $(-1)^{n+1}\chi_A(X)$, where $(X^{(n)})$ means the n-fold intersection number of X and $\chi_A(X)$ means the virtual arithmetic genus of X. In the next place let A and B be isogenous abelian varieties and let λ be a homomorphism from A onto B. If Y is a divisor on B, then two matrices $E_l(\lambda^{-1}(Y))$ and $E_l(Y)$ are combined by the relation $E_l(\lambda^{-}(Y))={}^tM_l(\lambda)\cdot E_l(Y)\cdot M_l(\lambda)$, where l is a prime number different from the characteristic of our geometry. This suggests to us that, if Y is positive non-degenerate, then $l(\lambda^{-1}(Y))$ is equal to $\nu(\lambda)\cdot l(Y)$. Actually a proof was given under an additional assumption in a recent paper [4] by Morikawa. In § 2 we shall show that this additional assumption can be omitted. The above equality plays an important rôle in the algebraic treatment of the theorem of Frobenius, and we shall discuss it in a forthcoming paper. Lastly the existence theorem of a basic polar divisor (in the sense of numerical equivalence) on a polarized abelian variety will be proved.

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§ 1. Arithmetic genera of abelian varieties

Let A be an abelian variety of dimension n and let X be any divisor on A. Then the set of points t of A satisfying $X_t \sim X$ is a subgroup[2)] of A and is denoted by $\mathfrak{G}_X$. If $\mathfrak{G}_X$ is finite, we shall call

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1) We shall use freely the notations and the results in Weil [8]. Numbers in brackets refer to the bibliography at the end.

2) We shall show later that the group $\mathfrak{G}_X$ is an algebraic subgroup,

X, following Morikawa [4], a non-degenerate divisor. In his paper [9], Weil proved that a positive divisor X is non-degenerate if and only if the complete linear system $|mX|$ is ample for sufficiently large m, and that every abelian variety can be embedded in a projective space. Throughout this paper we suppose that every abelian variety is in a projective space.

Let $\mathfrak{G}(A)$ be the additive group of all divisors on an abelian variety A. Then the set of divisors algebraically equivalent to zero is a subgroup of $\mathfrak{G}(A)$ and we shall denote it by $\mathfrak{G}_a(A)$. We shall say that a divisor X is numerically equivalent to zero if there exists an integer $m \neq 0$ such that $mX \equiv 0$ (mod. $\mathfrak{G}_a(A)$); obviously the set of divisors numerically equivalent to zero is a subgroup $\mathfrak{G}_n(A)$ of $\mathfrak{G}(A)$ containing $\mathfrak{G}_a(A)$. It is not so difficult to see that the numerical equivalence coincides with the equivalence $\equiv$ in Weil's sense (cf. Weil [8], [10]). It is well known that the residue group $\mathfrak{G}_n(A)/\mathfrak{G}_a(A)$ is a finite group of order $\sigma = p^f$, p being the characteristic. (cf. Weil [10]).

According to the theory of dual abelian varieties, we know that if X and X' are non-degenerate divisors such that $X \equiv X'$ (mod. $\mathfrak{G}_a(A)$), then $X \sim X'_t$ for some point t and whence $l(X) = l(X')$. In particular every maximal algebraic family $\{X\}$ containing a positive non-degenerate divisor X is complete in Matsusaka's sense (cf. Matsusaka [2]), i. e. each member of $\{X\}$ determines a complete linear system of the same dimension and every positive divisor algebraically equivalent to a member of $\{X\}$ necessarily belongs to $\{X\}$.

The constant term of the Hilbert characteristic function of A is called the arithmetic genus of A and is denoted by $\chi(A)$. Moreover, for every divisor X on A, we can define the virtual arithmetic genus $\chi_A(X)$ of X with respect to A as in Zariski [11]. Zariski's notation $p_a(X)$ is related to our $\chi_A(X)$ by $\chi_A(X) = 1 + (-1)^{n+1} p_a(X)$. It is well known that if two divisors X and Y are numerically equivalent to each other, then $\chi_A(X) = \chi_A(Y)$. (cf. Matsusaka [3]).

PROPOSITION 1. *Let X be a positive non-degenerate divisor on an abelian variety A. If a is an integer which is sufficiently large, then we have $l(aX) = l(Y)$ for every positive divisor Y such that $Y \equiv aX$* (mod. $\mathfrak{G}_n(A)$).

