On the Propagation of Upper Atmospheric Pressure Waves in the Westerly Current with Vertical Gradient. Part I.[1)]
(General Theory and Special Case of no Horizontal Stratification and no Friction)

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Introduction and Summary

With increasing altitudes many peculiar phenomena appear in the upper atmospheric pressure waves propagating in the westerly current with vertical gradient, such as displacements of high pressure- (or low pressure-) centers and the singularity in the vertical distribution of the upper wind. Among the physical properties of these upper air waves we are particularly interested in the phase changes associated with altitude and latitude. In fact the centers of cyclone and anticyclone developing in middle latitudes displace their positions with the altitude, that is, their phases change with the altitude. For these phase changes there are some explanations such as that of Holmbo's text book.[(1)] But this explanation contradicts the result of measurements reported by Saito.[(2)] From his statistical treatment of aerological data, he concluded that at about 4–5 km height there was a bend in the axis of the cyclone center, above which the center of the cyclone is not displaced, but there is a large change below it. We had also recognized these facts in the analysis of aerological data during the Pacific war.

As an attempt to interpret these phenomena from the theoretical point of view we shall propose here that these phenomena are connected with their phase change at a particular altitude in the westerly current with vertical gradient, at which altitude the velocity of the mean current coincides with that of the perturbation waves. These perturbation waves may consist of divergence and vorticity waves of the upper air wind.

In the present paper the author, in the first place, determined fundamental equations for these waves, seeking their particular solutions, for which the periods, wave lengths and propagation velocities are the same as those of the main cyclone- and anticyclone-waves effective in weather forecasting. From our discussion of this particular

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solution we can show that the characteristics of the above mentioned phase changes are well explained, and furthermore the types of the change of winds at this altitude are also discussed.

1. Theoretical Formulation of Fundamental Equations for Upper Air Waves (Divergence and Vorticity Waves of the Upper Air)

In our theoretical treatment of upper air waves we shall discuss, in the first place, the assumption about forces acting in the upper air, then show the fundamental equations of the motion, and lastly present the solutions of these equations by the method of approximation.

(A) Assumed forces:

We shall consider the following forces acting in the upper air.

(i) Pressure gradient force and barocline force:—We consider the usual pressure gradient forces as well as the forces arising from the effect of the earth's rotation on the barocline fields of density distribution.

(ii) Vertical components of acceleration, Coriolis and gradient forces:—We treat the case in which, as for the vertical component, the gravitational force is always in equilibrium with the vertical pressure gradients, while the vertical components of acceleration and Coriolis force are neglected.

(iii) Horizontal components of Coriolis force and their variation with altitude:—As for the horizontal motion, we take up not only Coriolis force but also its variation with latitude, as the stream-lines of air fluids to be discussed in this paper vary considerably in the direction of latitude, in accordance with Rossby's point of view for westerly perturbation waves. But the effects of spherical structure of the earth are not considered.

(iv) Frictional force:—For the lower atmosphere near the ground and the upper atmosphere, where the mean current coincides with the velocity of perturbation waves, we consider the effects of the eddy viscosity, by which the interesting phase changes of the centers of cyclone and anticyclone arc considerably modified.

(v) Polytropic change:—For the thermodynamic change of air we assume a polytropic change between the change of pressure and density including the effects of radiation.

(B) Fundamental Equations:

Fundamental equations of the upper air for our problem are the equations of motion, the equation of continuity and the equation of thermodynamic change. Considering the assumption mentioned above for forces acting, those equations are as follows:

(i) Equations of motion:

$$\frac{du}{dt}-2\lambda v=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\Delta u\,,$$
$$\frac{dv}{dt}+2\lambda u=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\Delta v\,, \tag{1}$$
$$0=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g\rho\,,$$

where λ is a horizontal component of the Coriolis force, u and v are horizontal and vertical components of wind velocity, g the gravitational constant, ν the coefficient of eddy viscosity, and p and ρ are the pressure and density of the air, respectively.

(ii) Equation of continuity:

$$\frac{d\rho}{dt}+\rho\nabla\cdot\boldsymbol{V}=0\,. \tag{2}$$

(iii) Equation for polytropic change:

$$\frac{d\rho}{dt}-\gamma\frac{dp}{dt}=0\,, \tag{3}$$

where γ is the piezotropic coefficient.

(C) Method of approximation:

To make analytically possible our theoretical treatment of upper air waves, we shall take the following step as the method of approximation for the atmospheric state.

