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Fourier Analytic Treatment of Some Problems on the Sums of Random Variables

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1. **Introduction.** A large amount of investigations in the theory of probability has been devoted to study behaviours of sums $S_n=\sum_{\nu=1}^n X_\nu$ $(n=1, 2, \cdots)$ of independent random variables X_ν. We have the central limit theorem in a refined form with an elaborate estimation of the degree of the Gaussian approximation. Under the existence of moments of X_ν up to a necessary order, the central limit theorem and sometimes the estimates of its error term can be used to deduce asymptotic properties of S_n. In many cases we can obtain the same result not appealing to the central limit theorem but directly from the distribution of X_ν.

The well known renewal theorem concerns non-negative variables X_ν (c.f. Feller [11], [12], Doob [8], Täcklind [16]). Blackwell [1], Erdös, Feller and Pollard [8] proved the theorem under the sole condition $E(X)=m\leq\infty$. Chung and Wolfowitz [4] formulated the theorem for X_ν with lattice distribution taking off its positivity, and proved it under the condition $m\leq\infty$. When $m<\infty$ and the distribution of X_ν is not of lattice type the theorem was proved by Chung and Pollard [5] under the additional condition

$$\overline{\lim_{|t|\to\infty}}\ |\varphi(t)|<1. \tag{1.1}$$

Afterward Blackwell [2] proved the theorem for the lattice and non-lattice X_ν without the assumption (1.1). The method of proof adopted by Chung and Pollard is different from others and interesting in the sense that it is purely analytic and uses only a formula involving the characteristic function $\varphi(t)$ of X_ν.

In this paper we introduce a method by means of which we can avoid difficulties caused by the unpleasant behaviour of $\varphi(t)$ near $t=\pm\infty$ when the equality holds instead of the inequality in (1.1). We can apply the method also to determine the limiting distribution of the numbers of S_n taking values in a finite interval. This problem was studied by Feller [11] for lattice variables, and by Chung and Kac [3], Kallianpur and Robbins [14] for variables belonging to the attraction domains of stable laws. When X_ν has the absolute moment of order $2+\delta$, $\delta>0$, we need Cramér's approximation [7], [10, p. 44], to the characteristic function of S_n, instead of the Cramér-Esseen refinenment [10] of Liapounov's theorem. This situation is exemplified in the simplified proofs of Theorem 3 due to Chung and Kac.

In the last section we consider a Gaussian stationary time series having the spectral density $f(\lambda)$. The limiting distribution depends on values taken by $f(\lambda)$ near $\lambda=0$. This is in a close connection with the growth of S_n, as $n\to\infty$. Indeed, as was shown in [15], the growth of S_n is much affected by the vanishing character of $f(\lambda)$ near $t=0$. A thorough investigation in this direction has been done by Hunt [13].

Throughout this paper we consider only non-lattice variables. The same method also applies to lattice variables with minor modifications and it will be sufficient to notice the following points specific to this case. The characteristic function $\varphi(t)$ is periodic, say with period $T>0$, and we may conveniently divide arising integrals involving $\varphi(t)$ into those over the intervals with length T. At the final step of evaluation we may make use of Poisson's summation formula.

2. Proof of the renewal theorem. To avoid difficulties stated in §1 caused by the behabiour near $t=\pm\infty$, we use convolution transforms with Fejér's kernel and the following simple lemma is required.

Lemma 1. *Let $W(x)$, $-\infty<x<\infty$, be a non-negative bounded measurable function such that $W(x)=O(x^{-2})$, as $|x|\to\infty$. Then we can choose a constant c_0 such that we have*

$$c_0\int_{-\infty}^{\infty}\frac{1}{1+x^2}K_\lambda(y-x)dx\geq W(y),\quad -\infty<y<\infty, \tag{2.1}$$

for all sufficiently large λ, where we put

$$K_\lambda(x)=\frac{\sin^2\lambda x/2}{\pi\lambda x^2/2}.$$

Proof. The integral of (2.1) is equal to

$$\frac{1}{2}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}e^{itx}e^{-|t|}dt\right)\frac{\sin^2\lambda(y-x)/2}{\pi\lambda(y-x)^2/2}dx$$

$$=\frac{1}{2}\int_{-\infty}^{\infty}e^{-|t|+ity}dt\int_{-\infty}^{\infty}e^{ity}\frac{\sin^2\lambda x/2}{\lambda x^2/2}dx$$

$$=\frac{1}{2}\int_{-\lambda}^{\lambda}e^{-|t|+ity}\left(1-\frac{|t|}{\lambda}\right)dt=\frac{1}{1+y^2}-\frac{(1-e^{-\lambda}\cos\lambda y)(1-y^2)-2ye^{-\lambda}\sin\lambda y}{\lambda(1+y^2)^2}$$

$$=\frac{1}{1+y^2}(1+O(\lambda^{-1})).$$

This proves the lemma.

Theorem 1. (Chung and Pollard) *Let $X_1, X_2,\dots$ be identically distributed independent random variables of non-lattice type, with mean* *value* $m=E(X_1)$, $0<m\leq\infty$, *and let* $N(a, a+h)$, $h>0$, *be the expected number of* S_n, $1\leq n<\infty$, *belonging to the interval* $(a, a+h)$. *Then* $N(a, a+h)<\infty$ *for all* a, h, *and*

$$(\alpha)\quad \lim_{a\to\infty} N(a, a+h)=\frac{h}{m} \quad \text{if } 0<m<\infty\,,$$

$$(\beta)\quad \lim_{a\to-\infty} N(a, a+h)=0 \quad \text{if } 0<m\leq\infty\,,$$

$$(\gamma)\quad \lim_{a\to\infty} N(a, a+h)=0 \quad \text{if } m=\infty\,.$$

Proof. We shall first prove (α). Define

$$(2.2)\qquad \begin{aligned} H(x)&=1 &&\text{if } |x|\leq h/2\,,\\ &=0 &&\text{if } |x|>h/2\,, \end{aligned}$$

Then we can write

$$(2.3)\qquad N(a, a+h)=E\left\{\sum_{\nu=1}^{\infty} H(S_\nu-a-h/2)\right\}.$$

From this we can expect that if we denote by $N(a, a+h)$ the right-hand member of (2.3) for a function H with sufficient regularities, there will hold

$$(2.4)\qquad \lim_{a\to\infty} N_H(a, a+h)=\frac{1}{m}\int_{-\infty}^{\infty} H(x)dx\,.$$

Consider now the formal Fourier transformations

$$(2.5)\qquad \int_{-\infty}^{\infty} H(x)e^{-itx}dx=h(t)\,,$$

$$(2.6)\qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} h(t)e^{itx}dt=H(x)\,.$$

We shall prove first that (2.4) is true if (i) $H(x)$ is a real-valued even function (ii) the integrals (2.5), (2.6) are absolutely convergent, and the equalities hold for all t and x, and (iii) $h(t)$ vanishes outside a finite interval $(-c, c)$. (i) is not essential and assumed only to avoid trivial difficulties. In the arguments of § 3 and § 4 it is not assumed.