PROOF. Let $\{M_0 = 0, \cdots, M_{\sigma-1}\}$ be a complete set of representatives of the residue group $\mathfrak{G}_n(A)/\mathfrak{G}_a(A)$. First we show that, if a is sufficiently large, then $l(aX + M_j) = \chi(A) - \chi_A(-aX - M_j)$ for $j = 0, 1, \cdots, \sigma - 1$. Let a_0 be a positive integer such that the complete linear system $|aX|$ is ample whenever a is not less than a_0. Then, by Zariski [11], Th. 5, we can find a positive integer $a(r)$ for each r, $0 \leqq r < a_0$, such that $l(aX + M_j) = \chi(A) - \chi_A(-aX - M_j)$ whenever $a \equiv r$ (mod. a_0) and a is not less than $a(r)$. Hence if a is greater than each $a(r)$, $0 \leqq r < a_0$, then $l(aX + M_j) = \chi(A) - \chi_A(-aX - M_j)$.

Now Y being a positive divisor in our proposition, there is j such that $Y \equiv aX+M_j$ (mod. $\mathfrak{G}_a(A)$). Then we have $l(Y)=l(aX+M_j)$. Since the virtual arithmetic genus is invariant by numerical equivalence, we have $l(aX+M_i)=l(aX+M_j)$ for $0 \leqq i, j \leqq \sigma-1$. Therefore $l(Y)=l(aX)$.

PROPOSITION 2. *Let A be an abelian variety of dimension n. Then the Hilbert characteristic function $\chi(A,m)$ is the polynomial $a_0/n!\cdot m^n$, where a_0 is the degree of A.*

PROOF. The Hilbert characteristic function $\chi(A,m)$ of A is given by the formula:

$$\chi(A,m)=a_0\binom{m}{n}+a_1\binom{m}{n-1}+\cdots+a_n\,,$$

where a_0 is equal to the degree of A. Let C be a generic hyperplane section of A. Then $l(mC)$ coincides with $\chi(A,m)$ for all sufficiently large m. Let a be a positive integer relatively prime to the characteristic p. Then, by Morikawa [4], Th. 4, we have $l((a\delta_A)^{-1}(C))=\nu(a\delta_A)\cdot l(C)$. Since $\nu(a\delta_A)=a^{2n}$ (cf. Weil [8], Th. 33, Cor. 1), we have $l((a\delta_A)^{-1}(C)) =a^{2n}l(C)$. On the other hand, by Weil [8] Prop. 31, $(a\delta_A)^{-1}(C)\equiv a^2C$ (mod. $\mathfrak{G}_n(A)$). Now applying Prop. 1, we know that whenever a is sufficiently large $l((a\delta_A)^{-1}(C))=l(a^2C)$. Whence we have proved that whenever a is sufficiently large and relatively prime to p, $l(a^2C)=l(C)\cdot a^{2n}$. Therefore the polynomial $\chi(A,m)$ must be of the form:

$$\chi(A,m)=a_0/n!\cdot m^n\,.$$

COROLLARY 1. *Notations being the same as in Prop. 2, we have $l(C) =1/n!\cdot \deg(C_{u_1}\cdot\cdots\cdot C_{u_n})$, where the points $u_1,\cdots,u_n$ are such that the intersection product $C_{u_1}\cdot\cdots\cdot C_{u_n}$ is defined.*

COROLLARY 2. *Let N be the dimension of the ambient projective space of an abelian variety A of dimension n. Then we have $n!\,(N+1)\leqq$ degree of A.*

PROOF. This follows immediately from the fact that $l(C)\geqq N+1$ and Prop. 2.

As a consequence of Prop. 2 we have the following

THEOREM 1. *The arithmetic genus of an abelian variety is zero.*

§ 2. The dimension of complete linear systems

Let X be any divisor on an abelian variety A of dimension n and let k be a field over which A is defined and X is rational. There are n points $u_1,\cdots,u_n$ such that the intersection product $X_{u_1}\cdot\cdots\cdot X_{u_n}$ is defined, and the degree of zero cycle $X_{u_1}\cdot\cdots\cdot X_{u_n}$ does not depend on the choice of such points. From now on we denote it by $(X^{(n)})$.

PROPOSITION 3. *Let X be a positive non-degenerate divisor on an abelian variety of dimension n. Then we have $l(X)=(X^{(n)})/n!$.*

PROOF. When the complete linear system $|X|$ is ample, our proposition is nothing else but Cor. 1 of Prop. 2. Whence if a is sufficiently large $l(aX)=((aX)^{(n)})/n!=(X^{(n)})a^n/n!$. Suppose now that a is relatively prime to p. Then similarly as in the proof of Prop. 2 we have $l(a^2X)=l(X)\cdot a^{2n}$. Hence, for such a, we have $l(X)\cdot a^{2n}=(X^{(n)})a^{2n}/n!$.

COROLLARY 1. *If a positive divisor X is reducible and at least one of its component is non-degenerate, then $l(X)\geqq 2$.*

PROOF. Suppose that $X=X_1+X_2$, $X_i>0$ and that X_1 is non-degenerate. Then the degree of $(X_1)_{u_1}\cdot\cdots\cdot(X_1)_{u_{n-1}}\cdot(X_2)_{u_n}$ is positive whenever the intersection product $(X_1)_{u_1}\cdot\cdots\cdot(X_1)_{u_{n-1}}\cdot(X_2)_{u_n}$ is defined. Therefore $(X^{(n)})\geqq(X_1^{(n)})+1$.

