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On Banach Limits in some Topological Vector Spaces

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The notion of Banach limits of sequences of real numbers is the generalization of that of the ordinary limits preserving the following properties;

"the sum of two convergent sequences is convergent and its limit is the sum of the limits of the two sequences",

"the scalar multiple of a convergent sequence and a real number is convergent and its limit is the scalar multiple of the limit of the sequence and the real number",

"the generalized limit of the bounded sequence lies between the lower and the upper limit of the sequence".

In this note, we shall show that the notion of Banach limits can be extended to the case where the sequences lie in some locally convex topological vector spaces and shall give its application to the theory of integrations of vector valued functions.

A *topological vector space* is a vector space, over the real space, with Hausdorff's topology making vector summation and scalar multiplication continuous. We shall say that the topological vector space with the fundamental system of neighbourhoods consisting of convex sets is *locally convex*. A locally convex topological vector space is briefly called a *locally convex space*. The *dual space* of a locally convex space E, denoted by E', means the space of all continuous linear functionals on E. A subset B of E is *bounded* if, for every neighbourhood U of $o\in E$, there exists a positive number λ such that $\lambda B\subset U$. The closed convex cover of a set $B\subset E$, $\overline{Co}\,B$, is the smallest closed convex set containing B; the *polar* of a bounded set $B\subset E, B^{\circ}$, is the subset of E' such that

$$B^{\circ}=\{x'\,;\,\sup_{x\in B}|x'(x)|\leqq 1,\, x'\in E'\}.$$

If we choose as a fundamental system of neighbourhoods the totality of B° where B runs over all bounded sets of E, we call this topology *strong* and its dual E' *strong dual*. We shall denote by E'' the dual space of the strong dual E'. A locally convex space E is said to be *semi-reflexive* if E and E'' are isomorphic as a vector space. Hereafter, we shall denote by the symbol E only the semi-reflexive locally convex space and by the symbol E' the strong dual of E.

A *directed sequence* in E, denoted by $\{x_\alpha\}_{\alpha\in A}$ or $\{x_\alpha\}$, is a function with its range in E defined on a certain directed set A the order of which is denoted by $<$ or $>$. A directed sequence is *bounded* if its range is bounded in E. We see at once that the class $\mathfrak{L}$ of all bounded directed sequences in E is a vector space as a functional space. Now, for brevity, we shall use the symbols X, X_β, and $x'(B)$ to denote the range of a bounded directed sequence $\{x_\alpha\}$, the set of elements x_α for all $\alpha>\beta$ i.e. $\{x_\alpha\,;\,\alpha>\beta\}$ and the image of $B\subset E$ by $x'\in E'$ respectively.

Let us begin with the following:

LEMMA 1. *The range of a bounded directed sequence $\{x_\alpha\}$ has the following properties* a)$\sim$c);

a) *the intersection of all the sets $\overline{Co}\,X_\beta$ for all $\beta\in A$ is not empty;*

$$\bigcap_{\beta\in A}\overline{Co}\,X_\beta\neq O\,,$$

b) *for every $x'\in E'$ and $\beta\in A$ the image of $\overline{Co}\,X_\beta$ by x' coincides with the closed convex cover of the image of X_β by x';*

$$x'(\overline{Co}\,X_\beta)=\overline{Co}\,x'(X_\beta)\,,$$

c) *let β be an arbitrary element of A, then the intersection of the inverse images of $x'(\overline{Co}\,X_\beta)$ for all $x'\in E'$ coincides with the set $\overline{Co}\,X_\beta$;*

$$\bigcap_{x'\in E'}\{x\,;\,x'(x)\in x'(\overline{Co}\,X_\beta)\}=\overline{Co}\,X_\beta\,.$$