Let $F(x)$ and $\varphi(t)$ be the distribution function and characteristic function of X_ν and let us write $G(x)=F(x+m)$. Then as in [5] if we introduce a convergence factor $0<\rho<1$, we can write

$$N_H(a-h/2, a+h/2)=\lim_{\rho\to1-0}\sum_{\nu=1}^{\infty}\rho^\nu E\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iat}h(t)e^{iS_\nu t}dt\right\}$$

$$(2.7)\qquad =\lim_{\rho\to1-0}\frac{1}{2\pi}\sum_{1}^{\infty}\rho^\nu\int_{-\infty}^{\infty} h(t)e^{-iat}\varphi^\nu(t)dt$$

$$=\lim_{\rho\to1-0}\int_{-c}^{c}\frac{\rho\varphi(t)}{1-\rho\varphi(t)}h(t)e^{-iat}dt\,.$$

Now if we write

$$\varphi(t)=e^{imt}\sigma(t),\ \sigma(t)=\int_{-\infty}^{\infty}\cos xtdG(x)+\int_{-\infty}^{\infty}\sin xtdG(x)$$
$$=u(t)+iv(t)\,,$$

we have the following relations

(2.8) $$u(t)=1+o(t),\quad v(t)=o(t),\quad u'(t)=o(1),\quad v'(t)=o(1),$$

$$Q(\rho,t)=|1-\rho\varphi(t)|^2=(1-\rho)^2+2(1-\rho)\rho(1-u(t))+\rho^2(1-u(t))^2$$

(2.9) $$+4\rho\sin^2\frac{mt}{2}+2\rho v(t)\sin mt+\rho^2v^2(t)$$

$$=(1-\rho)^2+4\rho\sin^2\frac{mt}{2}+(1-\rho)o(t)+o(t^2)\,,$$

(2.10) $$Q(t)=Q(1,t)=m^2t^2+o(t^2),\quad Q'(t)=2m^2t+o(t),$$

as $|t|\to 0$.

Take $\varepsilon>0$ sufficiently small and divide the integral in (2.7) into

$$\frac{1}{2\pi}\left[\int_{c\geq|t|\geq\varepsilon}+\int_{-\varepsilon}^{\varepsilon}\right]\frac{\rho\varphi(t)}{1-\rho\varphi(t)}h(t)e^{-iat}dt=I(a)+J(a)\,.$$

Then the Riemann-Lebesgue theorem gives

$$\lim_{a\to\infty}\lim_{\rho\to 1-0}I(a)=0$$

for every $\varepsilon>0$. Now, since $\varphi(-t)=\overline{\varphi(t)}$, $h(-t)=\overline{h(t)}=h(t)$

$$J(a)=\frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon}\mathcal{R}\left\{\frac{\rho\varphi(t)}{1-\rho\varphi(t)}h(t)e^{-iat}\right\}dt$$

(2.11) $$=\frac{\rho}{2\pi}\int_{-\varepsilon}^{\varepsilon}\frac{1-\rho}{Q(\rho,t)}\mathcal{R}\{\varphi(t)h(t)e^{-iat}\}dt$$

$$+\frac{\rho^2}{2\pi}\int_{-\varepsilon}^{\varepsilon}\frac{1}{Q(\rho,t)}\mathcal{R}\{(1-\overline{\varphi}(t))\varphi(t)h(t)e^{-iat}\}dt\,.$$

Putting $f(t)=\mathcal{R}(\varphi(t)h(t)e^{-iat})$, rewrite the first integral of (2.11) as

(2.12) $$\frac{\rho f(0)}{2\pi}\int_{-\varepsilon}^{\varepsilon}\frac{1-\rho}{Q(\rho,t)}dt+\frac{\rho}{2\pi}\int_{-\varepsilon}^{\varepsilon}\frac{1-\rho}{Q(\rho,t)}(f(t)-f(0))dt=J_1(a)+J_2(a)\,,$$

and recall (2.8) and (2.9), then

$$\frac{\rho f(0)}{2\pi}\int_{-\varepsilon}^{\varepsilon}\frac{1-\rho}{(1-\rho)^2+m^2(1-\eta)t^2}dt>J_1(a)$$

$$>\frac{\rho f(0)}{2\pi}\int_{-\varepsilon}^{\varepsilon}\frac{1-\rho}{(1-\rho)^2+m^2(1+\eta)t^2+\eta|t|(1-\rho)}dt\,,$$

where $\eta>0$ is a constant which can be made as small as we please with ε. Therefore, since $f(0)=h(0)$, on making $\rho\to 1-0$, we see that

$$(2.13)\qquad \frac{h(0)}{2m}(1+\eta')\geq \overline{\lim_{\rho\to1-0}}\, J_1(a)\geq \underline{\lim_{\rho\to1-0}}\, J_1(a)\geq \frac{h(0)}{2m}(1-\eta')$$

where η' is independent of a and $\eta'\to0$, as $\eta\to0$. Next, it is easy to show that

$$(2.14)\qquad \lim_{\rho\to1-0} J_2(a)=0\,.$$

The second integral of (2.11) can be divided into

$$\frac{\rho^2}{\pi}\int_0^\varepsilon \frac{\cos(m-a)t-\cos at}{Q(\rho,t)}u(t)h(t)dt+\frac{\rho^2}{\pi}\int_0^\varepsilon \frac{(1-u(t))\cos at}{Q(\rho,t)}u(t)h(t)dt$$
$$-\frac{\rho^2}{\pi}\int_0^\varepsilon \frac{v(t)\sin(m-a)t}{Q(\rho,t)}h(t)dt-\frac{\rho^2}{\pi}\int_0^\varepsilon \frac{v^2(t)}{Q(\rho,t)}\cos at\,h(t)dt$$
$$=I_1(a,\rho)+I_2(a,\rho)-I_3(a,\rho)-I_4(a,\rho)\,.$$

From (2.9), (2.10), we get

$$o<\frac{1}{Q(\rho,t)},\quad \frac{1}{Q(1,t)}<\frac{1}{\sin^2 mt/2}$$

for $|t|<\varepsilon$, ε small, and we observe that

$$\int_0^\varepsilon \frac{1-u(t)}{\sin^2 mt/2}dt\leq\frac{5}{m^2}\int_{-\infty}^\infty |x|\,dG(x)\int_0^\infty \frac{\sin^2 t}{t^2}dt<\infty\,.$$