COROLLARY 2. *If X is a reducible positive divisor on a simple abelian variety, then $l(X)\geqq 2$.*

PROOF. Since every positive divisor on a simple abelian variety is non-degenerate (cf. § 3), this follows immediately from Cor. 1.

Now we need the following

LEMMA 1. *Let A and B be isogenous abelian varieties and let λ be a homomorphism from A onto B. Then we have $\lambda\circ\lambda^{-1}(Y)=\nu(\lambda)Y$, where Y is any cycle on B.*

PROOF. We may assume that Y is a subvariety of B. Let Λ be the graph of λ. We set $\Lambda\cdot(A\times Y)=\sum_{j=1}^{r}X_j$, where each X_j is a subvariety of $A\times B$. Then we have $\lambda^{-1}(Y)=\sum_{j=1}^{r}pr_A X_j$ and $\lambda\circ\lambda^{-1}(Y)=\sum_{j=1}^{r}pr_B[\Lambda\cdot(pr_A X_j\times B)]=\sum_{j=1}^{r}[X_j:Y]Y$. On the other hand $\nu(\lambda)Y=pr_B[\Lambda\cdot(A\times Y)]=\sum_{j=1}^{r}[X_j:Y]Y$. Therefore $\nu(\lambda)$ is equal to $\sum_{j=1}^{r}[X_j:Y]$. Hence $\lambda\circ\lambda^{-1}(Y)=\nu(\lambda)\cdot Y$.

PROPOSITION 4. *Let Y be a positive non-degenerate divisor and let X be any divisor such that $X\equiv Y$ (mod. $\mathfrak{G}_n(A)$) on an abelian variety A. Then we have $l(X)\geqq 1$.*

PROOF. Let $\{M_0=0, M_1,\cdots, M_{\sigma-1}\}$ be a complete set of representatives of the residue group $\mathfrak{G}_n(A)/\mathfrak{G}_a(A)$. Since Y is positive non-degenerate, $aY+M_j$ is linearly equivalent to a positive divisor for $j=0,1,\cdots,\sigma-1$, whenever a is sufficiently large. Suppose now that a is relatively prime to p. Then $\{0, a^2M_1,\cdots, a^2M_{\sigma-1}\}$ is also a complete set of representatives of $\mathfrak{G}_n(A)/\mathfrak{G}_a(A)$, because the order σ of the group $\mathfrak{G}_n(A)/\mathfrak{G}_a(A)$ is a power of p. Therefore $a^2Y+a^2M_j\equiv a^2Y+M_{j'}$ (mod. $\mathfrak{G}_a(A)$) for some j'. Whence we have proved that if a is sufficiently large and is relatively prime to p, then $l((a\delta_A)^{-1}(Y+M_j))\geqq 1$ for $j=0,1,\cdots,\sigma-1$. Since $(a\delta_A)^{-1}(Y+M_j)$ is linearly equivalent to a divisor of the form $(a^2Y+$

$a^2M_i)_t$, there is a divisor Z such that $(a\delta_A)^{-1}(Y+M_j)\sim Z$ and $Z_s=Z$ for all s satisfying $as=0$.[3]

Let f be a function on A such that $(f)+Z>0$ and let k be a field over which A and f are defined and Z is rational; let x be a generic point of A over k. We set $\theta(x)=\Sigma f(x+s)$, where the summation is such that s runs over all points s satisfying $as=0$. Then obviously $(\theta)+Z>0$, and we set $Z'=(\theta)+Z$. Since $Z'_s=Z'$ for all s satisfying $as=0$, there exists a positive divisor Y_1 such that $Z'=(a\delta_A)^{-1}(Y_1)$. By Lemma 1, $(a\delta_A)\circ(a\delta_A)^{-1}(Y_1)=a^{2n}Y_1$ and $(a\delta_A)\circ(a\delta_A)^{-1}(Y+M_j)=a^{2n}(Y+M_j)$, where $n=\dim A$. Therefore we have $a^{2n}(Y+M_j)\equiv a^{2n}Y_1$ (mod. $\mathfrak{G}_a(A)$) and hence $Y+M_j\equiv Y_1$ (mod. $\mathfrak{G}_a(A)$). The suffix being arbitrary, this completes the proof.

COROLLARY. *Let X and Y be non-degenerate divisor on an abelian variety A such that $X\equiv Y$* (mod. $\mathfrak{G}_n(A)$). *Then we have $l(X)=l(Y)$.*

THEOREM 2. *If X is a non-degenerate divisor on an abelian variety A of dimension n, then we have $l(aX)=l(X)\cdot a^n$ for all positive integers a.*

PROOF. When $l(X)\geqq 1$, our assertion follows immediately from Prop. 3. Therefore we have only to prove that if $l(X)$ is zero, then $l(aX)$ is also zero for all positive integers a.