PROOF. The closed convex cover of a bounded set in E is also bounded [3, p. 5]. It is therefore weakly bounded. A closed convex set is also weakly closed [3, p. 67] and moreover a bounded closed convex set in E is weakly compact [3, p. 88]. From these known facts it follows that $\overline{Co}\,X$ is weakly compact and $\overline{Co}\,X_\beta$ is a weakly closed set in $\overline{Co}\,X$. The class, $\{\overline{Co}\,X_\beta\,;\,\beta\in A\}$, of these sets has the finite intersection property. Hence, we have $\bigcap_{\beta\in A}\overline{Co}\,X_\beta\neq O$. Since every $\overline{Co}\,X_\beta$ is also weakly compact and every $x'\in E'$ is weakly continuous, the relation b)

$$x'(\overline{Co}\,X_\beta)=\overline{Co}(x'(X_\beta))$$

holds. Now, we shall show

$$\bigcap_{x'\in E'}\{x\,;\,x'(x)\in x'(B)\}=B$$

for every closed convex set $B\subset E$. From the fact that the closed convex set of a locally convex space is equal to the intersection of all closed half-spaces which contain the given set [2, p. 73], we obtain

$$\begin{aligned} B &= \bigcap_{x'\in E'} \{x \,;\, x'(x) \leqq \sup_{y\in B} x'(y)\} \\ &= \bigcap_{x'\in E'} \{x \,;\, \inf_{y\in B} x'(y) \leqq x'(x) \leqq \sup_{y\in B} x'(y)\} \\ &= \bigcap_{x'\in E'} \{x \,;\, x'(x) \in [\inf_{y\in B} x'(y), \sup_{y\in B} x'(y)]\} . \end{aligned}$$

In the real field we have

$$[\inf_{y\in B} x'(y), \sup_{y\in B} x'(y)] = \overline{Co}\, x'(B) = x'(B) .$$

Then we get

$$B = \bigcap_{x'\in E'} \{x \,;\, x'(x) \in x'(B)\}$$

which proves Lemma 1.

Now, we shall prove the following Theorem 1 which gives the definition of Banach limits in a space E.

THEOREM 1. *To each bounded directed sequence* $\{x_\alpha\}$ *in* E *we can determine an element* $\operatorname{Lim}_\alpha x_\alpha$ *in* E, *so that the following conditions* (1.1) *and* (1.2) *may be satisfied*;

$$\operatorname{Lim}_\alpha \{a x_\alpha + b y_\alpha\} = a \operatorname{Lim}_\alpha x_\alpha + b \operatorname{Lim}_\alpha y_\alpha , \tag{1.1}$$

$$\operatorname{Lim}_\alpha x_\alpha \in \bigcap_{\beta\in A} \overline{Co}\{x_\alpha \,;\, \alpha > \beta\} . \tag{1.2}$$

PROOF. Let $\{x_\alpha\}$ be an arbitrary bounded directed sequence. For every $x'\in E'$, the sequence $\{x'(x_\alpha)\}$ of real numbers is bounded. Since Banach limit is defined for every bounded sequence of real numbers, there exists a Banach limit $\operatorname{Lim}_\alpha x'(x_\alpha)$. Since this $\operatorname{Lim}_\alpha x'(x_\alpha)$ depends on x' and $\{x_\alpha\}$, we shall write

$$l(x', \{x_\alpha\}) = \operatorname{Lim}_\alpha x'(x_\alpha) .$$

Then, we shall show that $l(x', \{x_\alpha\})$ has the following properties;

(i) for every fixed $x'\in E'$, $l(x', \{x_\alpha\})$ is linear as a functional defined on $\mathfrak{L}$,

(ii) for every fixed $\{x_\alpha\}\in\mathfrak{L}$, $l(x', \{x_\alpha\})$ is linear and continuous as a functional defined on E',

and

$$l(x', \{x_\alpha\}) \in [\underline{\lim}_\alpha x'(x_\alpha), \overline{\lim}_\alpha x'(x_\alpha)] . \tag{iii}$$

These properties are almost obvious and hence we shall prove only the continuity of l as a linear functional defined on E', that is, to each positive number $\varepsilon > 0$, there exists a neighbourhood U' such that $x'\in U'$

implies

$$|l(x', \{x_\alpha\})| \leqq \varepsilon .$$

Since the range X of $\{x_\alpha\}$ is bounded, we put

$$U' = \left(\frac{1}{\varepsilon} X\right)^\circ .$$

Thus this U' is the required neighbourhood.

