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On the Transition Probability Functions of the Markov Process

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1. Introduction. The stochastic equation has been an important article in the recent probability theory and proved useful especially to reveal functional dependences between different stochastic processes, or construct stochastic processes from other basic stochastic processes. Among others we may mention important contributions due to S. Bernstein [1], P. Lévy [9], and K. Itō [6]. As is well known certain types of parabolic partial differential equations define Markov processes. But, as shown by these authors, stochastic equations are sometimes more convenient to construct Markov processes. In this paper we shall show that a considerably general class of Markov processes defined by stochastic equations has transition probability functions satisfying parabolic differential equations. This fundamental fact seems not yet fully established, while differentiation of the family of operators associated with the transition probabilities has been discussed [7].

Suppose that $m(t,x)$, $\sigma(t,x)$, $-\infty<x<\infty$, $t_0\leq t\leq t_1$, are continuous functions of (t,x) satisfying the uniform Lipschitz condition

$$|m(t,x)-m(t,x')|+|\sigma(t,x)-\sigma(t,x')|\leq c|x-x'| \tag{1.1}$$

with c independent of t and x. Define a Markov process $z(t)$ by the equation due to K. Itō,

$$z(t)=z_0+\int_{t_0}^{t} m(\tau,z)d\tau+\int_{t_0}^{t}\sigma(\tau,z)dB(\tau)\,, \tag{1.2}$$

where B(t) is the normalized Wiener process $B(t_0)=0$, $E(\varDelta B(t))^2=\varDelta t$. Since in the following arguments the length of the t-interval is immaterial, we shall hereafter restrict our considerations to the unit interval $0\leq t\leq 1$. As is well known (1.2) can be reduced to the simpler equation

$$y(t)=y_0+\int_0^t a(\tau,y)d\tau+x(t) \tag{1.3}$$

by means of the transformation

$$y(t)=\varPhi(t,z(t))\,,\qquad \varPhi(t,z)=\int_0^z\frac{d\zeta}{\sigma(\tau,\zeta)}\,,$$

where $x(t)$ is another normalized Wiener process,

$$a(t,y)=-\frac{\varphi_t'(t,y)}{\sigma(t,\varphi(t,y))}+\frac{m(t,\varphi(t,y))}{\sigma(t,\varphi(t,y))}-\sigma_\varphi'(t,\varphi(t,y))$$

and $z=\varphi(t,y)$ is the inverse function of $y=\Phi(t,z)$. For considerably wide and practically important classes $a(t,y)$ also satisfies the Lipschitz condition

$$(1.4) \qquad |a(t,x)-a(t,x')| \leq c_0|x-x'| .$$

Therefore we are sufficed to consider simply the class of processes defined by (1.3). For instance, under the conditions imposed on $m(t,x)$, $\sigma(t,x)$ in Feller [4] $a(t,x)$ satisfies the above Lipschits condition and, in addition, becomes *bounded*. In this respect the following Theorem 2 will be a partial improvement of Feller's result. It will also be noteworthy that if the transition probabilities $F(s,x;t,y)=\Pr\{y(t)\leq y|y(s)=x\}$ satisfy

$$(C_1) \qquad \int_{|y-x|>\delta} dF(s,x;t,y)=o(t-s) ,$$

$$(C_2) \qquad \int_{|y-x|\leq\delta} (y-x)dF(s,x;t,y)=m(s,x)(t-s)+o(t-s) ,$$

$$(C_3) \qquad \int_{|y-x|\leq\delta} (y-x)^2dF(s,x;t,y)=\sigma^2(s,x)(t-s)+o(t-s) ,$$

with o-terms having suitable regularities, then $y(t)$ satisfies (1.1). Hence if, in addition, σ and m satisfy differentiability conditions required in Theorem 2, $F(s,x;t,y)$ is also differentiable and satisfies

$$(1.5) \qquad \left\{\frac{\partial}{\partial s}+m(s,x)\frac{\partial}{\partial x}+\frac{1}{2}\sigma^2(s,x)\frac{\partial^2}{\partial x^2}\right\}F(s,x;t,y)=0 .$$

This amounts to the fact that we can construct the fundamental solution of (1.5) by means of the solution of (1.2), i.e. by the probability measure on the function space.