Suppose now that $l(aX)\geqq 1$ for convenient positive integer a. Here we may assume that a is relatively prime to p. Then also $l(a^2X)\geqq 1$. Similarly as in the proof of Prop. 4, we can show that $l(X)\geqq 1$.

Now we can restate Prop. 3 in a slightly better form as follows:

THEOREM 3. *Let X be a non-degenerate divisor on an abelian variety A of dimension n. If aX is congruent to a positive divisor modulo $\mathfrak{G}_n(A)$ for suitable positive integer a, then we have $l(X)=(X^{(n)})/n!$.*

There arises now a following problem: Suppose that X is non-degenerate and the n-fold intersection number $(X^{(n)})$ is not less than 1. Then, is $l(X)+l(-X)$ equal to $(X^{(n)})/n!$? This problem seems to be plausible, but the writer do not know the proof. When the dimension $n=2$, then this follows immediately from the theorem of Riemann-Roch.

THEOREM 4. *Let A and B be isogenous abelian varieties and let λ be a homomorphism from A onto B. If Y is any non-degenerate divisor on B, then we have $l(\lambda^{-1}(Y))=\nu(\lambda)\cdot l(Y)$.*

PROOF. When $l(Y)=0$, then $l(\lambda^{-1}(Y))=0$. In fact, suppose that $l(\lambda^{-1}(Y))\geqq 1$. Then $\lambda^{-1}(Y)$ is linearly equivalent to a positive divisor X on A. Therefore $\lambda\cdot\lambda^{-1}(Y)\equiv\lambda(X)$ (mod. $\mathfrak{G}_a(A)$). Since $\lambda\circ\lambda^{-1}(Y)=\nu(\lambda)Y$ by Lemma 1, it follows from Th. 2 and Th. 3 that $l(Y)\geqq 1$.

Now suppose that $l(Y)\geqq 1$. Then we may assume that Y is positive. Let k be a field over which A, B and λ are defined and Y is rational; let $u_1,\cdots,u_n$ be n independent generic points of A over k, where n is the dimension of A. We set $\lambda(u_j)=v_j$, $j=1,\cdots,n$. We can

3) Cf. Weil [8], p. 160.

readily see that

$$(\lambda^{-1}(Y))_{u_1}\cdot\cdots\cdot(\lambda^{-1}(Y))_{u_n}=\lambda^{-1}(Y_{v_1})\cdot\cdots\cdot\lambda^{-1}(Y_{v_n})$$
$$=\lambda^{-1}(Y_{v_1}\cdot\cdots\cdot Y_{v_n})\,,$$

hence

$$l(\lambda^{-1}(Y))=\deg((\lambda^{-1}(Y))_{u_1}\cdot\cdots\cdot(\lambda^{-1}(Y))_{u_n})/n!$$
$$=\nu(\lambda)\deg(Y_{v_1}\cdot\cdots\cdot Y_{v_n})/n!$$
$$=\nu(\lambda)l(Y)\,.$$

COROLLARY. *If X is a non-degenerate divisor on an abelian variety A such that $l(X)=1$. Then we have $\mathfrak{G}_X=0$.*

PROOF. Since $\mathfrak{G}_X$ is invariant with respect to numerical equivalence, X may be assumed to be positive. Then, for all points t of $\mathfrak{G}_X$, we have $X_t=X$. By Chow [1] there exist a quotient abelian variety $A/\mathfrak{G}_X$ and a separable homomorphism λ from A onto $A/\mathfrak{G}_X$ whose kernel coincides with $\mathfrak{G}_X$. Then by Weil [8], Prop. 33, we can find a positive non-degenerate divisor Y on $A/\mathfrak{G}_X$ such that $X=\lambda^{-1}(Y)$. By Th. 4 we have $1=l(X)=l(\lambda^{-1}(Y))=\nu(\lambda)l(Y)$. This shows that $\mathfrak{G}_X=0$.

§ 3. Some remarks on degenerate divisors

Let A be an abelian variety and let $\hat{A}$ be the dual abelian variety of A (i. e. the Picard variety of A). Then, as is well known, A and $\hat{A}$ are isogenous abelian varieties. Now let X be any divisor on A. Then mapping the point u of A to the point $\hat{u}$ of $\hat{A}$ corresponding to the linear class of X_u-X, we can define a homomorphism φ_X from A into $\hat{A}$. We shall call φ_X the homomorphism attached to X. We can readily see that the group $\mathfrak{G}_X$ defined in § 2 coincides with the kernel of φ_X. Thus the group $\mathfrak{G}_X$ is an algebraic subgroup and there is an abelian subvariety C such that $\mathfrak{G}_X$ is the union of a finite number of subvarieties C_a. We shall call C the abelian subvariety attached to X. When the dimension of the abelian subvariety attached to X is i, then we shall call X i-degenerate.