Therefore, for each $\{x_\alpha\}$, $l(x', \{x_\alpha\})$, as a continuous linear functional defined on E', is the element of E''. Now by the condition of semi-reflexivity, there exists, for every fixed $\{x_\alpha\} \in \mathfrak{L}$, an element $x(\{x_\alpha\})$ in E such that

$$x'(x(\{x_\alpha\})) = l(x', \{x_\alpha\})$$

for all $x' \in E'$. Let us define a $\operatorname*{Lim}_\alpha x_\alpha$ for $\{x_\alpha\} \in \mathfrak{L}$, by

$$\operatorname*{Lim}_\alpha x_\alpha = x(\{x_\alpha\}) .$$

We shall show that $\operatorname*{Lim}_\alpha x_\alpha$ satisfies the conditions (1.1) and (1.2). In the first place, by the property (i) of l and the linearity of $x' \in E'$ we obtain the equation

$$\begin{aligned} x'(\operatorname*{Lim}_\alpha\{ax_\alpha + by_\alpha\}) &= ax'(\operatorname*{Lim}_\alpha x_\alpha) + bx'(\operatorname*{Lim}_\alpha y_\alpha) \\ &= x'(a \operatorname*{Lim}_\alpha x_\alpha + b \operatorname*{Lim}_\alpha y_\alpha) . \end{aligned}$$

Generally, in the locally convex space for a given $x \neq o$, there exists a continuous linear functional x' such that $x'(x) \neq 0$ [2, p. 102]. Then we have

$$\operatorname*{Lim}_\alpha\{ax_\alpha + by_\alpha\} = a \operatorname*{Lim}_\alpha x_\alpha + b \operatorname*{Lim}_\alpha y_\alpha .$$

Next, we shall prove (1.2). From the property (iii) of l, we can conclude that $l(x', \{x_\alpha\})$ belongs to the closed interval $[\underline{\lim}_\alpha x'(x_\alpha), \overline{\lim}_\alpha x'(x_\alpha)]$ for each $x' \in E'$ and $\{x_\alpha\} \in \mathfrak{L}$. In the real space, the closed interval $[\underline{\lim}_\alpha x'(x_\alpha), \overline{\lim}_\alpha x'(x_\alpha)]$ is expressed by $\bigcap_{\beta \in A} \overline{Co}\{x'(x_\alpha) ; \alpha \succ \beta\}$, using the term of the closed convex cover. Then by Lemma 1, b) we have

$$l(x', \{x_\alpha\}) \in \bigcap_{\beta \in A} x'(\overline{Co}\, X_\beta) .$$

In other words, for every $\beta \in A$ and $x' \in E'$, $x'(\operatorname*{Lim}_\alpha x_\alpha)$ is always contained in $x'(\overline{Co}\, X_\beta)$. Therefore, we have

$$\operatorname*{Lim}_\alpha x_\alpha \in \bigcap_{x' \in E'} \{x ; x'(x) \in x'(\overline{Co}\, X_\beta)\} .$$

Then, by Lemma 1, c) we have

$$\operatorname*{Lim}_{\alpha} x_\alpha \in \overline{Co}\, X_\beta .$$

Since β is arbitrary, it follows

$$\operatorname*{Lim}_{\alpha} x_\alpha \in \bigcap_{\beta \in A} \overline{Co}\, X_\beta .$$

Thus we have proved the theorem.

We shall call $\operatorname*{Lim}_{\alpha} x_\alpha$ the *Banach limit* of $\{x_\alpha\}$.

If a bounded directed sequence converges in the sense of Moore-Smith, then the limit coincides with its Banach limit in our sense. In other words, this Banach limit is a generalization of Moore-Smith's. And the concept of Banach limits in a semi-reflexive locally convex space is an extension of Banach limits for the real space.

Now as an application of Banach limits, we shall show that there exists a linear mapping from the space of the all bounded functions with ranges in E into the space of the all completely additive set functions with ranges in E.