2. Differentiability of the transition probabilities. We shall prove the following two theorems.

Theorem 1. *Let* $a(t,x)$, $-\infty<x<\infty$, $0\leq t\leq 1$, *be continuous and satisfy* (1.4), *and* $F(s,x;t,y)$ *the transition probabilities of the process determined by* (1.3). *Then* $F(s,x;t,y)$ *is given by*

$$(2.1.1) \qquad F(s,y_0;t,x)=\int_{-\infty}^{x} A(s,y_0;t,y)B(s,y_0;t,y)dy ,$$

$$A=E\{L(y(\cdot))|y(t)=y\} ,$$

$$B=\frac{1}{\sqrt{2\pi(t-s)}}e^{-(y-y_0)^2/2(t-s)} ,$$

$$(2.1.2) \qquad L(y(\cdot))=\exp\left[\int_s^t a(\tau,y)\,dy(\tau)-\frac{1}{2}\int_s^t a^2(\tau,y)d\tau\right] ,$$

where E denotes the expectation taken under the probability measure associated with the Wiener process $y(t)$ conditioned with $y(s)=y_0$, $y(t)=y$.

Theorem 2. *Let* $a(t,x)$, $-\infty<x<\infty$, $0\leq t\leq 1$, *be continuous in* (t,x), *continuously twice differentiable in* x, *with bounded* $a'_x(t,x)$ *and*

$a''_{xx}(t,x)=O(|x|^k)$ for some $k>0$. Then the transition probability $F(s,x;t,y)$ is continuously twice differentiable in x, once in t, and satisfy

$$\left\{\frac{\partial}{\partial s}+a(s,x)\frac{\partial}{\partial x}+\frac{1}{2}\frac{\partial^2}{\partial x^2}\right\}F(s,x;t,y)=0\ . \tag{2.2}$$

To prove these theorems we require several lemmas.

Lemma 1. *Let $z(t)$ be the solution of* (1.2), *$f(t,x)$ be continuous, then for any real λ*

$$E\exp\left[\lambda\int_0^t f(\tau,z)dB(\tau)-\frac{\lambda^2}{2}\int_0^t f^2(\tau,z)d\tau\right]\leq 1\ . \tag{2.4}$$

This is a generalization of Lemma 10 in the previous paper [12], where we assumed m, σ and f to be bounded. As will be clear from the proof of Lemma 10, the boundedness of m, σ is necessary only for (5.9) of Lemma 10 but superfluous for (5.10) [12].

Proof. Define

$$\begin{aligned} f_N(\tau,z(\tau)) &= f(\tau,z(\tau)) && \text{if} \quad |f(\tau,z(\tau))|\leq N\ ,\\ &= N && \text{if} \quad f(\tau,z(\tau))>N\ ,\\ &= -N && \text{if} \quad f(\tau,z(\tau))<-N\ ,\\ & N>0\ . \end{aligned}$$

Then Lemma 10 [12] and the above observation give us

$$E\exp\left[\lambda\int_0^t f_N(\tau,z)dB(\tau)-\frac{\lambda^2}{2}\int_0^t f_N(\tau,z)d\tau\right]=1\ .$$

On making $N\to\infty$, and appealing to Fatou's lemma we get (2.4).

Lemma 2. *Let $B(t)$, $0\leq t\leq 1$, be a normalized Wiener process as in* (1.2), *$\Delta=\Delta(t_0,t_1,\ldots\ldots,t_n)$ a division of the interval $(0,t)$, $0=t_0<t_1<\cdots<t_n=t$, and put $B_\nu=B(t_\nu)$. Then for $\lambda t^2<1/4$*

$$E\,(\exp[\lambda\sum_{\nu=1}^{n}B_{\nu-1}^2\Delta t_{\nu-1}])\leq c(\lambda,t)<\infty\ ,$$

where $c(\lambda,t)$ is independ of Δ.