From now on, throughout this §, we assume that X is positive. We can see that the set-theoretic intersection $X\cap C$ is empty whenever the intersection product $X\cdot C$ is defined and that, for each component Y of X, $Y_u=Y$ holds for all $u\in C$. (cf. Weil [9]).

According to Chow [1], we can construct a quotient abelian variety A/C and a separable homomorphism λ from A onto A/C whose kernel is exactly C. Then similarly as in the proof of Weil [8], Prop. 33, there is a positive divisor Y on A/C such that $X=\lambda^{-1}(Y)$. And it can be shown that Y is non-degenerate; in fact if $Y_{\lambda u}\sim Y$, then $X_u=\lambda^{-1}(Y_{\lambda u})$

$\sim\lambda^{-1}(Y)=X$, and we can conclude that $Y_{\lambda u}\sim Y$ for only finite number of points λu of A/C.

Since Y is non-degenerate, the connected component $\{Y\}$ containing Y is an (irreducible) variety in the algebraic system of positive divisors to which Y belongs. (Note the fact that any maximal algebraic family containing a non-degenerate divisor is complete.) On the other hand we can readily see that the abelian subvariety attached to X depends only on the class of X modulo $\mathfrak{G}_n(A)$. Therefore it follows that the connected component $\{X\}$ containing X is an (irreducible) variety in the algebraic system of positive divisors to which X belongs, and the dimension of $\{X\}$ is equal to that of $\{Y\}$. Thus each member of $\{X\}$ is of the form $\lambda^{-1}(Y')$, $Y'\in\{Y\}$, hence we have $X'\sim X_t$, $t\in A$, where X' is any member of $\{X\}$. From this we have $l(X')=l(X)$ for any two members X', X of $\{X\}$, and the dimension of $\{X\}$ is $\dim A-i+l(X)-1$. Since the dimension of $\{Y\}$ is $\dim A-i+l(Y)-1$, we have $l(X)=l(Y)$. Now we can state the following

THEOREM 5. *Let X be a positive i-degenerate divisor on an abelian variety A and C be the abelian subvariety attached to X of dimension i. If the intersection product $X_{u_1}\cdot\cdots\cdot X_{u_{n-i}}\cdot X_{u_{n-i+1}}$ is defined, then the i-cycle $X_{u_1}\cdot\cdots\cdot X_{u_{n-i}}$ is of the form ΣC_u, $u\in A$, and the number of the components is equal to $(n-i)!\cdot l(X)$, and $X_{u_1}\cdot\cdots\cdot X_{u_{n-i}}\cdot X_{n-i+1}$ is zero, where $n=\dim A$.*

COROLLARY. *A and X being as in Th. 5, we have $l(aX)=l(X)a^{n-i}$ for all positive integers a.*

§ 4. Virtual arithmetic genera of divisors

LEMMA 2. *Let V be a non-singular projective variety of dimension n, and let X and Y be any two divisors on V. Then the function $\chi_V(mX+Y)$[4] of m is a polynomial in m of degree not greater than n for all integers m.*

PROOF. When V is an algebraic curve, then $\chi_V(mX+Y)=\deg(mX+Y)=\deg(X)m+\deg(Y)$. Therefore, in this case, our assertion is trivial. We now proceed by the induction on the dimension n.

Let k be a field over which V is defined and X, Y are rational. Let C_t be a generic hypersurface section of degree t over k. Then it is well known that if t is sufficiently large, then the complete linear system $|X+C_t|$ is ample (cf. Matsusaka [2], Lemma 2). Now let E be a generic member of $|X+C_t|$ over K, where K is a field containing k over which the variety C_t is defined. As is well known the modular properties

$$\chi_V((m+1)X+Y+2C_t)=\chi_V(mX+Y+C_t)+\chi_V(X+C_t)$$
$$-\chi_E((mX+Y+C_t)\cdot E)$$

4) $\chi_V(mX+Y)$ means the virtual arithmetic genus of $mX+Y$ on V.