The Riemann-Lebesgue theorem is then applicable to I_2 and I_4, thus having

$$\lim_{a\to\infty}\lim_{\rho\to1-0}(I_2+I_4)=0\,.$$

Next

$$(2.15)\qquad \lim_{\rho\to1-0} I_3(a,\rho)=\frac{1}{\pi}\int_0^\varepsilon \frac{\sin at\, v(t)}{Q(\rho,t)}dt+o(1)=I_3(a)+o(1)\,,$$

as $a\to\infty$, and

$$I_3(a)=\frac{1-\cos a\varepsilon}{\pi a}\,\frac{v(\varepsilon)}{Q(\varepsilon)}-\frac{1}{2\pi}\int_0^\varepsilon \frac{1}{a}\left(\frac{\sin at/2}{t/2}\right)^2\frac{v'(t)Q(t)-Q'(t)v(t)}{Q^2(t)}t^2dt\,.$$

But since from (2.10) we get

$$\frac{v'(t)Q(t)-Q'(t)v(t)}{Q^2(t)}t^2=o(1)\,,\quad |t|\to0\,,$$

we have

$$(2.16)\qquad I_3(a)=O\left(\frac{1}{a}\right)+\int_0^\infty\left(\frac{\sin t/2}{t/2}\right)^2 o(1)dt\to0\,,\quad a\to\infty\,.$$

Finally

$$(2.17)\qquad \lim_{\rho\to1-0} I_1(a,\rho)=\frac{2}{\pi}\int_0^\varepsilon \frac{\sin at\sin mt/2}{Q(t)}h(0)dt+o(1)=I_1(a)+o(1)\,,$$

as $a\to\infty$, and

$$I_1(a)=\frac{2(1-\cos a\varepsilon)}{\pi a}\frac{\sin m\varepsilon/2h(0)}{Q(\varepsilon)}$$
$$-\frac{2}{\pi}h(0)\int_0^{\varepsilon}\frac{2\sin^2 at/2}{at^2}\frac{m/2\cos mt/2-\sin mt/2\,Q'(t)}{Q^2(t)}t^2dt\,.$$

But from (2.10) we get

$$\frac{\frac{m}{2}\cos\frac{mt}{2}Q(t)-\sin\frac{mt}{2}Q'(t)}{Q^2(t)}t^2=-\frac{1}{2m}+o(1)\,,\quad |t|\to 0\,.$$

Therefore

$$I_1(a)=O(a^{-1})+\frac{h(0)}{2m}\frac{1}{\pi}\int_0^{\varepsilon}\frac{1}{a}\left(\frac{\sin at/2}{t/2}\right)^2(1+o(1))dt \tag{2.18}$$
$$\to\frac{h(0)}{2m}\,,\ \text{as } a\to\infty\,.$$

(2.13) - (2.18) give us $N_H(a, a+h)<\infty$ and

$$\lim_{a\to\infty}N_H(a, a+h)=\frac{h(0)}{m}=\frac{1}{m}\int_{-\infty}^{\infty}H(x)dx\,.$$

It remains to show that the above relation holds also for (2.2). First we define an even non-negative continuous function $H_0(x)$, $-\infty<x<\infty$, such that it vanishes outside a finite interval $(-k, k)$ and

$$0<H_0(x)-H(x)\qquad \text{for}\quad |x|\leq h/2\,,$$
$$\int_{-\infty}^{\infty}(H_0(x)-H(x))dx<\eta\,,$$

with sufficiently small $\eta>0$, and the let us define

$$H_0^*(x)=\int_{-\infty}^{\infty}H_0(x-y)K_\lambda(y)dy\,.$$

Then

$$0<\int_{-\infty}^{\infty}(H_0^*(x)-H(x))dx=\int_{-\infty}^{\infty}(H_0(x)-H(x))dx\,, \tag{2.19}$$

and if λ is sufficiently large

$$H(x)\leq H_0^*(x)\,,\quad -\infty<x<\infty\,. \tag{2.20}$$

Next take $X>h>0$ sufficiently large and define $H_1(x)$, $H_2(x)$ such that

$$H_1(x)=0 \qquad \text{if}\quad |x|\geq X,$$
$$=H_0^*-H(x) \qquad \text{if}\quad |x|<X,$$
$$H_0^*(x)-H(x)=H_1(x)+H_2(x),\ -\infty<x<\infty\,.$$

Then by the definition, for $|x|\geq X$ we have

$$H_2(x)=H_0{}^*(x)$$
$$\leq c_0\int_{-k}^{k} K_\lambda(x-y)dy=c_0\int_{-k-x}^{k-x} K_\lambda(y)dy$$
$$\leq \frac{2c_0}{\pi\lambda}\int_{-k-x}^{k-x}\frac{dy}{y^2}\leq\frac{c_1}{\lambda(1+x^2)},$$

where c_0, c_1 are constants independent of x and λ. Since $H_2(x)$ vanishes interior to $(-X, X)$ we have by Lemma 1

$$H_2(x)\leq\frac{c_2}{\lambda}\int_{-\infty}^{\infty}\frac{1}{1+y^2}K_{\lambda'}(x-y)dy=\overline{H}_2(x)$$

with a constant c_2 and sufficiently large $\lambda'>0$. Further we define a continuous even function $H_3(x)$ such that it vanishes outside a finite interval,

$$0<H_3(x)-H_1(x),\quad |x|\leq X,$$

and

$$\int_{-\infty}^{\infty}(H_3(x)-H_1(x))dx\leq\eta,$$

and let

$$H_3{}^*(x)=\int_{-\infty}^{\infty}H_3(x-y)K_\lambda(y)dy.$$

Then, if λ is sufficiently large, (2.19) and (2.20) hold with $H_0{}^*(x)$, $H(x)$ replaced by $H_3{}^*(x)$ and $H_1(x)$.

Now by (2.20)

$$(2.21)\qquad 0<N_H(a, a+h)\leq N_{H_0{}^*}(a,a+h),$$

$$(2.22)\qquad \begin{aligned}0<N_{H_0{}^*}(a, a+h)-N_H(a, a+h)&=N_{H_0{}^*-H_0}(a, a+h)\\ &=N_{H_2}(a, a+h)+N_{H_1}(a, a+h)\leq N_{\overline{H}_2}(a, a+h)+N_{H_3{}^*}(a, a+h).\end{aligned}$$

$H_0{}^*(x)$, $\overline{H}_2(x)$, $H_3{}^*(x)$ and their Fourier transforms $h_0^*(t)$, $\overline{h}_2(t)$, $h_3^*(t)$ satisfy (i)-(iii). Therefore by the results just proved applied to (2.21) $N_H(a, a+h)<\infty$, and we have

$$\lim_{a\to\infty}(N_{\overline{H}_2}(a, a+h)+N_{H_3{}^*}(a, a+h))=\frac{1}{m}\left(\int_{-\infty}^{\infty}\overline{H}_2(x)dx+\int_{-\infty}^{\infty}H_3{}^*(x)dx\right)$$
$$\leq O(\lambda^{-1})+\frac{1}{m}\int_{-\infty}^{\infty}H_1(x)dx+\frac{\eta}{m}\leq O(\lambda^{-1})+\frac{\eta}{m}+\frac{1}{m}\int_{-\infty}^{\infty}(H_0{}^*(x)-H(x))dx$$
$$\leq O(\lambda^{-1})+\frac{2\eta}{m},$$

and

$$\lim_{a\to\infty}N_{H_0{}^*}(a, a+h)=\frac{1}{m}\int_{-\infty}^{\infty}H_0{}^*(x)dx\leq\frac{h}{m}+\frac{\eta}{m}.$$

Hence

$$0<\overline{\lim_{a\to\infty}}\, N_H(a, a+h)-\underline{\lim_{a\to\infty}}\, N_H(a, a+h)\leq O(\lambda^{-1})+\frac{2\eta}{m}\,.$$

But since the left-hand side of the above ineguality is independent of λ and η, taking $\lambda^{-1}+\eta$ sufficiently small, we see that there exists

$$\lim_{a\to\infty} N_H(a, a+h)=\frac{h}{m}\,.$$

This proves (α) of the theorem.

