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On the Poisson Distribution Derived from Independent Random Walks

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1. Introduction. J.L. Doob has shown in his book [1] that a Poisson distribution of points over the real line is invariant under independent displacements of the points when the displacements are governed by the same probability law. This shows that when free Brownian particles are distributed "at random" at a time $t=t_0$, the same will be true for $t>t_0$. The object of this note is to clarify this situation for a system of independent particles whose movements are governed by a probability law satisfying fairly general conditions.

2. The theorem. Let $F(x)$ be the distribution function of a non-negative random variable, $F(-0)=0$, $F(\infty)=1, \cdots x_{-1}, x_0, x_1, x_2, \cdots$ be points on the real line such that $\cdots<x_{-1}<x_0<x_1<\cdots$, $x_0\equiv 0$, and let $\{x_i-x_{i-1};\ i=0,1,2,\cdots\}$ be independent random variables with common distribution function $F(x)$. These points are said to be *distributed at random according to* $F(x)$. The equi-distant distribution of points $x_i-x_{i-1}=d$, where $d>0$ is a constant, is included in this case with $F(d+0)-F(d-0)=1$. Consider a system of particles which start from initial random positions x_n and change their positions independently each other with temporally homogeneous independent displacements. The coordinate $Y_n(t)$ of the n-th particle at a time t is then represented in the form

$$Y_n(t)=x_n+X_n(t),\ X_n(0)=0,\ t\geq 0\,.$$

In the following we confine ourselves to the discrete time parameter $t=0,1,2,\cdots$, the continuous case requiring no essential change of the arguments. We shall prove the

Theoreme. *Let* $\{x_n,\ n=0,\ \pm 1,\ \pm 2,\cdots\}$ *be the initial positions of the particles* P_n *distributed at random according to* $F(x)$ *with finite mean* m:

$$0<m=\int_0^\infty x dF(x)<\infty\,.$$

Suppose that $P_n, n=0, \pm 1, \pm 2,\cdots$, *start from* $x_n, n=0, \pm 1, \pm 2\cdots$, *and move in such a way that the movements are independent each other, the displacements* $X_n(t)-X_n(t-1)$, $t=1,2,\cdots$, *of* P_n *are independent and obey the same non-lattice probability law* $G(x)$, *and* $\{X_n(t),\ n=0,\ \pm 1,\cdots\}$ *is independent of* $\{x_n,\ n=0,\ \pm 1,\cdots\}$.

Then if we denote by $N_I(t)$ the number of particles lying in an interval $I=(a, b)$ at t, we have

$$\lim_{t\to\infty} \Pr\{N_I(t)=k\} = e^{-\mu}\frac{\mu^k}{k!},\ k=0, 1, 2, \cdots,\ \mu=(b-a)/m.$$

Proof. Let us put

$$(2.1)\qquad H(x)=1 \quad \text{if } a\leq x\leq b, \qquad =0 \quad \text{otherwise}.$$

Then we can write

$$N_I(t) = \sum_{n=-\infty}^{\infty} H(Y_n(t)).$$

We shall first prove that

$$(2.2)\qquad E\{N_I(t)\} < \infty \qquad \text{for all } t\geq 0.$$

and

$$(2.3)\qquad \lim_{t\to\infty} E\{N_I(t)\} = \mu.$$

For this purpose we introduce a non-negative smooth function $H(x)$ with its Fourier transform $h(t)$ satisfying (ii), (iii) of § 2, [2]. Suppose first that $F(x)$ is a non-lattice distribution and write

$$\varphi(u)=\int_{-\infty}^{\infty} e^{iux}dF(x),$$

$$\psi(u)=\int_{-\infty}^{\infty} e^{iux}dG(x).$$

Then, since $\{x_n, n=0, \pm 1, \cdots\}$, $\{X_0(t), t\geq 0\}$, $\{X_1(t), t\geq 0\}$, $\{X_{-1}(t), t\geq 0\}, \cdots$ are independent and

$$\begin{aligned} E\{e^{iux_n}\} &= E\{\exp[iu \textstyle\sum_1^n (x_j-x_{j-1})]\} \\ &= \varphi^n(u) \qquad \text{if } n\geq 0, \\ &= \varphi^n(-u) \qquad \text{if } n\leq 0, \end{aligned}$$

we have

$$\begin{aligned} E\{H(Y_n(t))\} &= E\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iu(x_n+X_n(t))} h(u)\,du\right\} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi^n(u)\psi^t(u)h(u)du \qquad \text{if } n\geq 0, \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi^n(-u)\psi^t(u)h(u)du \qquad \text{if } n\leq 0. \end{aligned}$$