(i) In the 0th approximation the atmosphere stratified horizontally and vertically flows stationarily in the west-east direction forming a longitudinal flow. Its velocity increases with altitude up to the troposphere as $U=U_0'z+U_0$. Its order of magnitude is 10^3, as that of the barotropic coefficient is $\Gamma=\frac{d\rho_0}{dp_0}\sim 10^{-10}$.

(ii) In the first approximation the following perturbation waves are superposed on the air currents. Their average values are

period	3 days,	$(T=2.4\times10^5)$,
wave length	3000 km,	$\left(k=\frac{2\pi}{\lambda}=2\times10^{-8}\right)$,
propagation velocity	20 km/sec,	$(c=2\times10^3)$.

2. Treatment of Fundamental Equations

In this section we shall treat the above mentioned fundamental equations by the method of approximation step by step. Physical quantities—presssure, density and upper wind velocity—are divided into two parts—the 0th and first approximation terms.

$$
\begin{aligned}
p(x, y, z, t) &= p_0(y, z) + p_1(x, y, z, t) , \\
\rho(x, y, z, t) &= \rho_0(y, z) + \rho_1(x, y, z, t) , \\
u(x, y, z, t) &= U(y, z) + u_1(x, y, z, t) , \\
v(x, y, z, t) &= \qquad\qquad v_1(x, y, z, t) , \\
w(x, y, z, t) &= \qquad\qquad w_1(x, y, z, t) .
\end{aligned} \tag{4}
$$

In the 0th approximation after substituting the 0th term of Eq. (4) into Eq. (1), we obtain the following equations of motion

$$
\begin{aligned}
2\lambda U(z) &= -\frac{1}{\rho_0}\frac{\partial p_0}{\partial y} , \\
0 &= -\frac{1}{\rho_0}\frac{\partial p_0}{\partial z} - g .
\end{aligned} \tag{5}
$$

The equation of continuity is trivial. The equation of a stratified layer is

$$
\frac{d\rho_0}{dp_0} = \Gamma . \tag{6}
$$

From Eqs. (5) and (6) components of the density gradient are

$$
\begin{aligned}
\frac{\partial \rho_0}{\partial z} &= \frac{d\rho_0}{dp_0}\frac{\partial p_0}{\partial z} = -g\rho_0\Gamma , \\
\frac{\partial \rho_0}{\partial y} &= \frac{d\rho_0}{dp_0}\frac{\partial p_0}{\partial y} = -2U\rho_0\Gamma .
\end{aligned} \tag{7}
$$

The effects of stratified layer appear in two parts—horizontal and vertical stratifications. In this paper we shall assume that vertical stratification prevails and take it into considerations.

In the first approximation substituting Eq. (4) into Eq. (1) and eliminating the 0th order quantities and second order terms, as the latter are infinitesimal quantities, we obtain the following equations

$$
\begin{aligned}
\frac{\partial u}{\partial t} + U(z)\frac{\partial u}{\partial x} - 2\lambda v &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2} , \\
\frac{\partial v}{\partial t} + U(z)\frac{\partial v}{\partial y} + 2\lambda u &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \frac{\rho}{\rho_0} + \nu\frac{\partial^2 v}{\partial z^2} , \\
0 &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - \frac{\rho}{\rho_0}g .
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + U(z)\frac{\partial \rho}{\partial x} + \frac{\partial \rho_0}{\partial y}v + \frac{\partial \rho_0}{\partial z}w + \rho_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 , \\
&\frac{\partial \rho}{\partial t} + U(z)\frac{\partial \rho}{\partial x} + \frac{\partial \rho_0}{\partial z}w = \gamma\left(\frac{\partial p}{\partial t} + U(z)\frac{\partial p}{\partial y} + \frac{\partial p_0}{\partial y}v + \frac{\partial p_0}{\partial z}w\right) .
\end{aligned} \tag{9}
$$

The effects of friction are considered only in the direction of z-axis and all the suffixes are omitted.

Now making the rotation and divergence for Eqs. (6), we have

$$\begin{aligned}&\frac{\partial\zeta}{\partial t}+U\frac{\partial\zeta}{\partial x}+2\lambda\chi+2\frac{\partial\lambda}{\partial y}v-\nu\frac{\partial^2\zeta}{\partial z^2}+\frac{2\lambda U}{\rho_0}\frac{\partial\rho}{\partial x}=0\,,\\&\frac{\partial\chi}{\partial t}+U\frac{\partial\chi}{\partial x}-2\lambda\zeta+2\frac{\partial\lambda}{\partial y}u+\frac{1}{\rho_0}\Delta p-\nu\frac{\partial^2\chi}{\partial z^2}+\frac{2\lambda U}{\rho_0}\frac{\partial\rho}{\partial y}=0\,,\end{aligned}\tag{10}$$

where $\zeta=\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}$ and $\chi=\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}$.