The lemma fails to hold for large values of λt^2. In view of the result of Cameron and Martin [2] (c.f. also Kac [8], Lévy [10]) we are naturally led to the above formulation as its discrete counterpart. Hereafter we shall denote by c_i numerical constants.

Proof. Writing

$$B_\Delta(t)=B(t_{\nu-1}) \qquad \text{for} \qquad t_{\nu-1}\leq t<t_\nu$$

we get

$$E\exp[\lambda\sum_{\nu=1}^{n}B_{\nu-1}^2\Delta t_{\nu-1}]\leq\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\left\{E\left(\int_0^t B_\Delta^2(\tau)d\tau\right)^{2k}\right\}^{1/2}$$

$$\leq\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\left(t^{2k-1}\int_0^t EB_\Delta^{4k}(\tau)d\tau\right)^{1/2}\leq\sum_{k=0}^{\infty}\frac{(\lambda t^2)^k}{k!\sqrt{2k+1}}\left(\frac{4k!}{2k!2^{2k}}\right)^{1/2}$$

$$\leq c_1\sum_{k=0}^{\infty}\frac{(4\lambda t^2)^k}{k}=c(\lambda,t)\ ,$$

with an absolute constant c_1. This proves the lemma.

Lemma 3. *Let $B(t)$, $0\leq t<\infty$, be the normalized Wiener process, then for any real λ*

$$E\exp\left[\lambda\int_0^\infty\frac{|B(u)|}{(1+u)^2}du\right]\leq c_2e^{c_2\lambda^2},\tag{2.5.1}$$

$$E\exp\left[\lambda\int_0^\infty\frac{B^2(u)}{(1+u)^3}du\right]<\infty\ .\tag{2.5.2}$$

Proof. We proceed as in the proof of Lemma 2. First the left-hand side of (2.5.1) is equal to

$$\sum_0^\infty\frac{\lambda^n}{n!}E\left(\int_0^\infty\frac{|B(u)|}{(1+u)^2}du\right)^n\leq\sum_0^\infty\frac{\lambda^n}{n!}\left\{E\left(\int_0^\infty\frac{|B(u)|}{(1+u)^2}du\right)^{2n}\right\}^{1/2}.\tag{2.6}$$

Second

$$E\left(\int_0^\infty\frac{|B(u)|}{(1+u)^2}du\right)^{2n}\tag{2.7}$$

$$\leq E\left(\int_0^\infty\frac{B(u)^{2n}}{(1+u)^{2+2(2n-1)\varepsilon}}du\right)\left(\int_0^\infty\frac{du}{(1+u)^{2(1-\varepsilon)}}\right)^{2n-1},$$

and $EB(u)^{2n}=2n!u^n/n!2^n$. If we choose $\varepsilon=1/4$, the right-hand member of (2.7) is dominated by

$$\frac{2n!}{n!\,2^n}\left(\int_0^\infty\frac{du}{(1+u)^{3/2}}\right)^{2n}\sim 2^{3n+1/2}n^ne^{-n}\ .$$

Hence the general term $c_n(\lambda)$ of the right-hand side of (2.6) satisfies

$$c_n(\lambda)\sim\frac{\lambda^ne^{n/2}}{\sqrt{2\pi}}n^{-(n+1)/2}2^{n+n/2+1/4}\sim\frac{2^{1/4}}{\sqrt{2\pi}}\lambda^n\frac{(\sqrt{8})^n}{\sqrt{n!}n^{1/4}}\ .$$

Substituting this into (2.6) we obtain (2.5.1), and similarly with (2.5.2).