$$\chi_V((m+1)X+Y+2C_t)=\chi_V((m+1)X+Y+C_t)+\chi_V(C_t)$$
$$-\chi_{C_t}(((m+1)X+Y+C_t)\cdot C_t)$$

hold, where $\chi_E((mX+Y+C_t)\cdot E)$, $\chi_{C_t}(((m+1)X+Y+C_t)\cdot C_t)$ mean the virtual arithmetic genera of divisors $(mX+Y+C_t)\cdot E$, $((m+1)X+Y+C_t)\cdot C_t$ on non-singular varieties E and C respectively. From these two equalities we have

$$\chi_V((m+1)X+Y+C_t)-\chi_V(mX+Y+C_t)=\chi_V(X+C_t)-\chi_V(C_t)$$
$$+\chi_{C_t}(((m+1)X+Y+C_t)\cdot C_t)-\chi_E((mX+Y+C_t)\cdot E)\,,$$

and again applying the modular properties

$$\chi_V((m+1)X+Y)-\chi_V(mX+Y)=\chi_V(X)-\chi_{C_t}(X\cdot C_t)$$
$$+\chi_{C_t}(((m+1)X+Y+C_t)\cdot C_t)+\chi_{C_t}(((m+1)X+Y)\cdot C_t)$$
$$-\chi_{C_t}((mX+Y)\cdot C_t)-\chi_E((mX+Y+C_t)\cdot E)\,.$$

Now fix t. Then by induction assumptions, the last three terms of the right hand side are polynomials in m of degrees not greater than $n-1$, and the first two terms are constants. Whence the difference $\chi_V((m+1)X+Y)-\chi_V(mX+Y)$ is a polynomial in m of degree not greater than $n-1$. Then we can easily see that the function $\chi_V(mX+Y)$ is a polynomial in m of degree not greater than n. The proof of our Lemma is thereby completed.

In particular, when V is a non-singular projective surface, we have

$$\chi_V(mX+Y)=-(X,X)\binom{m}{2}+(\chi_V(X)-(X,Y))m+\chi_V(Y)\,.$$

Now we shall come back to abelian varieties.

PROPOSITION 5. *Let X be any divisor on an abelian variety A of dimension n. Then we have $\chi_A(-X)=(-1)^n\chi_A(X)$.*

PROOF. By the theorem of Riemann-Roch in Serre [7] and by Th. 1 in § 1, we have

$$l(X)-h^1(X)+h^2(X)-\cdots+(-1)^nh^n(X)=-\chi_A(-X)\,.$$

Since the zero cycle is a canonical cycle on A, it follows from duality theorem that $h^s(-X)=h^{n-s}(X)$, $1\leqq s\leqq n-1$, and $h^n(-X)=l(X)$ (cf. Serre [6]). Whence we have $\chi_A(-X)=(-1)^n\chi_A(X)$.

THEOREM 6. *Let X be any divisor on an abelian variety A of dimension n. Then we have $\chi_A(X)=(-1)^{n+1}\cdot(X^{(n)})/n!$.*

PROOF. Let C be a generic hyperplane section of A. By Zariski [11], Th. 5, we have $l(mC)=-\chi_A(-mC)$ for all sufficiently large m.

Since $\chi_A(-mC)$ is a polynomial in m for all integers m and $l(mC)=l(C)m^n$, we have $-\chi_A(-mC)=l(C)\cdot m^n$ for all positive integers m; in particular, setting $m=1$, we have $-\chi_A(-C)=l(C)$. By Prop. 3 and Prop. 5 we can conclude that $\chi_A(C)=(-1)^{n+1}(C^{(n)})/n!$.

Now suppose that X is positive non-degenerate. Then the complete linear system $|mX|$ is ample for all sufficiently large m. Therefore, by the above arguments, we have $\chi_A(mX)=(-1)^{n+1}(X^{(n)})\cdot m^n/n!$ for all sufficiently large m and hence for all integers m; in particular $\chi_A(X)=(-1)^{n+1}\cdot(X^{(n)})/n!$.

Finally let X be any divisor on A. Then the complete linear system $|X+mC|$ is ample, if m is sufficiently large. Therefore for such m we have

$$\begin{aligned}\chi_A(X+mC)&=(-1)^{n+1}\cdot((X+mC)^{(n)})/n!\\&=(-1)^{n+1}\cdot(X^{(n)})/n!+g(m)\,,\end{aligned}$$

where $g(m)$ denotes a polynomial in m that has no costant term. Thus the polynomial $\chi_A(X+mC)$ must coincides with $(-1)^{n+1}(X^{(n)})/n!+g(m)$. We set $m=0$. Then we have the desired formula:

$$\chi_A(X)=(-1)^{n+1}(X^{(n)})/n!\,.$$

COROLLARY 1. *Let X be a positive divisor on an abelian variety A. Then X is degenerate if and only if $\chi_A(X)=0$.*

COROLLARY 2. *A, X being as in Th. 6, we have $\chi_A(mX)=\chi_A(X)\cdot m^n$ for all integers m.*

COROLLARY 3. *If X is a positive non-degenerate divisor on an abelian variety A of dimension n, then we have $l(X)=(-1)^{n+1}\chi_A(X)$.*

§ 5. Existence of basic polar divisors on polarized abelian varieties

Let $X_1,\cdots,X_r$ be a set of divisors on an abelian variety A. If there exist integers $a_1,\cdots,a_r$, not all zero, such that $\sum_{j=1}^{r}a_jX_j\equiv 0$ (mod. $\mathfrak{G}_n(A)$), then they are said to be numerically dependent. The theorem of Severi-Néron asserts that the residue group $\mathfrak{G}(A)/\mathfrak{G}_n(A)$ is a finitely generated free abelian group (cf. Néron [5]). First we shall give a proof without making use of this theorem.