Let $\mathfrak{M}$ denote σ-algebra of subsets of a space S and m a completely additive positive measure on $\mathfrak{M}$. The totality of finite divisions of S is a directed set with the following order $\prec$ or $\succ$. The order relation $\Delta_1 \prec \Delta_2$ means Δ_2 is a subdivision of Δ_1. A function defined on S is *bounded* if its range is bounded in E. The class of all bounded functions defined on S with range in E is denoted by $\mathfrak{B}$. Then $\mathfrak{B}$ is a vector space.

THEOREM 2. *To each bounded function x defined on S with range in E and to each set σ in $\mathfrak{M}$, we can determine an element $I(x, \sigma)$ in E so that the following conditions* (2.1)$\sim$(2.3) *may be fulfilled*;

$$I(x, \sigma) \in m(\sigma)\overline{Co}\, x(\sigma) , \tag{2.1}$$

(2.2) *for every fixed $\sigma \in \mathfrak{M}$, $I(x, \sigma)$ is a linear mapping from $\mathfrak{B}$ into E,*
(2.3) *for every fixed $x \in \mathfrak{B}$, $I(x, \sigma)$ is a completely additive set function defined on $\mathfrak{M}$.*

PROOF. From Zermelo's axiom of choice, we can take a function of choice φ defined on $\mathfrak{M}$. For an arbitrary $x \in \mathfrak{B}$ and $\Delta\,; S = \sigma_1 \cup \cdots \cup \sigma_{n_\Delta}$ we shall define an element in E as follows

$$J_\Delta(x) = \sum_{j=1}^{n_\Delta} m(\sigma_j) x(\varphi(\sigma_j)) .$$

We immediately obtain the following property (2.4) because $\varphi(\sigma_j)$ is depends only on the set $\sigma_j \in \mathfrak{M}$;
(2.4) *For every fixed Δ, $J_\Delta(x)$ is a linear mapping from $\mathfrak{B}$ into E.* $\{J_\Delta(x)\}$ is a directed sequence. Moreover, since $x \in \mathfrak{B}$ is bounded, $\{J_\Delta(x)\}$ is also

bounded. Then using Theorem 1 we can define the Banach limit for $\{J_\Delta(x)\}$. Let χ_σ denote the characteristic function of σ. Then, for an arbitrary $x \in \mathfrak{B}$ and $\sigma \in \mathfrak{M}$, we shall define the product of χ_σ and x as follows

$$\chi_\sigma x(s) = \chi_\sigma(s) x(s)$$

for every $s \in S$. $\chi_\sigma x \in \mathfrak{B}$ is obvious. We put

$$I(x, \sigma) = \operatorname*{Lim}_{\Delta} J_\Delta(\chi_\sigma x) .$$

Now we shall show that $I(x, \sigma)$ satisfies the properties (2.1)$\sim$(2.3). Let Δ_σ be the division; $S = \sigma \cup \sigma^c$. For every subdivision of Δ_σ, we have always

$$J_\Delta(\chi_\sigma x) \in m(\sigma) \overline{Co}\, x(\sigma) .$$

Using the property (1.2) of Banach limits, we obtain

$$I(x, \sigma) = \operatorname*{Lim}_{\Delta} J_\Delta(\chi_\sigma x) \in \bigcap_{\Delta'} \overline{Co}\{J_\Delta(\chi_\sigma x) ; \Delta \succ \Delta'\}$$

$$\subset \bigcap_{\Delta' \succ \Delta_\sigma} \overline{Co}\{J_\Delta(\chi_\sigma x) ; \Delta \succ \Delta'\} \subset m(\sigma) \overline{Co}\, x(\sigma) .$$

Thus (2.1) is proved. The proof of (2.2) is almost obvious from (2.4) and (2.1). The proof of (2.3) is established by the usual method using the boundedness of $x \in \mathfrak{B}$, the condition (2.1), the complete additivity of the measure m and the finite additivity of $I(x, \sigma)$ for every fixed $x \in \mathfrak{B}$ which is easily obtained.

The correspondence I in Theorem 2 gives not only the linear mapping from $\mathfrak{B}$ into the family of completely additive set functions defined on $\mathfrak{M}$ with ranges in E, as mentioned before, but also completely additive set mapping from $\mathfrak{M}$ into E'.

References

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