Proof of Theorem 1. To construct the solution, we shall set up, as in [11] [12], the difference equations

$$y_1=y_0+a(t_0,y_0)\Delta t+\Delta x_0$$
$$y_2=y_1+a(t_1,y_1)\Delta t+\Delta x_1$$
$$\cdots\qquad\cdots$$
$$y_n=y_{n-1}+a(t_{n-1},y_{n-1})\Delta t+\Delta x_{n-1}\ ,$$

where $x_\nu=x(t_\nu)$, $\Delta x_{\nu-1}=x_\nu-x_{\nu-1}$, and $\Delta=\Delta(t_0,t_1,\cdots,t_n)$ is an equidistant division of the interval $(s,1)$ i.e. $s=t_0<t_1<\cdots<t_n=1$, $\Delta t=(1-s)/n$. We can now write

$$(2.8)\quad \Pr\{y_n\leq x\}=(\Delta t2\pi)^{-n/2}\int_{-\infty}^{\infty} e^{-(y_1-a(t_0,y_0)\Delta t-y_0)^2/2\Delta t}dy_1$$
$$\cdot\int_{-\infty}^{\infty} e^{-(y_2-a(t_1,y_1)\Delta t-y_1)^2/2\Delta t}dy_2$$
$$\cdots\int_{-\infty}^{\infty} e^{-(y_{n-1}-a(t_{n-2},y_{n-2})\Delta t-y_{n-2})^2/2\Delta t}dy_{n-1}$$
$$\cdot\int_{-\infty}^{x} e^{-(y_n-a(t_{n-1},y_{n-1})\Delta t-y_{n-1})^2/2\Delta t}dy_n$$
$$=(\Delta t2\pi)^{-n/2}\iint_{-\infty}^{\infty}$$
$$\cdots\int e^{-(y_1-y_0)^2/2\Delta t-(y_2-y_1)^2/2\Delta t-\cdots-(y_n-y_{n-1})^2/2\Delta t}$$
$$\cdot L(\Delta)\varphi_x(y_n)dy_1dy_2\cdots dy_n,$$

where

$$(2.9)\quad L(\Delta)\equiv L(\Delta,y(\cdot),a)$$
$$=\exp\left[\sum_{\nu=0}^{n-1}a(t_\nu,y_\nu)\Delta y_\nu-\frac{1}{2}\sum_{\nu=0}^{n-1}a^2(t_\nu,y_\nu)\Delta t\right],$$

and

$$\varphi_x(u)=1,\qquad u\leq x,$$
$$=0,\qquad u>x.$$

Then according to the form of the weight function in (2.8), y_ν can be seen as the coordinates of a Wiener process $y(t)$, $0\leq t\leq 1$, $y(s)=y_0$, $E(\Delta y(t))^2=\Delta t$. From this point of view $\Pr\{y_n\leq y|y(s)=y_0\}$ has been proved to be given by an average of the Wiener functional $L(\Delta,y(\cdot),a)\varphi_x(y(1))$. Making $\Delta t\to 0$ this average will be expected to converge to

$$E\{L(y(\cdot))\varphi_x(y(1))\},$$
$$L(s,y_0,y(\cdot))\equiv L(y(\cdot))$$
$$=\exp\left[\int_s^1 a(\tau,y)dy(\tau)-\frac{1}{2}\int_s^1 a^2(\tau,y)d\tau\right].$$

This is true, when at least $1-s$ is sufficiently small. Indeed by Lemma 2

$$E\{L^2(\Delta,y(\cdot),a)\}\leq\{EL(\Delta,y(\cdot),4a)\}^{1/2}$$
$$\cdot\{E\exp[6\sum_{\nu=0}^{n-1}a^2(t_\nu,y_\nu)\Delta t]\}^{1/2}$$
$$\leq\{E\exp c_3[1+y_0^2+\sum_{\nu=1}^{n-1}B^2(t_\nu)\Delta t]\}^{1/2}\leq c_4e^{c_4y_0^2}c^{1/2}(c_3,1-s),$$

where $1-s$ is so small that we have $c_3(1-s)^2<1/4$. Also by Lemma 3 and Lemma 7 in [12]

$$\text{l.i.m.}\ L(\Delta,y(\cdot),a)=L(y(\cdot)).$$

Hence by Lemma 4 in [12] we get

$$E\{L(\Delta, y(\cdot), a)\varphi_x(y(1))\} \to E\{L(y(\cdot))\varphi_x(y(1))\}, \ \Delta t \to 0 ,$$