PROPOSITION 6. *Let A and B are isogenous abelian varieties and let λ be a homomorphism from A onto B. If $Y_1,\cdots,Y_r$ are divisors on B which are numerically independent, then $\lambda^{-1}(Y_1),\cdots,\lambda^{-1}(Y_r)$ are also numerically independent on A.*

PROOF. Suppose that $\sum_{j=1}^{r}a_j\lambda^{-1}(Y_j)\equiv 0$ (mod. $\mathfrak{G}_n(A)$). Then we have $\sum_{j=1}^{r}a_j\lambda\circ\lambda^{-1}(Y_j)\equiv 0$ (mod. $\mathfrak{G}_n(B)$), and hence $\nu(\lambda)\sum_{j=1}^{r}a_jY_j\equiv 0$ (mod. $\mathfrak{G}_n(B)$).

This shows that $a_1=\cdots=a_r=0$.

LEMMA 3. *Let $\mathfrak{G}$ be a Z-module, Z being the ring of rational integers, with the following properties*: (i) *$\mathfrak{G}$ has no element of order finite*; (ii) *There are r elements $g_1,\cdots,g_r$ in $\mathfrak{G}$ which are linearly independent over Z*; (iii) *With the previous elements $g_1,\cdots,g_r$ if we denote by $(g_1,\cdots,g_r)$ the submodule of $\mathfrak{G}$ generated by $g_1,\cdots,g_r$, then there exists a positive integer c such that, for any element g of $\mathfrak{G}$, cg belongs to the submodule $(g_1,\cdots,g_r)$. Then $\mathfrak{G}$ is a finitely generated free Z-module.*

PROOF. Let $\mathfrak{H}$ be a submodule such that h belongs to $\mathfrak{H}$ if and only if $c'h$ belongs to the submodule (g_1) for suitable integer $c'\neq 0$. We shall show that $\mathfrak{H}$ is generated by a single element. If $\mathfrak{H}$ is not finitely generated, then we can find a strictly ascending infinite chain of submodules $(g_1)\subset(g_{1,1})\subset\cdots\subset(g_{1,n})\subset\cdots\subset\mathfrak{H}$. We set $g_1=c_n g_{1,n}$. Here we may assume that each c_n is positive. Then the set $\{c_n\}$ is not bounded. On the other hand $cg_{1,n}$ belongs to $(g_1,\cdots,g_r)$ and hence we can write $cg_{1,n}=a_1g_1+\cdots+a_rg_r$, $a_j\in Z$. From these we have $(a_1c_n-c)g_1+\cdots=0$, and by property (ii) $a_1c_n=c$. Since c is a constant, this is a contradiction. Whence we have proved that there is an element h of $\mathfrak{G}$ such that $\mathfrak{H}=(h)$. And we can readily see that the residue module $\mathfrak{G}/(h)$ also has the properties (i), (ii) and (iii).

Now we proceed by the induction on r. The case $r=1$ is already settled by above arguments. Since $\mathfrak{G}/(h)$ is finitely generated by the induction assumption, $\mathfrak{G}$ is also finitely generated.

PROPOSITION 7. *The group $\mathfrak{G}(A)/\mathfrak{G}_n(A)$ is a finitely generated free abelian group.*

PROOF. There exists an abelian variety B such that $A\times B$ is isogenous to a Jacobian variety J. By Weil [8], we know that the group $\mathfrak{G}(J)/\mathfrak{G}_n(J)$ is finitely generated free abelian group. Now let $Z_1,\cdots,Z_\rho$ be a set of divisors on J which represent a base of $\mathfrak{G}(J)/\mathfrak{G}_n(J)$; let λ and μ be two homomorphisms from $A\times B$ onto J and from J onto $A\times B$ such that $\mu\circ\lambda=\nu(\lambda)\delta_{A\times B}$. Then for any divisor X on $A\times B$ we have $\mu^{-1}(X)\equiv\sum_{j=1}^{\rho}a_jZ_j$ (mod. $\mathfrak{G}_n(J)$) and hence $\lambda^{-1}\circ\mu^{-1}(X)\equiv\sum_{j=1}^{\rho}a_j\lambda^{-1}(Z_j)$ (mod. $\mathfrak{G}_n(A\times B)$). Since we have $\lambda^{-1}\circ\mu^{-1}(X)=(\mu\circ\lambda)^{-1}(X)=(\nu(\lambda)\delta_{A\times B})^{-1}(X)\equiv\nu(\lambda)^2X$ (mod. $\mathfrak{G}_n(A\times B)$), and $\lambda^{-1}(Z_1),\cdots,\lambda^{-1}(Z_\rho)$ are numerically independent on $A\times B$ by Prop. 6, we can apply Lemma 3 to $\mathfrak{G}(A\times B)/\mathfrak{G}_n(A\times B)$. Thus we know that $\mathfrak{G}(A\times B)/\mathfrak{G}_n(A\times B)$ is a finitely generated free abelian group.