We shall next prove (β) and (γ). The proof of (β) for $0<m<\infty$ is contained in the above. We have only to note that in the evaluation of (2.13) $J_1(a)\to -h(0)/2m$, as $a\to\infty$, when $-\infty<m<0$. This proves that $N(a, a+h)\to 0$, as $a\to\infty$, when $-\infty<m<0$, a statement equivalent to (β).

I failed to prove the remaining part of the theorem by means of an analytical method as in the above and was obliged to employ the result obtained by Chung and Fuchs [6].

First we observe that

$$\frac{1}{2}[N_H(a-h/2, a+h/2)+N_H(-a-h/2, -a+h/2)]$$

$$=\lim_{\rho\to 1-0}\frac{1}{2\pi}\int_{-c}^{c}\frac{\rho\varphi(t)}{1-\rho\varphi(t)}h(t)\cos at\,dt$$

$$=\lim_{\rho\to 1-0}\frac{1}{2\pi}\int_{-c}^{c}\mathcal{R}\Big(\frac{1}{1-\rho\varphi(t)}\Big)h(t)\cos at\,dt$$

$$-\frac{1}{2\pi}\int_{-c}^{c}h(t)\cos at\,dt\,.$$

Obviously from the strong law of large numbers, when $m=\infty$, S_n has no recurrent value. Therefore according to [6] the function

$$\mathcal{R}\Big(\frac{1}{1-\varphi(t)}\Big)>0$$

is integrable over any finite interval $(-c, c)$, and

$$\frac{1}{2}[N_H(a-h/2, a+h/2)+N_H(-a-h/2, -a+h/2)]$$

$$=\frac{1}{2\pi}\int_{-c}^{c}\mathcal{R}\Big(\frac{1}{1-\varphi(t)}\Big)h(t)\cos at\,dt-\frac{1}{2\pi}\int_{-c}^{c}h(t)\cos at\,dt$$

$$\to 0,\ \text{as}\ a\to\infty\,.$$

This means that as $a\to\infty$

$$\text{(2.23)}\qquad \begin{aligned} &N_H(a-h/2, a+h/2)\to 0\,,\\ &N_H(-a-h/2, -a+h/2)\to 0\,. \end{aligned}$$

Now we may take

$$H(x)=\frac{\sin^2 \lambda x/2}{\lambda x^2/2} \qquad (\lambda>0).$$

Then since $H(x)>0$ for $\left(-\frac{2\pi}{\lambda}, \frac{2\pi}{\lambda}\right)$, (2.23) means that ($\beta$) and ($\gamma$) hold for $m=\infty$. This completes the proof.

3. Fluctuation of S_n. Also in this section, with an intention to clarify the process of analysis, we shall treat only non-lattice variables. We shall prove

Theorem 2. *Let $X_1, X_2, \ldots$ be identically distributed independent non-lattice variables with mean 0, variance σ^2, and for some $\delta>0$ satisfy*

$$\beta_{2+\delta}=E(|X_1|^{2+\delta})<\infty .$$

Let N_n be the number of $S_\nu=\sum_{j=1}^{\nu} X_j$, $1\le\nu\le n$, belonging to the finite interval $(-h/2, h/2)$, $h>0$.

Then as $n\to\infty$

$$\Pr\left\{N_n\le\frac{\sqrt{n}\,h}{\sigma}x\right\}\to\sqrt{\frac{2}{\pi}}\int_0^x e^{-u^2/2}du, \quad x>0,$$

(truncated normal distribution).

Proof. Let us write

$$N_n=\sum_{\nu=1}^{n} H(S_\nu)=\frac{1}{2\pi}\sum_{\nu=1}^{n}\int_{-\infty}^{\infty} h(t)e^{itx}dt$$

and we shall first obtain, as in § 1 for a special function $H(x)$ satisfying (ii), (iii) there, the limiting distribution of N_n by the method of moments. In the following proof the condition that $H(x)$ is a real-valued even function is not used. We have

$$E\left(\frac{N_n}{\sqrt{n}}\right)=\frac{1}{\sqrt{n}\,2\pi}\sum_{\nu=1}^{n}\int_{-c}^{c} h(t)\varphi^\nu(t)dt$$

$$=\frac{1}{\sqrt{n}\,2\pi\sigma}\sum_{\nu=1}^{n}\frac{1}{\sqrt{\nu}}\int_{-c\sigma\sqrt{\nu}}^{c\sigma\sqrt{\nu}} h\left(\frac{t}{\sigma\sqrt{\nu}}\right)\varphi^\nu\left(\frac{t}{\sigma\sqrt{\nu}}\right)dt .$$

Writing

$$\int_{-c\sigma\sqrt{\nu}}^{c\sigma\sqrt{\nu}} h\left(\frac{t}{\sigma\sqrt{\nu}}\right)\varphi^\nu\left(\frac{t}{\sigma\sqrt{\nu}}\right)dt$$

(3.1)

$$=\left(\int_{-\varepsilon\sqrt{\nu}}^{\varepsilon\sqrt{\nu}}+\int_{c\sigma\sqrt{\nu}\ge|t|\ge\varepsilon\sqrt{\nu}}\right)h\left(\frac{t}{\sigma\sqrt{\nu}}\right)\varphi^\nu\left(\frac{t}{\sigma\sqrt{\nu}}\right)dt,$$

we apply the estimate [10]

$$\left|\varphi\left(\frac{t}{\sigma\sqrt{n}}\right)-e^{-t^2/2}\right|\le\gamma_{2+\delta}\frac{|t|^{2+\delta}}{n^{\delta/2}}e^{-t^2/4} \quad \text{for} \quad |t|\le c_{2+\delta}\sqrt{n},$$

where $\gamma_{2+\delta}$ and $c_{2+\delta}$ are constants depending on σ and $\beta_{2+\delta}$ but independent of n and t. Take $\varepsilon<c_{2+\delta}$ sufficiently small, and remember that $h(t)$ is uniformly continuous, $-\infty<t<\infty$, and indeed we can make $|h(t)-h(0)|=O(|t|)$. Then the first term of the right-hand side of (3.1) is