Following Rossby we take into consideration the latitude change of Coriolis force as follows

$$2\frac{\partial\lambda}{\partial y}=2\omega\frac{\partial\sin\varphi}{\partial(a\varphi)}=v_ck^2\,,\tag{11}$$

where

$$v_c=\frac{2\omega a\cos^3\varphi}{\kappa^2}\quad\text{and}\quad\kappa=ka\cos\varphi\,.$$

Then we assume that these perturbation waves are simple monochromatic waves propagating for positive x-axis (eastwards), and for positive y-axis (northwards) with exponentially increasing amplitudes such as

$$\begin{aligned}u&=-\{ikA(z)+lB(z)\}e^{ly+ik(x-ct)}\,,\\v&=\{-lA(z)+ikB(z)\}e^{ly+ik(x-ct)}\,,\\w&=C(z)e^{ly+ik(x-ct)}\\p&=\rho_0(y,z)D(z)e^{ly+ik(x-ct)}\,,\\\rho&=\rho_0(y,z)E(z)e^{ly+ik(x-ct)}\,,\end{aligned}\tag{12}$$

and investigate the distribution in the vertical direction. Then the rotation and divergence waves of the wind velocity are

$$\begin{aligned}\zeta&=-(k^2-l^2)B(z)e^{ly+ik(x-ct)}\,,\\\chi&=(k^2-l^2)A(z)e^{ly+ik(x-ct)}\,.\end{aligned}\tag{13}$$

Introducing Eqs. (7) and (8) into Eqs. (5) and (6), we have the following five equations for determining five quantities A, B, C, D and E as functions of the altitude variable z.

$$\begin{aligned}&ik\{-(U-c)(k^2-l^2)+k^2v_c\}B+\nu(k^2-l^2)B''\\&\qquad-\{k^2v_cl-2\lambda(k^2-l^2)\}A+2ik\lambda UE=0\,,\\&ik\{(U-c)(k^2-l^2)-k^2v_c\}A-\{k^2v_cl-2\lambda(k^2-l^2)\}B\\&\qquad-\nu(k^2-l^2)A''-(k^2-l^2)D+2l\lambda UE=0\,,\end{aligned}$$

$$
\begin{aligned}
&D'-g\Gamma D+gE=0\,, && (14)\\
&ik(U-c)E-\lambda U\Gamma(-lA+ikB)-g\Gamma C+C'+(k^2-l^2)A=0\,, &&\\
&ik(U-c)E-\lambda U\Gamma(-lA+ikB)-g\Gamma C &&\\
&\qquad=\gamma\{ik(U-c)D-\lambda U\Gamma(-lA+ikB)-g\Gamma C\}\,. &&
\end{aligned}
$$

where the prime denotes the differentiation.

3. The Case of no Horizontal Stratification

We shall consider the case of no horizontal stratification to investigate some features of our formulation. Neglecting the effects of stratification, that is, the term $\lambda U\Gamma$, we have from Eq. (4)

$$
\begin{aligned}
&-ik\Big(U-c-v_c\frac{k^2}{k^2-l^2}\Big)B+\Big(2\lambda-v_c\frac{k^2 l}{k^2-l^2}\Big)A+\nu B''=0\,, &&\\
&ik\Big(U-c-v_c\frac{k^2}{k^2-l^2}\Big)A+\Big(2\lambda-v_c\frac{k^2 l}{k^2-l^2}\Big)B-\nu A''-D=0\,, &&\\
&D'-g\Gamma D+gE=0\,, && (15)\\
&ik(U-c)E-g\Gamma C+C'+(k^2-l^2)A=0\,, &&\\
&ik(U-c)E-g\Gamma C=\gamma\{ik(U-c)D-gC\}\,. &&
\end{aligned}
$$

From Eq. (15) we shall seek an equation containing $A(z)$ only. From the first and second of Eq. (15) we have a relation between A and B.

$$
\begin{aligned}
&\nu^2 A'''-2ik\Big(U-c-v_c\frac{k^2}{k^2-l^2}\Big)\nu A''-ikU'\nu A' &&\\
&\qquad-\Big\{k^2\Big(U-c-v_c\frac{k^2}{k^2-l^2}\Big)^2-\Big(2\lambda-v_c\frac{k^2 l}{k^2-l^2}\Big)^2\Big\}A && (16)\\
&\qquad=-\nu D''+ik\Big(U-c-v_c\frac{k^2 l}{k^2-l^2}\Big)D\,. &&
\end{aligned}
$$

On the other hand from the third, fourth and fifth of Eq. (15) we have another relation between A and D.

$$
A=\frac{ik}{g^2(\gamma-\Gamma)}\,\frac{1}{l^2-k^2}\{(U-c)D''+(U'-g\Gamma\overline{U-c})D'+U'g(\gamma-\Gamma)D\}. \quad (17)
$$

A differential equation may be obtained for a single variable D or A, but it needs not to write down here as it is too complicated.