Therefore if we write

$$N_{H,\rho}(t) = \sum_{-\infty}^{\infty} \rho^{|n|} H(Y_n(t))$$

we have

$$E\{N_{H,\rho}(t)\}=\frac{1}{2\pi}\int_{-c}^{c}\left\{1+\frac{\rho\varphi(u)}{1-\rho\varphi(u)}+\frac{\rho\varphi(-u)}{1-\rho\varphi(-u)}\right\}h(u)\psi^t(u)du$$

$$(2.4)\qquad =\frac{1}{\pi}\int_{-c}^{c}\frac{1-\rho}{Q(\rho,u)}h(u)\psi^t(u)du$$

$$+\frac{\rho}{\pi}\int_{-c}^{c}\frac{1-a(u)}{Q(\rho,u)}h(u)\psi^t(u)du-\frac{1}{2\pi}\int_{-c}^{c}h(u)\psi^t(u)du$$

$$=I(\rho,t)+J(\rho,t)+K(t),$$

where we put

$$a(u)=\int_0^\infty \cos uxdF(x),\quad Q(\rho,u)=|1-\rho\varphi(u)|^2.$$

The analysis used in §2, [2] is applicable to (2.3). First by (2.8)–(2.13) there we have

$$\lim_{\rho\to 1-0} I(\rho,t)=\frac{h(0)}{m}$$

and also

$$\lim_{\rho\to 1-0} J(\rho,t)=\frac{1}{\pi}\int_{-c}^{c}\frac{1-a(u)}{Q(u)}h(u)\psi^t(u)du,$$

where $Q(u)=Q(1,u)$. Hence, if we remember that $|\psi(u)|<1$ for $u\neq 0$, we have

$$\lim_{t\to\infty}\lim_{\rho\to 1-0} J(\rho,t)=0,\quad \lim_{t\to\infty}K(t)=0,$$

and so

$$(2.5)\qquad \lim_{\rho\to 1-0} E(N_{H,\rho}(t))=E(N_H(t))<\infty,$$

$$(2.6)\qquad \lim_{t\to\infty} E(N_H(t))=\frac{h(0)}{m}=\frac{1}{m}\int_{-\infty}^{\infty}H(x)dx.$$

Reduction of (2.5), (2.6) to those with the $H(x)$ in (2.1), that is (2.2) and (2.3), can be done in the same manner as in [2]. We define continuous non-negative functions $H_0(x)$, $H_0^*(x)$, $H_1(x)$, $H_2(x)$, $\overline{H}_2(x)$, and $H_3^*(x)$ in such a way: $H_0(x)$ vanishes outside a finite interval,

$$o<H_0(x)-H(x)\qquad \text{for}\quad x\in I=(a,b),$$

$$\int_{-\infty}^{\infty}(H_0(x)-H(x)dx<\eta;$$

put

$$H_0^*(x)=(H_0*K_\lambda)(x)\quad (*\text{ on the right means convolution});$$

for an interval $J\supset I$ we let

$$H_1(x)=0\qquad \text{if}\quad x\bar{\in} J,$$

$$=H_0^*(x)-H(x)\qquad \text{if}\quad x\in J,$$

$$H_2(x)=(H_0^*(x)-H(x))-H_1(x);$$

let

$$0 < H_3(x) - H_1(x) \qquad \text{for} \quad x \in J,$$
$$\int_{-\infty}^{\infty} (H_3(x) - H_1(x))dx < \eta\,;$$

and put

$$H_3^*(x) = (H_3 * K_\lambda)(x)\,;$$

$\overline{H}_2(x)$ is the same as in [2]. The condition that $H_0(x)$, $H_1(x)$ etc. are even functions is not essential to the argument of reduction and it applies to the present case having

$$E\{N_I(t)\} < \infty \qquad \text{for all } t \geq 0\,;$$

and

$$\lim_{t\to\infty} E\{N_I(t)\} = \frac{1}{m}\int_{-\infty}^{\infty} H(x)dx = \mu.$$

This proves (2.2) and (2.3).

Now

$$\begin{aligned} &E\{\exp[izN_I(t)]\} \\ &\quad = E\{\exp[iz\sum_{-\infty}^{\infty} N_I(Y_n(t))]\} \\ &\quad = \prod_{-\infty}^{\infty} E\{\exp[izN_I(Y_n(t))]\} = \prod_{-\infty}^{\infty} \{1 + p_n(t)(e^{iz}-1)\}, \end{aligned}$$

where

$$p_n(t) = \Pr\{H(Y_n(t)) = 1\} = E\{H(Y_n(t))\}.$$

If we note that for sufficiently large λ there holds

$$H_0^*(x) > H(x), \quad -\infty < x < \infty,$$

we can easily obtain

$$p_n(t) \leq E\{H_0^*(Y_n(t))\} \leq \frac{1}{2\pi}\int_{-\infty}^{\infty} |\, h_0^*(u)\psi^t(u)\,|\, du\,, \tag{2.7}$$