$$h(0)\int_{-\varepsilon\sqrt{\nu}}^{\varepsilon\sqrt{\nu}} e^{-t^2/2}dt+O\Big(\frac{1}{\nu^{\delta/2}}\Big)\int_0^\infty t^3e^{-t^2/4}dt+O(\varepsilon)=\sqrt{2\pi}\,h(0)+o(1)+O(\varepsilon),$$

where $o(1)$ tends to 0, as $\nu\to\infty$. On the other hand since $\left|\varphi\Big(\frac{t}{\sigma\sqrt{\nu}}\Big)\right|<\rho$ for $\varepsilon\sqrt{\nu}\leq t\leq c\sigma\sqrt{\nu}$ with a constant ρ, $0<\rho<1$, the second term of (3.1) is

$$O(\rho^\nu\sqrt{\nu}),\quad \nu\to\infty.$$

Therefore

$$E\Big(\frac{N_n}{\sqrt{n}}\Big)=\frac{\sqrt{2\pi}}{\sigma 2\pi}h(0)\frac{1}{\sqrt{n}}\sum_{\nu=1}^{n}\frac{1}{\sqrt{\nu}}+\frac{1}{\sqrt{n}}\sum_{\nu=1}^{n}\frac{1}{\sqrt{\nu}}(o(1)+O(\varepsilon)),$$

and hence

$$\lim_{n\to\infty}E\Big(\frac{N_n}{\sqrt{n}}\Big)=\frac{h(0)}{\sigma\sqrt{2\pi}}\int_0^1\frac{dt}{\sqrt{t}}.$$

Next

$$(3.2)\qquad E\Big(\frac{N_n}{\sqrt{n}}\Big)^2=\frac{1}{n}\sum_{k+l\leq n}2E(H(S_k)H(S_{k+l}))+\frac{1}{n}\sum_{k=1}^{n}E(H(S_k))^2.$$

Now, since $h(t)$ is uniformly continuous

$$E(H(S_k)H(S_{k+l}))=\Big(\frac{1}{2\pi}\Big)^2E\Big\{\int_{-\infty}^{\infty}e^{isS_k s}h(s)ds\int_{-\infty}^{\infty}e^{itS_{k+l}t}h(t)dt\Big\}$$

$$=\Big(\frac{1}{2\pi}\Big)^2\iint_{-\infty}^{\infty}\varphi^k(s+t)\varphi^l(t)h(s)h(t)ds\,dt$$

$$=\Big(\frac{1}{2\pi\sigma}\Big)^2\frac{1}{\sqrt{kl}}\Big\{\int_{|t|\leq\varepsilon\sqrt{l}}\varphi^l\Big(\frac{t}{\sigma\sqrt{l}}\Big)h\frac{t}{\sigma\sqrt{l}}\Big)dt+O(\sqrt{l}\,\rho^l)\Big\}$$

$$\cdot\Big\{\int_{|s|\leq\varepsilon\sqrt{k}}\varphi^k\Big(\frac{s}{\sigma\sqrt{k}}\Big)h\Big(\frac{s}{\sigma\sqrt{k}}-\frac{t}{\sigma\sqrt{l}}\Big)ds+O(\sqrt{k}\,\rho^k)\Big\}$$

$$=\Big(\frac{1}{2\pi\sigma}\Big)^2\frac{1}{\sqrt{kl}}(h(0))^2\int_{|t|\leq\varepsilon\sqrt{l}}\varphi^k\Big(\frac{t}{\sigma\sqrt{l}}\Big)dt\int_{|s|\leq\varepsilon\sqrt{k}}\varphi^k\Big(\frac{s}{\sigma\sqrt{k}}\Big)ds(1+\eta)$$

$$=\Big(\frac{h(0)}{\sqrt{2\pi}\,\sigma}\Big)^2\frac{1}{\sqrt{kl}}\big(1+\eta'\big),$$

where η' is small with ε, k^{-1}, l^{-1}. Hence the first term of (3.2) is

$$2\Big(\frac{h(0)}{\sqrt{2\pi}\,\sigma}\Big)^2\frac{1}{n}\sum_{k+l\leq n}\frac{1}{\sqrt{kl}}+o(1).$$

On the other hand, since $H^2(x)$ also satisfies (ii), (iii), the second term of (3.2) vanishes when $n\to\infty$. Therefore

$$\lim_{n\to\infty} E\Big(\frac{N_n}{\sqrt{n}}\Big)^2 = 2!\Big(\frac{h(0)}{\sqrt{2\pi}\,\sigma}\Big)^2 \iint\limits_{0\le t_1\le t_2\le 1} \frac{dt_1dt_2}{\sqrt{t_1(t_2-t_1)}}.$$

In the same way we have

$$\lim_{n\to\infty} E\Big(\frac{N_n}{\sqrt{n}}\Big)^m = m!\Big(\frac{h(0)}{\sqrt{2\pi}\,\sigma}\Big)^m \iint\limits_{0\le t_1\le t_2\le\cdots\le t_m\le 1}\cdots\int \frac{dt_1dt_2\cdots dt_m}{\sqrt{t_1(t_2-t_1)\cdots(t_m-t_{m-1})}}$$

$$=\mu_m\Big(\frac{h(0)}{\sigma}\Big)^m,\quad \mu_m=\sqrt{\frac{2}{\pi}}\int_0^\infty u^m e^{-u^2/2}du.$$

This means that

$$\Pr\Big\{N_n\le\sqrt{n}\,\frac{h(0)}{\sigma}x\Big\}\to\sqrt{\frac{2}{\pi}}\int_0^x e^{-u^2/2}du. \tag{3.3}$$

If we use the idea introduced in the latter half of the proof of Theorem 1, it is easy to complete the proof of the theorem. Write

$$\frac{N_n}{\sqrt{n}}=\frac{\sum_1^n H_0^*(S_\nu)}{\sqrt{n}}-\frac{\sum_1^n\{H_0^*(S_\nu)-H(S_\nu)\}}{\sqrt{n}}$$

$$=\frac{\sum_1^n H_0^*(S_\nu)}{\sqrt{n}}-\frac{\sum_1^n H_1(S_\nu)}{\sqrt{n}}-\frac{\sum_1^n H_2(S_\nu)}{\sqrt{n}},$$

and take $\eta>0$, λ^{-1} small as in § 2, then

$$E\left\{\frac{\sum_1^n H_1}{\sqrt{n}}\right\}\le E\left\{\frac{\sum_1^n H_3^*}{\sqrt{n}}\right\}=O\Big(\int_{-\infty}^\infty H_3(x)dx\Big)+o(1)$$

$$\le O\Big(\int_{-\infty}^\infty H_1(x)dx\Big)+O(\eta)+o(1)$$

$$=O(\eta)+o(1)\le c_3\eta,\quad \text{as } n\to\infty,$$

with some $c_3>0$. Also

$$E\left\{\frac{\sum_1^n H_2}{\sqrt{n}}\right\}\le E\left\{\frac{\sum \overline{H}_2}{\sqrt{n}}\right\}=O(\lambda^{-1})+o(1)\le c_3\eta.$$