4. The Case of no Horizontal Stratification and no Friction

For this case the equation to be satisfied by D is

$$
\Big\{k^2\Big(U-c-v_c\frac{k^2}{k^2-l^2}\Big)^2-\Big(2\lambda-v_c\frac{k^2 l}{k^2-l^2}\Big)^2\Big\}
$$

$$\times\{(U-c)D''+[U'-g\Gamma(U-c)]D'+[U'g(\gamma-\Gamma)+(U-c)g^2\gamma\Gamma]D\} \quad (18)$$
$$=g^2(\gamma-\Gamma)(k^2-l^2)\left(U-c-v_c\frac{k^2}{k^2-l^2}\right)D\,.$$

Assuming a simple relation for the mean westerly current as $U=U_0'z+U_0$, we obtain a differential equation for D

$$(z-a+b)(z-a-b)\{(z-c)D''+(d_1z+d_2)D'+(e_1z+e_2)D\}+(f_1z+f_2)D=0\,, \quad (19)$$

where

$$\begin{aligned} a&=\frac{-1}{U_0'}\left(U_0-c-v_c\frac{k^2}{k^2-l^2}\right), & b&=\frac{1}{kU_0'}\left(2\lambda-v_c\frac{k^2l}{k^2-l^2}\right), \\ d_1&=-U_0'g\Gamma\,, & d_2&=U'-g\Gamma(U_0-c)\,, \\ e_1&=U_0'g^2\gamma\Gamma\,, & e_2&=U_0'g(\gamma-\Gamma)+(U_0-c)g^2\gamma\Gamma\,, \\ f_1&=-\frac{k^2-l^2}{k^2}g^2\frac{(\gamma-\Gamma)}{U_0'^2}\,, & f_2&=-\frac{k^2-l^2}{k^2}g^2(\gamma-\Gamma)\left(U_0-c-v_c\frac{k^2}{k^2-l^2}\right). \end{aligned} \quad (20)$$

In this differential equation for D there are three regular singular points ($z=a+b,\ a-b,\ c$), and one irregular singular point ($z=\infty$). We shall call these singular points as the singular heights of D, its outline is shown in the Fig. 1. As easily seen, the height of $z=a$, which is singular when the effect of Coriolis force is neglected, is split out into two singular heights ($z=a+b,\ a-b$).

$$v_c'=v_c\frac{k^2}{k^2-l^2}$$

$$v_a=\frac{2\lambda}{k}-v_c\frac{kl}{k^2-l^2}$$

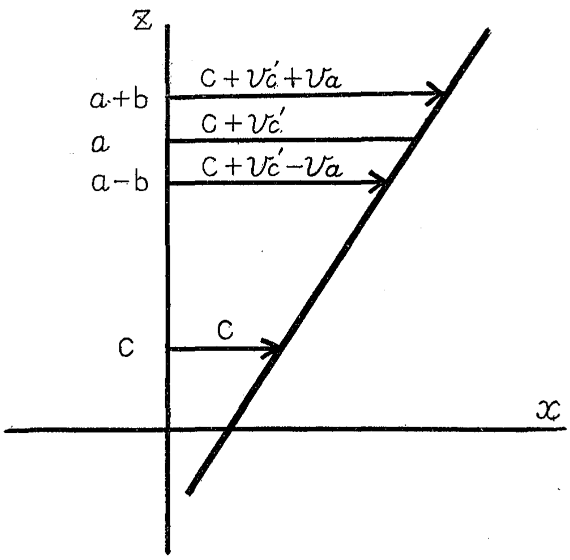

Fig. 1. Singular Heights (Velocities of each perturbation wave are indicated by their arrows).