as was to be proved. But according to Theorem 1 [11] [12] y_n converges in the mean to the solution of (1.3) and therefore we obtain

$$\text{(2.10)} \qquad \Pr\{y(1) \leq x\} = E\{L(y(\cdot))\varphi_x(y(1))\} .$$

If we rewrite the right-hand side of (2.10) we have finally

$$\text{(2.11)} \qquad \begin{aligned} &\Pr\{y(1) \leq x\} = \int_{-\infty}^{x} A(y_0, y)B(y_0, y)dy , \\ &A(s, y_0; 1, y) \equiv A(y_0, y) = E\{L(y(\cdot)) | y(1) = y\} , \end{aligned}$$

with

$$B(s, y_0; 1, y) \equiv B(y_0, y) = \frac{1}{\sqrt{2\pi(1-s)}} e^{-(y-y_0)^2/2(1-s)} ,$$

which completes the proof.

Proof of Theorem 2. First observe that under the condition $y(1)=y$ $y(t)$ becomes a Markov process which satisfies the stochastic equations

$$\text{(2.12)} \qquad \begin{aligned} &y(t) = y_0 + \int_s^t \frac{y - y(\tau)}{1-\tau} d\tau + \eta(t) - \eta(s) , \\ &y(t) = \frac{1-t}{1-s} y_0 + \frac{t-s}{1-s} y + X(t) , \qquad X(t) = \frac{1-t}{\sqrt{1-s}} \xi\left(\frac{t-s}{1-t}\right) , \end{aligned}$$

where $\xi(t)$, $\eta(t)$ are normalized Wiener processes, $X(t)$ is independent of y_0, y and a conditional Wiener process with $X(s)=X(t)=0$ (see Lévy [9], Doob [3]). (2.12) gives also a connection between $\xi(t)$, $\eta(t)$ of the form

$$\eta(t) - \eta(s) = \frac{1}{\sqrt{1-s}} \int_s^t (1-\tau) d\xi\left(\frac{\tau - s}{1-\tau}\right) .$$

If we substitute (2.12) into (2.1.2.) and differentiate the latter with respect to y_0, we get

$$\text{(2.13)} \qquad \left| \frac{\partial}{\partial y_0} L(y(\cdot)) \right| \leq P_0 P_1 P_2 , \qquad \text{(2.14)} \qquad |L(y(\cdot))| \leq P_1 P_2 ,$$

$$\text{(2.15)} \qquad \begin{aligned} P_0 = c_4 \Big\{ &(y_0^2 + y^2)^{1/2} + (1-s)^{1/2} \int_0^1 \left| \xi\left(\frac{\tau}{1-\tau}\right) \right| d\tau \\ &+ (1-s)^{-1} \left| \int_s^1 a_y'(\tau, y)(1-\tau) d\eta(\tau) \right| \Big\} , \end{aligned}$$

$$\begin{aligned} P_1 = \exp c_4 \Big[&y_0^2 + y^2 + (1-s)^{1/2}(y_0^2 + y^2 + 1)^{1/2} \int_0^1 \left| \xi\left(\frac{\tau}{1-\tau}\right) \right| d\tau \\ &+ (1-s) \int_0^1 (1-\tau) \xi^2\left(\frac{\tau}{1-\tau}\right) d\tau \Big] , \end{aligned}$$

and

$$P_2=\exp\left[\int_s^1 a(\tau, y)d\eta(\tau)-\frac{1}{2}\int_s^1 a^2(\tau, y)d\tau\right].$$

From these estimates we can deduce that

$$(2.16)\qquad \begin{aligned} &E\{|L(y(\cdot))|\}\leq c_5 e^{c_5(y_0^2+y^2)},\\ &E\left\{\left|\frac{\partial}{\partial y_0}L(y(\cdot))\right|\Big|\, y(1)=y\right\}\leq c_5 e^{c_5(y_0^2+y^2)}.\end{aligned}$$