Therefore

$$\Pr\left\{\frac{\sum_1^n H_0^*(S_\nu)}{\sqrt{n}}\le x\right\}\le\Pr\left\{\frac{N_n}{\sqrt{n}}\le x\right\}\le\Pr\left\{\frac{\sum_1^n H_0^*(S_\nu)}{\sqrt{n}}\le x+\delta\right\}+\delta,$$

where $\delta>0$ is small with η. We can now apply (3.3) to both extreme terms in the above inequality, having

$$\sqrt{\frac{2}{\pi}}\int_0^{\sigma x/h_0^*(0)} e^{-u^2/2}du \leq \overline{\lim_{n\to\infty}} \Pr\left\{\frac{N_n}{\sqrt{n}} \leq x\right\} \leq \sqrt{\frac{2}{\pi}}\int_0^{\sigma(x+\delta)/h_0^*(0)} e^{-u^2/2}du + \delta .$$

Making $\eta \to 0$ we obtain the required relation.

We shall next prove Chung's theorem on changes of sign of S_n.

Theorem 3. *Let N_n be the number of S_ν, $1 \leq \nu \leq n$, such that $S_\nu \geq 0$, $S_{\nu+1} \leq 0$. Then under the condition of theorem 1*

$$\lim_{n\to\infty} \Pr\left\{N_n \leq \frac{\beta_1}{2\sigma}\sqrt{n}\,x\right\} = \sqrt{\frac{2}{\pi}}\int_0^x e^{-u^2/2}du ,$$

where β_1 is the first absolute moment $\beta_1 = E(|x_1|)$.

Proof. To simplify notations we consider the case $\sigma = 1$. If we put

$$(3.4) \qquad \begin{aligned} U(x) &= 1 \quad \text{if} \quad x \geq 0 \\ &= 0 \quad \text{if} \quad x < 0 \end{aligned} \;; \qquad \begin{aligned} V(x) &= 1 \quad \text{if} \quad x \leq 0 \\ &= 0 \quad \text{if} \quad x > 0 \end{aligned} \;;$$

we can write

$$N_n = \sum_{\nu=1}^n U(S_\nu)V(S_{\nu+1}) .$$

Introducing the auxiliary functions $U_a(x)$ and $V_a(x)$:

$$U_a(x) = 1 \quad \text{if} \quad 0 \leq x \leq a, \; = 0 \quad \text{otherwise};$$

$$V_a(x) = 1 \quad \text{if} \quad -a \leq x \leq 0, \; = 0 \quad \text{oterwise}, \; a > 0,$$

we write

$$(3.5) \qquad N_n = \sum_{\nu=1}^n U_a(S_\nu)V_a(S_{\nu+1}) + R_n ,$$

where

$$R_n = \sum_{\nu=1}^n U(S_\nu)\{V(S_{\nu+1}) - V_a(S_{\nu+1})\} + \sum_{\nu=1}^n V_a(S_{\nu+1})\{U(S_\nu) - U_a(S_\nu)\} .$$

We can see that if a is large R_n is small compared with the first term of (3.5). Indeed if we put $F_k(x) = \Pr\{S_k \leq x\}$ $(F(x) \equiv F_1(x))$

$$E\{U(S_k)(V(S_{k+1}) - V_a(S_{k+1}))\} = \int_0^\infty \Pr\{x + X_{k+1} \leq -a\}\, dF_k(x)$$

$$\leq \int_{|y| \geq a}^\infty y^2 dF(y) \int_0^\infty \frac{dF_k(x)}{(x+a)^2} ,$$

and if $a > 1$, in view of the proof of Theorem 2

$$\int_0^\infty \frac{dF_k(x)}{(x+a)^2} \leq E\left(\frac{1}{1+S_k^2}\right) = O\left(\frac{1}{\sqrt{k}}\right).$$

Therefore

$$E\left(\frac{R_n}{\sqrt{n}}\right)=o(1),$$

where $o(1)$ means a term tending to 0, as $a\to\infty$, uniformly in n.

Consider again formal Fourier transformations

$$u_a(t)=\int_{-\infty}^{\infty}U_a(x)e^{-itx}dx,\quad U_a(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}u_a(t)e^{itx}dt,$$

and similarly for $V_a(x)$ and its transform, $v_a(t)$. As in the proof of Theorem 1 and Theorem 2, we approximate $U_a(x)$ by functions with regularities required there. Thus we consider the limiting distribution of $\sum_{\nu=1}^{n}U(S_\nu)V(S_{\nu+1})$, with U, V satisfying (ii), (iii) of § 2.

First as in the proof of Theorem 1

$$(3.6)\qquad E\{U(S_k)V(S_{k+1})\}=\left(\frac{1}{2\pi}\right)^2\iint_{-\infty}^{\infty}u(s)v(t)\varphi^k(s+t)\varphi(t)ds\,dt$$

$$=\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}v(t)\varphi(t)dt\int_{-\infty}^{\infty}\frac{1}{\sqrt{k}}\varphi^k\left(\frac{s}{\sqrt{k}}\right)u\left(\frac{s}{\sqrt{k}}-t\right)ds$$

$$=\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}v(t)\varphi(t)\frac{1}{\sqrt{k}}\{\sqrt{2\pi}+o(1)\}u(-t)dt$$

$$=\left(\frac{1}{2\pi}\right)^2\sqrt{2\pi}\frac{1}{\sqrt{k}}\int_{-\infty}^{\infty}v(t)u(-t)\varphi(t)\,dt+o\left(\frac{1}{\sqrt{k}}\right),$$

where we should note that the integrals are to be performed over a finite domain, since $u(t)$, $v(t)$ vanish outside some finite intervals. Next

$$(3.7)\qquad E\{U(S_k)V(S_{k+1})U(S_{k+1})V(S_{k+l+l})\}$$

$$=\left(\frac{1}{2\pi}\right)^4\iiiint_{-\infty}^{\infty}\varphi^k(s_1+t_1+s_2+t_2)\varphi(t_1+s_2+t_2)$$

$$\varphi^{l-1}(s_2+t_2)\varphi(t_2)u(s_1)v(t_1)u(s_2)v(t_2)ds_1dt_1ds_2dt_2$$

$$=\left(\frac{1}{2\pi}\right)^4\frac{1}{\sqrt{kl}}\int_{-\infty}^{\infty}\varphi(t_2)v(t_2)dt_2\int_{-\infty}^{\infty}\varphi^{l-l}\left(\frac{s_2}{\sqrt{l}}\right)u\left(\frac{s_2}{\sqrt{l}}-t_2\right)ds_2$$

$$\cdot\int_{-\infty}^{\infty}\varphi(t_1)v\left(t_1-\frac{s_2}{\sqrt{l}}\right)dt_1\int_{-\infty}^{\infty}\varphi^k\left(\frac{s_1}{\sqrt{k}}\right)u\left(\frac{s_1}{\sqrt{k}}-t_1\right)ds_1$$

$$\sim\frac{1}{\sqrt{kl}}\left\{\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\varphi(t)v(t)u(-t)dt\cdot\sqrt{2\pi}\right\}^2,\quad k,\,l\to\infty.$$