where $h_0^*(u)$, the Fourier transform of $H_0^*(x)$, vanishes outside a finite interval. Therefore we get

$$p_n(t) \to 0\,, \quad \text{as } t \to \infty\,, \tag{2.8}$$

uniformly in n. So that

$$\begin{aligned} E\{\exp[izN_I(t)]\} &= \exp[\sum_{-\infty}^{\infty} \log\{1 + p_n(t)(e^{iz}-1)\}] \\ &= \exp[\sum_{-\infty}^{\infty} p_n(t)(e^{iz}-1) + o(1)\sum_{-\infty}^{\infty} p_n(t)] \\ &\to \exp[\mu(e^{iz}-1)]\,, \qquad t \to \infty\,, \end{aligned}$$

by (2.3) and (2.8). This proves the theorem for non-lattice $F(x)$.

When $F(x)$ is a lattice distribution with maximum span $d > 0$, then $\varphi(u)$ has $2\pi/d$ but no other smaller number as a period. We can write now, for the smooth function $H(x)$ introduced in the above

$$
\begin{aligned}
&E\{N_{I,\rho}(t)\}\\
&=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1-\rho a(u)}{Q(\rho,u)}h(u)\psi^t(u)du-\frac{1}{2\pi}\int_{-\infty}^{\infty}h(u)\psi^t(u)du\\
&=\frac{1}{\pi}\int_{-\pi/d}^{\pi/d}\sum_{\nu=-\infty}^{\infty}h\left(u+\frac{2\nu\pi}{d}\right)\psi^t\left(u+\frac{2\nu\pi}{d}\right)\frac{1-\rho}{Q(\rho,u)}du \qquad (2.9)\\
&\qquad+\frac{\rho}{\pi}\int_{-\pi/d}^{\pi/d}\sum_{\nu=-\infty}^{\infty}h\left(u+\frac{2\nu\pi}{d}\right)\psi\left(u+\frac{2\nu\pi}{d}\right)\frac{1-a(u)}{Q(\rho,u)}du\\
&\qquad-\frac{1}{2\pi}\int_{-\infty}^{\infty}h(u)\psi^t(u)du\\
&=I(\rho,t)+J(\rho,t)+K(t)\,.
\end{aligned}
$$

We get as before

$$\lim_{\rho\to 1-0} I(\rho,t)=\frac{1}{m}\sum_{-\infty}^{\infty}h\left(\frac{2\nu\pi}{d}\right)\psi^t\left(\frac{2\nu\pi}{d}\right), \tag{2.10}$$

$$\lim_{\rho\to 1-0} J(\rho,t)=\frac{1}{\pi}\int_{-\pi/d}^{\pi/d}\sum_{\nu=-\infty}^{\infty}h\left(u+\frac{2\nu\pi}{d}\right)\psi^t\left(u+\frac{2\nu\pi}{d}\right)\frac{1-a(u)}{Q(u)}du\,, \tag{2.11}$$

$$\lim_{t\to\infty} K(t)=0\,. \tag{2.12}$$

It should be noted that the summations in (2.9)-(2.11) are really extended over a finite number of integers. Hence the right-hand members of (2.10) and (2.11) are finite and

$$E(N_H(t))<\infty \qquad \text{for all } t\geq 0,$$

$$\lim_{t\to\infty}\lim_{\rho\to 1-0} I(\rho,t)=\frac{h(0)}{m},\quad \lim_{t\to\infty}\lim_{\rho\to 1-0} J(\rho,t)=0,$$

so that

$$\lim_{t\to\infty} E(N_I(t))=\frac{h(0)}{m}\,.$$

The remaining part of the proof is the same as in the non-lattice case. We thus have proved the theorem.

Remarks. (i) From the above arguments it is clear that $X_n(t)-X_n(t-1)$, $t=1,2,\cdots$, need not depend on the same distribution, but we have only to require the relation

$$\psi(t,u)=E\{e^{iX_n(t)u}\}\to 0,\ \text{as } t\to\infty.$$

(ii) Suppose $H(x)$ be a Riemann-integrable function vanishing outside a finite interval. Then using the above results we can easily deduce that

$$
\begin{aligned}
&\lim_{t\to\infty} E(e^{izN_H(t)})\\
&=\exp\left\{\int_{-\infty}^{\infty}(e^{izH(x)}-1)dx\right\}=\exp\left\{\int_{-\infty}^{\infty}(e^{izu}-1)dn(u)\right\}, \qquad (2.13)
\end{aligned}
$$

where

$$n(u) = \text{meas.}\{x | H(x) \leq u\}.$$

(2.13) is the characteristic function of an infinitely divisible distribution.

References

[1] J.L. Doob, *Stochastic Processes*, John Wiley, New York, 1953.

[2] G. Maruyama, Fourier analytic treatment of some problems on the sums of random variables, this issue of this Report.

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