Indeed first we have

$$E(P_0^3)\leq c_6\left\{|y_0|^3+|y|^3+E\left(\int_0^1\left|\xi\left(\frac{\tau}{1-\tau}\right)\right|d\tau\right)^3 + (1-s)^{-3}\left(E\left|\int_s^1 a_y'(\tau, y)(1-\tau)d\eta(\tau)\right|^4\right)^{3/4}\right\},$$

and

$$E\left(\int_0^1\left|\xi\left(\frac{\tau}{1-\tau}\right)\right|d\tau\right)^3\leq c_7$$

by Lemma 3. Second by the formula (5.17) of [12]

$$\begin{aligned} E\left|\int_s^1 a_y'(\tau, y)(1-\tau)d\eta(\tau)\right|^4 &=3E\left(\int_s^1 a_y'^2(1-\tau)^2 d\tau\right)^2\\ &\quad+6E\left\{\left(\int_s^1 a_y'(1-\tau)d\eta\right)^2\left(\int_s^1 a_y'^2(1-\tau)^2 d\tau\right)\right\}\\ &\leq c_8\left\{(1-s)^6+(1-s)^3\int_s^1 E(a_y'(\tau, y)(1-\tau))^2 d\tau\right\}\leq c_9(1-s)^6.\end{aligned}$$

Hence

$$E(P_0^3)\leq c_{10}(1+|y_0|^3+|y|^3).$$

Next from Lemma 3

$$\begin{aligned} E(P_1^3)\leq e^{3c_4(y_0^2+y^2)}&\left\{E\exp 6c_4\left[(1+y_0^2+y^2)^{1/2}\int_0^\infty\frac{|\xi(u)|}{(1+u)^2}du\right]\right\}^{1/2}\\ &\cdot\left\{E\exp\left[6c_4\int_0^\infty\frac{|\xi(u)|^2}{(1+u)^3}du\right]\right\}^{1/2}\leq c_{11}e^{c_{11}(y_0^2+y^2)}.\end{aligned}$$

Also from Lemma 1 and Lemma 3

$$\begin{aligned} E(P_2^3)&\leq\left\{E\exp\left[\int_s^1 6a(\tau, y)d\eta(\tau)-18\int_s^1 a^2(\tau, y)d\tau\right]\right\}^{1/2}\\ &\quad\cdot\left\{E\exp\left(15\int_s^1 a^2(\tau, y)d\tau\right)\right\}^{1/2}\\ &\leq c_{12}\left\{E\exp c_{12}\left(y_0^2+y^2+\int_0^\infty\frac{\xi^2(u)}{(1+u)^3}du\right)\right\}^{1/2}\leq c_{13}e^{c_{13}(y_0^2+y^2)}.\end{aligned}$$

Combining these we get (2.16). In quite a similar manner we can obtain

(2.17) $$E\left\{\left|\frac{\partial^2}{\partial y_0^2}L(y(\cdot))\right|\Big|y(1)=y\right\}\leq c_{14}e^{c_{14}(y_0^2+y^2)} .$$

By means of (2.13), (2.14), and (2.15) it is now immediate to apply the Lebesgue criteria of dominated convergence to prove that $A(y_0, y)$ is continuously differentiable twice with respect to y_0 and we have

(2.18) $$\begin{aligned}\frac{\partial}{\partial y_0}A&=\frac{\partial}{\partial y_0}E\left\{L(y(\cdot))|y(1)=y\right\}=E\left\{\frac{\partial}{\partial y_0}L(y(\cdot))|y(1)=y\right\} ,\\ \frac{\partial^2}{\partial y_0^2}A&=\frac{\partial^2}{\partial y_0^2}E\left\{L(y(\cdot))|y(1)=y\right\}=E\left\{\frac{\partial^2}{\partial y_0^2}L(y(\cdot))|y(1)=y\right\} .\end{aligned}$$

Thus prepared we are now in a position to show that $F(s, y_0; t, y)$ is continuously differentiable twice with respect to y_0, and