In view of (3.4), we may confine ourselves to those $U(x)$, $V(x)$ which satisfy $V(-x)=U(x)$. Then $\varphi(t)v(t)u(-t)$ is the Fourier transform of $W(x)=(U*U*F)(x)$ ($*=$convolution)

$$\varphi(t)v(t)u(-t)=\int_{-\infty}^{\infty}e^{itx}W(x)dx\,,$$

and

(3.8) $$\int_{-\infty}^{\infty}\varphi(t)v(t)u(-t)dt=2\pi W(0)\,.$$

Therefore, from (3.6) and (3.8) we get first

$$\lim_{n\to\infty}E\Big(\frac{N_n}{\sqrt{n}}\Big)=\frac{W(0)}{\sqrt{2\pi}}\int_0^1\frac{dt}{\sqrt{t}}=W(0)\mu_1\,.$$

Second

$$E\Big(\frac{N_n}{\sqrt{n}}\Big)^2=\frac{1}{n}\sum_{k+l\leq n}2E\{U(S_k)V(S_{k+1})U(S_{k+l})V(S_{k+l+1})\}$$

$$+\frac{1}{n}\sum_{k=1}^{n}E\{U(S_k)V(S_{k+1})\}^2,$$

of which the second term is $o(1/\sqrt{n})$, and the first term is from (3.7)

$$\Big(\frac{W(0)}{\sqrt{2\pi}}\Big)^2 2!\,\frac{1}{n}\sum_{1\leq\alpha\leq\beta\leq n}\frac{1}{\sqrt{\alpha(\beta-\alpha)}}+o(1)\,,\quad n\to\infty.$$

Therefore

$$\lim_{n\to\infty}E\Big(\frac{N_n}{\sqrt{n}}\Big)^2=(W(0))^2\mu_2.$$

In the same way, in general it is easy to show that

$$\lim_{n\to\infty}E\Big(\frac{N_n}{\sqrt{n}}\Big)^m=(W(0))^m\mu_m.$$

Thus we have

$$\lim_{n\to\infty}\Pr\{N_n\leq\sqrt{n}\,W(0)x\}=\sqrt{\frac{2}{\pi}}\int_0^x e^{-u^2/2}du\,.$$

Now from the argument in the proof of Theorem 2, reduction to the case (3.4) is obvious if we use (3.5) and note that

$$W_a(x)\equiv\int_{-\infty}^{\infty}dF(y)\int_0^{\infty}U_a(x-y-z)U_a(z)dz\,,$$

$$\lim_{a\to\infty}W_a(0)=\int_{-\infty}^{\infty}dF(x)\int_0^{\infty}U(-x-y)U(y)dy=-\int_{-\infty}^{0}xdF(x)\,,$$

$$\int_0^{\infty}xdF(x)+\int_{-\infty}^{0}xdF(x)=0\,,$$

$$\beta_1=\int_0^{\infty}xdF(x)-\int_{-\infty}^{0}xdF(x)=-2\int_{-\infty}^{0}xdF(x)\,,$$

and hence

$$\lim_{a\to\infty}W_a(0)=\frac{\beta_1}{2}\,.$$

We have thus proved the theorem.

3. Gaussian stationary time series. We shall now apply the method of moments to study fluctuation of $S_n \equiv \sum_1^n X_\nu$, in which $\{X_\nu\}$ is a Gaussian stationary time series. The same method will also apply to the class of other stationary sequences, when the behaviour of the characteristic function of S_n is known.

Theorem 4. *Suppose that $X_1, X_2, \dots$ is a Gaussian (real) stationary time series with mean 0, autocorrelation coefficient ρ_k, and spectral density $f(\lambda)(2\pi)^{-1}$, $-\pi \leq \lambda \leq \pi$, continuous in a neighbourhood of $\lambda=0$, $\infty > f(0) \neq 0$,*

$$\rho_k = E(X_{n+k}X_n) = (2\pi)^{-1}\int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda)d\lambda . \tag{4.1}$$

Let N_n be the number of $S_\nu = \sum_1^\nu X_k$, $1 \leq \nu \leq n$, belonging to a finite interval $(-h/2, h/2)$, $h>0$.

Then

$$\lim_{n\to\infty} \Pr\left\{N_n \leq \frac{h}{\sqrt{f(0)}}\sqrt{n}\,x\right\} = \left(\frac{2}{\pi}\right)^{1/2}\int_0^x e^{-u^2/2}du .$$

Proof. To prove the statement, appealing to the device used in Theorem 2, it is sufficient to show that

$$E\left\{\frac{1}{\sqrt{n}}\sum_1^n H(S_k)\right\}^m \to \left(\frac{h(0)}{\sqrt{f(0)}}\right)^m \mu_m$$

for $H(x)$ of the class of functions in § 2, with

$$\int_{-\infty}^{\infty} H(x)e^{-itx}dx = h(t), \quad (2\pi)^{-1}\int_{-\infty}^{\infty} h(t)e^{itx}dt = H(x).$$

Now

$$E\left\{\frac{1}{\sqrt{n}}\sum_1^n H(S_k)\right\} = \frac{1}{\sqrt{n}}\sum_1^n (2\pi)^{-1}\int_{-\infty}^{\infty} h(t)e^{-q_k t^2/2}dt ,$$

where from (4.1)

$$q_k \equiv \operatorname{Var}\left\{\sum_{\nu=1}^k S_\nu\right\} = (2\pi)^{-1}\int_{-\pi}^{\pi}\left(\frac{\sin k\lambda/2}{\sin \lambda/2}\right)^2 f(\lambda)d\lambda$$

$$\sim k f(0), \quad k\to\infty .$$

Therefore we have

$$\lim_{n\to\infty} E\left\{\frac{1}{\sqrt{n}}\sum_1^n U(S_k)\right\} = \frac{h(0)}{\sqrt{f(0)}}\mu_1 .$$