(A) Approximate Solution of D in the Neighborhood of Singular Heights

In the neighborhood of the points ($z=a+b,\ a-b$) the equation to be satisfied by D is

$$z(D''+dD'+eD)+fD=0\,, \quad (21)$$

where

$$d=d_1(a\pm b)+d_2/a\pm b-c \qquad e=e_1(a\pm b)+e_2/a\pm b-c$$
$$f=\pm\frac{f_1(a\pm b)+f_2}{2b(a\pm b-c)}$$

We shall solve Eq. (21) by the method of Frobenius. Putting

$$D(z)=\sum_{i=0}^{\infty}a_i z^{\rho+i}\,, \tag{22}$$

its indicial equation becomes

$$\rho(\rho-1)=0\,. \tag{23}$$

As the difference of the roots of Eq. (23) is an integer, we have a solution containing a logarithmic term such as
for $z>0$

$$\begin{aligned} D_1(z)&=z+a_2z^2+a_3z^3+\cdots\cdots\,, \\ D_2(z)&=1+b_1z+b_2z^2+\cdots\cdots \qquad -fD_1(z)\log z\,, \end{aligned} \tag{24}$$

for $z<0$ we have the same D_1 as $z>0$, but for D_2 we obtain as follows

$$D_2(z)=1+b_1z+b_2z^2+\cdots\cdots \qquad -fD_1(z)(\log z\pm i\pi)\,, \tag{25}$$

where the double sign is meant as

$$\begin{cases} + & \text{for a weak cyclone,} \\ - & \text{for a developing cyclone.} \end{cases}$$

(B) Phase Change of Pressure Waves with Altitude (Case of a developing cyclone)

For the general pressure waves the vertical distribution of the density D is determined as the linear combination of D_1 and D_2, their ratio being fixed by the boundary conditions. In this section we shall consider a simple case, in which there exists only D_2, and make clear its feature for the phase change. The details for the boundary problem will be discussed in our later paper. The characteristics of phase change in the neighborhood of the singular height ($z=0$) will be fully discussed by our present treatment for D_2 only.

We have as the approximate solution for $D_1(z)$ and $D_2(z)$

$z>0$

$$\begin{cases} D_1(z)=z\,, \\ D_2(z)=1-fz\log z\,, \end{cases}$$

$z<0$ (26)

$$\begin{cases} D_1(z)=z\,, \\ D_2(z)=1-fz(\log z-i\pi)\,. \end{cases}$$

To make clear the phase change in $z<0$, we put as $D_2=A_0e^{i\varphi}$, then

$$\varphi=\tan^{-1}\frac{\pi fz}{1-\pi fz}\doteqdot\tan^{-1}\pi fz\,. \tag{27}$$

For this case the pressure wave propagating towards the positive x-axis takes the following form:

for $z>0$ $\qquad p=\rho_0 D_2(z)e^{ly+ik(x-ct)}$,

for $z<0$ $\qquad p=\rho_0 A_0(z)e^{ly+i\{k(x-ct)+\varphi\}}$,

The phase of pressure wave retarded by πfz in $z<0$, as seen in Eq. (28). The position of the equal phase proceeds in the direction of positive x, according as z becomes large, as z is negative. This method of approximation can be applied up to the order of magnitude $fz\sim{}^1/_2$, as the order of magnitude of e is 10^{-6}, the phase change arises linearly up to the height $z=5\times10^5$ ($=5$ km). The characteristics of these pressure waves can be said as follows.

Though there happens no change of phase in the pressure waves above the singular height ($z>0$), *the position with an equal phase displaces forwards for the developing cyclone and backwards in the weak cyclone below the singular height* ($z<0$).

When the order of magnitude of fz becomes 1, the orders of D_1 and D_2 become the same as 1 and the phase change becomes stationary for the larger values than $\varphi=\tan^{-1}\pi=72°$. This fact is shown in Fig. 2 for the developing cyclone.

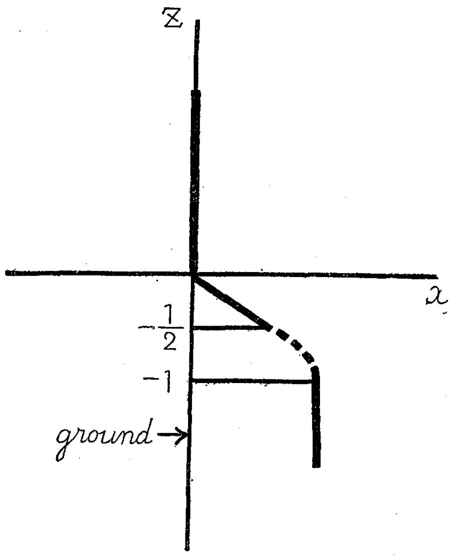

Fig. 2. Phase change for the developing cyclone.

Reference

(1) Holmboe-Forsythe-Gustin: Dynamic Meteorology, p. 262, (1945).
(2) Saito: Report of Aerological Congression held at Tateno, Japan (1947).

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