(2.19) $$\begin{aligned}\frac{\partial}{\partial y_0}F(s,y_0;1,x)&=\int_{-\infty}^{x}\frac{\partial}{\partial y_0}(AB)dy ,\\ \frac{\partial^2}{\partial y_0^2}F(s,y_0;1,x)&=\int_{-\infty}^{x}\frac{\partial^2}{\partial y_0^2}(AB)dy .\end{aligned}$$

In fact in view of (2.16), (2.17) and (2.18) the integrals in the right-hand members of (2.19) are convergent uniformly in y_0, at least when $1-s$ is sufficiently small, and hence the orders of integration and differentiation are exchangeable, thus proving (2.19). The above arguments also apply, without essential changes, to $F(s, y_0; t, y)$ if only $t-s$ is sufficiently small, obtaining (2.19) with 1, s replaced by t, s.

Now we shall pass on to differentiation with respect to s. First we shall set up another expression approximating $A(y_0, y)$ which is slightly different from (2.9) but more convenient to calculate $\partial A(s, y_0;\ 1, y)/\partial s$. If we define

$$L^*(\Delta, y(\cdot))=e^{\alpha(t_1, y_0)(y_1-y_0)-\alpha^2(t_1, y_0)\Delta s/2}\,L(t_1, y_1, y(\cdot)) ,$$
$$\Delta s=t_1-s ,$$

we can write

(2.20) $$\begin{aligned}A^*_\Delta(s, y_0)&\equiv E\{L^*(\Delta, y(\cdot))|y(1)=y\}\\ &=\int_{-\infty}^{\infty}\varphi(y_0, y_1, \Delta s)C(t_1, y_1)dy_1 ,\end{aligned}$$

where

$$\varphi(y_0, y_1, \Delta s)=\frac{1}{\sqrt{2\pi\Delta s}}\,e^{-(y_1-y_0-\alpha(s, y_0)\Delta s)^2/2\Delta s} ,$$

$$\alpha(s, y_0)=\frac{y-y_0}{1-s} ,$$

and

$$C(t_1, y_1)=e^{\alpha(t_1, y_0)(y_1-y_0)-\alpha^2(t_1, y_0)\Delta s/2}\,A(t_1, y_1, y) .$$

By the continuity of $A(t_1, y_1, y)$ already established we can easily show that

$$A^*_\Delta(s, y_0) \to A(s, y_0)\,, \qquad t_1 \downarrow s\,.$$

Differentiation of (2.20) gives

$$\frac{\partial}{\partial s} A^*_\Delta(s, y_0) = \frac{1}{2\Delta s}\int \varphi C dy_1 - \frac{1}{2(\Delta s)^2}\int (y_1 - y_0)^2 \varphi C dy_1 \tag{2.21}$$

$$+\frac{\alpha^2}{2}\int \varphi C dy_1 + \frac{1}{(1-s)^2}\int (y_1 - y_0)^2 \varphi C dy_1$$

$$+\frac{1}{2}a^2(t_1, y_0)\int \varphi C dy_1\,.$$

Now if we use the estimates for $C'_y(s, y)$, $C''_{yy}(s, y)$ which will be easily obtained from (2.16), (2.17), and (2.18) we can write

$$\frac{1}{\Delta s}\int \varphi(y_0, y_1, \Delta s) C(s, y_1) dy_1 = \frac{1}{\Delta s\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-u^2/2} C(s, u\sqrt{\Delta s} + \alpha \Delta s + y_0) du$$

$$= \frac{1}{\Delta s\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-u^2/2}\Big\{C(s, y_0) + (u\sqrt{\Delta s} + \alpha\Delta s) C'_y(s, y_0)$$

$$+\frac{1}{2}(u\sqrt{\Delta s} + \alpha \Delta s)^2 C''_{yy}(s, y_0 + \theta u\sqrt{\Delta s} + \theta\alpha\Delta s)\Big\} du$$

$$= \frac{C(s, y_0)}{\Delta s} + \alpha C'_y(s, y_0) + \frac{1}{2} C''_{yy}(s, y_0) + o(1)\,,$$

$$0 < \theta < 1\,,$$

where $o(1)$ represents a term tending to 0 uniformly in s. In a similar manner

$$\frac{1}{(\Delta s)^2}\int (y_1 - y_0)^2 \varphi C dy_1 = \frac{C}{\Delta s} + \alpha^2 C + 3\alpha C'_y + \frac{3}{2} C''_{yy} + o(1)\,.$$