To calculate the second moment of N_n first we observe that

$$E\{U(S_k)U(S_{k+l})\} = (2\pi)^{-2}\iint_{-\infty}^{\infty} E\{e^{iS_k s + iS_{k+l}t}\}u(s)u(t)dsdt$$

$$= \iint_{-\infty}^{\infty} \exp\{-Q(s,t)/2\}u(s)u(t)dsdt$$

$$= \int_{-\infty}^{\infty} u(t)dt\int_{-\infty}^{\infty} \exp\{-Q(s-t,t)/2\}u(s-t)ds ,$$

where from (4.1)

$$Q(s,t)\equiv(s+t)^2(2\pi)^{-1}\int_{-\pi}^{\pi}\left(\frac{\sin k\lambda/2}{\sin\lambda/2}\right)^2 f(\lambda)d\lambda$$

$$+t^2(2\pi)^{-1}\int_{-\pi}^{\pi}\left(\frac{\sin l\lambda/2}{\sin\lambda/2}\right)^2 f(\lambda)d\lambda$$

$$+2(s+t)t(2\pi)^{-1}\int_{-\pi}^{\pi}\overline{\frac{e^{ik\lambda}-1}{e^{i\lambda}-1}e^{i\lambda}}\,\frac{e^{i(k+1)\lambda}(e^{il\lambda}-1)}{e^{i\lambda}-1}f(\lambda)d\lambda$$

and

$$Q\left(\frac{s}{\sqrt{k}}-\frac{t}{\sqrt{l}},\ \frac{t}{\sqrt{l}}\right)=f(0)s^2+f(0)t^2$$

$$+\frac{st}{\pi\sqrt{kl}}\int_{-\pi}^{\pi}\frac{\sin k\lambda/2\sin l\lambda/2}{\sin^2\lambda/2}\cos\frac{k+l}{2}\lambda(f(\lambda)-f(0))d\lambda+o(1),$$

as $k\to\infty$, $l\to\infty$.
But

$$\left|\frac{1}{\sqrt{kl}}\int_{-\pi}^{\pi}\frac{\sin k\lambda/2\sin l\lambda/2}{\sin^2\lambda/2}\cos\frac{k+l}{2}\lambda(f(\lambda)-f(0))d\lambda\right|$$

$$\leq\left\{\int_{-\pi}^{\pi}\frac{\sin^2 k\lambda/2}{k\sin^2\lambda/2}|f(\lambda)-f(0)|\,d\lambda\right\}^{1/2}\left\{\int_{-\pi}^{\pi}\frac{\sin^2 l\lambda/2}{l\sin^2\lambda/2}|f(\lambda)-f(0)|\,d\lambda\right\}^{1/2}$$

$$\to 0,\quad \text{as } k\to\infty,\ l\to\infty.$$

Recalling that $h(t)$ vanishes outside a finite interval we get

$$E\{H(S_k)H(S_{k+l})\}\sim\left(\frac{1}{\sqrt{f(0)2\pi}}h(0)\right)^2\frac{1}{\sqrt{kl}}.$$

This gives

$$E\left\{\frac{1}{\sqrt{n}}\sum_{k=1}^{n}H(S_k)\right\}^2=\frac{1}{n}\sum_{k+l\leq n}2!\,E\{H(S_k)H(S_{k+l})\}+\frac{1}{n}\sum_{k=1}^{n}E\{H(S_k)\}^2$$

$$=\frac{2!}{n}\left(\frac{1}{\sqrt{f(0)2\pi}}h(0)\right)^2\sum_{k+l\leq n}\frac{1}{\sqrt{kl}}+o(1)\to\left(\frac{h(0)}{\sqrt{f(0)}}\right)^2\mu_2$$

In the same way we have

$$E\left\{\frac{1}{\sqrt{n}}\sum_{1}^{n}H(S_k)\right\}^m\to\left(\frac{h(0)}{\sqrt{f(0)}}\right)^m\mu_m$$

as was to be proved.

When $f(0)=0$ or $f(0)=\infty$, the normalization factor $\sqrt{n}$ must be altered. For instance when $f(\lambda)=|\lambda|^{\alpha}$, $-1<\alpha<1$, it can be shown that $N_n/n^{(1+\alpha)/2}$ has a limiting distribution, when $n\to\infty$. It seems not easy to give a closed expression for this limiting distribution. If we form an appropriate average sequence from the original time series:

$$Y_n=\sum_{\nu=0}^{p} c_\nu X_{n-\nu}$$

we can let Y_n have the spectral density $g(\lambda)$ satisfying the condition of Theorem 4. Consider X_{2n}, $n=1, 2,\dots$, then the autocorrelation coefficient becomes

$$\begin{aligned}\rho_k'&=E\{X_{2(n+k)}X_{2n}\}=\rho_{2k}\\&=(2\pi)^{-1}\int_{-\pi}^{\pi} e^{2\pi ik\lambda} f(\lambda)d\lambda=\frac{1}{2}(2\pi)^{-1}\int_{-2\pi}^{2\pi} e^{ik\lambda} f(\lambda/2)d\lambda\\&=\frac{1}{2}(2\pi)^{-1}\left(\int_{-\pi}^{\pi}+\int_{\pi}^{2\pi}+\int_{-2\pi}^{-\pi}\right)e^{ik\lambda}f(\lambda)d\lambda\\&=\frac{1}{2}(2\pi)^{-1}\left(\int_{-\pi}^{\pi} e^{ik\lambda}f(x)d\lambda+\int_{-\pi}^{0} e^{ik\lambda}f(\lambda/2+\pi)d\lambda\right.\\&\qquad\qquad\left.+\int_{0}^{\pi} e^{ik\lambda}f(\lambda/2-\pi)d\lambda\right)\\&=(2\pi)^{-1}\int_{-\pi}^{\pi} e^{ik\lambda}g(\lambda)d\lambda,\end{aligned}$$

where

$$g(\lambda)=\frac{1}{2}\{f(\lambda)+f(\pi-|\lambda|/2)\}.$$

When $f(\lambda)=|\lambda|^\alpha$, $\alpha>0$ we get $g(0)=\frac{1}{2}f(\pi)=\frac{1}{2}\pi^\alpha$, that is the subsequence $\{X_{2n}\}$ satisfies the condition of Theorem 4 and the number of return of $S_n'=\sum_1^n X_{2\nu}$ to a finite interval has the turncated normal distribution in the limit. On the other hand when $f(\lambda)=|\lambda|^\alpha$, $-1<\alpha<0$, the time series $X_{\nu+1}-X_\nu$, $\nu=1, 2,\dots$ has the spectral density $|e^{i\lambda}-1|^2|\lambda|^\alpha=2\sin^2\frac{\lambda}{2}|\lambda|^\alpha$, and hence $X_{2\nu+1}-X_{2\nu}$, $\nu=1, 2,\dots$ has the continuous spectral density $g(\lambda)$, for which $g(0)=\sin^2\frac{\pi}{2}\cdot\pi^\alpha=\pi^\alpha$. Therefore the number of return to a interval of $S_n'=\sum_1^n(-1)^\nu X_\nu$ has the truncated normal distribution in the limit. Thus the number of return to a finite interval of a particle with stationary Gaussian velocity obey, in the limit, the trucated normal distribution, if its spectral density $f(\lambda)$ satisfies $f(0)\neq 0$.

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