Let $Y_1,\cdots,Y_\rho$ be a set of divisors on $A\times B$ which represent a base of $\mathfrak{G}(A\times B)/\mathfrak{G}_n(A\times B)$. We set $X_j=Y_j\cdot(A\times Q)$, $j=1,\cdots,\rho$, where Q is a point of B such that the intersection product $Y_j\cdot(A\times Q)$ is defined for each j. We can choose a maximal set of numerically independent divisors $X_{i_1},\cdots,X_{i_\tau}$ among $X_1,\cdots,X_\rho$. Now let $\bar{X}$ be any divisor on A.

Then we can write $\bar{X}\times B\equiv\sum_{j=1} b_jY_j$ (mod. $\mathfrak{G}_n(A\times B)$). Hence we have $\bar{X}\times Q=(\bar{X}\times B)\cdot(A\times Q)\equiv\sum_{j=1}^{\rho} b_jX_j$ (mod. $\mathfrak{G}_n(A\times Q)$). Therefore there exists a constant positive integer c such that $c(\bar{X}\times Q)\equiv\sum_{j=1}^{\tau} b_j'X_{i_j}$ (mod. $\mathfrak{G}_n(A\times Q)$). Again we can apply Lemma 3 to $\mathfrak{G}(A)/\mathfrak{G}_n(A)$. This completes the proof.

Thus the group $\mathfrak{G}(A)/\mathfrak{G}_n(A)$ is a free abelian group with ρ generaters. This number ρ is called the *Picard number* of A. Prop. 6 asserts that if A and B are isogenous abelian varieties, then the Picard number of A is equal to that of B.

Let X_0 be a positive non-degenerate divisor on A; let $\mathfrak{C}(X_0)$ be the set of positive divisors X such that $mX\equiv m'X_0$ (mod. $\mathfrak{G}_a(A)$) for convenient positive integers m, m'. We shall say that the class $\mathfrak{C}(X_0)$ determines a polarization on A. And each divisor of $\mathfrak{C}(X_0)$ is called a polar divisor. If there exists a polar divisor Y such that for any polar divisor X we have $X\equiv mY$ (mod. $\mathfrak{G}_n(A)$) for suitable positive integer m, then we shall call Y a basic polar divisor in the sense of numerical equivalence.

THEOREM 7. *There exists a basic polar divisor in the sense of numerical equivalence.*

PROOF. Let $Z_1,\cdots, Z_\rho$ be a set of divisors which represent a base of $\mathfrak{G}(A)/\mathfrak{G}_n(A)$. Then we can write $X_0\equiv\sum_{j=1}^{\rho} a_jZ_j$ (mod. $\mathfrak{G}_n(A)$). Let d be the greatest common divisor of $a_1,\cdots, a_\rho$. We set $a_j=da_j'$, $j=1,\cdots,\rho$, and $Y'=\sum_{j=1}^{\rho} a_j'Z_j$. Since $dY'\equiv\sum_{j=1}^{\rho} a_jZ_j\equiv X_0$ (mod. $\mathfrak{G}_n(A)$) and $X_0>0$, we have, by Prop. 4, $l(dY')\geqq 1$. Then Th. 2 asserts that there is a positive divisor Y such that $Y\sim Y'$.

Now we can show that Y is a basic polar divisor in the sense of numerical equivalence. Clearly Y belongs to the class $\mathfrak{C}(X_0)$, because $dY\equiv X_0$ (mod. $\mathfrak{G}_n(A)$). Let X be any polar divisor; then we can write $X\equiv\sum_{j=1}^{\rho} b_jZ_j$ (mod. $\mathfrak{G}_n(A)$). There are two positive integers m, m' such that $mX\equiv m'Y$ (mod. $\mathfrak{G}_n(A)$). Here we may assume that m, m' are relatively prime to each other. Then since $\sum_{j=1}^{\rho} mb_jZ_j\equiv\sum_{j=1}^{\rho} m'a_j'Z_j$ (mod. $\mathfrak{G}_n(A)$), we have $mb_j=m'a_j'$ for $j=1,\cdots,\rho$. This shows that m is a common divisor of $a_1',\cdots, a_\rho'$ and hence we have $m=1$.

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