Also we have by definition

$$C'_y(s, y_0) \to a(s, y_0) A(s, y_0) + \frac{\partial}{\partial y_0} A(s, y_0)\,,$$

$$C''_{yy}(s, y_0) \to a^2(s, y_0) A(s, y_0) + 2a(s, y_0)\frac{\partial}{\partial y_0} A(s, y_0) + \frac{\partial^2}{\partial y_0^2} A(s, y_0)\,, \quad t_1 \downarrow s.$$

Hence the first two terms in (2.21) contribute

$$-\frac{\alpha^2}{2} C - \alpha C'_y - \frac{1}{2} C''_{yy} + o(1)\,,$$

and the remaining three terms do

$$\frac{\alpha^2}{2} A(s, y_0) + \frac{1}{2} a^2(s, y_0) A(s, y_0) + o(1)\,,$$

either uniformly in s. So that

$$(2.22) \qquad \frac{\partial}{\partial s}A^*_\Delta(s, y_0)=U(s, y_0)+o(1)\,, \qquad U(s, y_0)$$
$$=-(\alpha aA)(s, y_0)-(\alpha A'_y)(s, y_0)-(aA'_y)(s, y_0)-\frac{1}{2}A''_{yy}(s, y_0)$$

uniformly in s of any finite interval. Therefore the operations $\Delta s\to 0(t_1\downarrow s)$ and $\partial/\partial s$ are exchangeable, obtaining

$$(2.23) \qquad \frac{\partial}{\partial s}A(s, y_0)=\lim_{t_1\to s}\frac{\partial}{\partial s}A^*_\Delta(s, y_0)=U(s, y_0)\,.$$

Hence by (2.1), for at least an every small $1-s<\delta$, say, we have

$$(2.24) \qquad \frac{\partial}{\partial s}F(s, y_0; 1, x)=\int_{-\infty}^{x}(A'_sB+AB'_s)dy$$
$$=-a(s, y_0)\int_{-\infty}^{x}\frac{\partial}{\partial y_0}(AB)dy-\frac{1}{2}\int_{-\infty}^{x}\frac{\partial^2}{\partial y_0^2}(AB)dy$$
$$=\left\{-a(s, y_0)\frac{\partial}{\partial y_0}-\frac{1}{2}\frac{\partial^2}{\partial y_0^2}\right\}F(s, y_0; 1, x)\,,$$

where the change of the orders of differentiation in s and integration in y is assured by (2.16), (2.17), (2.18), and (2.23). The same arguments apply also without essential changes to $F(s, y_0; t, x)$ if only $t-s<\delta$. Finally we have to prove that (2.3) holds in general for arbitrary s and t.

Choose u such that $s<u<t$, $u-s<\delta$, and write the Chapman-Kolmogorov equation in our notations as

$$F(s, y_0; t, x)=\int_{-\infty}^{\infty}A(s, y_0; u, y_1)B(s, y_0; u, y_1)F(u, y_1; t, x)dy_1\,.$$

Then, since $0\leq F(u, y_1; t, x)\leq 1$, by the same arguments as in (2.24) we can write

$$\frac{\partial}{\partial s}(s, y_0; t, x)=\int_{-\infty}^{\infty}(A'_sB+AB'_s)F(u, y_1; t, x)dy_1$$
$$=\left\{-a(s, y_0)\frac{\partial}{\partial y_0}-\frac{1}{2}\frac{\partial^2}{\partial y_0^2}\right\}\int_{-\infty}^{\infty}(AB)F(u, y_1; t, x)dy_1$$
$$=\left\{-a(s, y_0)\frac{\partial}{\partial y_0}-\frac{1}{2}\frac{\partial^2}{\partial y_0^2}\right\}F(s, y_0; t, x)\,,$$

as was to be proved.

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