On the Theorem of Castelnuovo-Enriques

Teruhisa Matsusaka (松阪輝久)

Department of Mathematics, Faculty of Science,
Ochanomizu University, Tokyo

Introduction. The classical theorem of Castelnuovo-Enriques asserts that the one dimensional Betti number of a sufficiently general divisor on a non-singular algebraic Variety $V^n(n\geqq 3)$ is the same as that of V^n or the base for 1-cycles on such divisor also forms the base for 1-cycles on V. The algebraic equivalent of the above theorem may be formulated as follows:

$V^n(n\geqq 3)$ *and its sufficiently general divisor have the same Picard (Albanese) Variety up to an isomorphism.*

The aim of the present note is to prove the theorem of Castelnuovo-Enriques in the above formulation.

Let X be a divisor on the product $\Gamma\times V^n$ of a non-singular Curve Γ with an algebraic Variety V^n in a projective space and k be a common field of definition for Γ, V over which X is rational. We shall say that the totality $\mathfrak{A}$ of V-divisors of the form $X(u)$ defined by

$$(u\times V)\cdot X=u\times X(u)$$

is a *one-dimensional algebraic family* defined by Γ and X or Γ and X defines $\mathfrak{A}$. A field such as k shall be refered to as *a field of definition* for $\mathfrak{A}$ or we shall say that $\mathfrak{A}$ *is defined* over k. The one dimensional algebraic family on V which we shall treat in this paper is a linear pencil of the special kind. When V is absolutely locally normal, and when the complete linear system on V is sufficiently ample, then one can extract from the complete linear system a linear pencil such that it contains a Variety. Moreover, when V is non-singular, we may assume that it contains also a non-singular Variety. Our interest will be concentrated to the linear pencil having this property.

Assume that $\mathfrak{A}$ is a pencil, V is an algebraic Surface such that it has a base Point at a simple Point of V and that a generic divisor $X(u)$ corresponding to a generic Point u of Γ over a common field of definition k for V and Γ is a non-singular Curve. By Chow's result on Jacobian Varieties (cf. [C]-1, [C]-2, or [M]-4, § 2)[1], there is a symmetric function Ψ defined on the product of sufficiently many factors equal to $X(u)$ on the Jacobian Variety J of $X(u)$ defined over $k(u)$ immersed

---

We shall use the same terminology and convention as Weil's books "Foundations of Algebraic Geometry" and "Variétès Abeliennes et Courbes algébriques.

[1] Letters in brackets refer to the bibliography at the end.

into a projective space such that when we write Ψ as

$$\Psi=\sum\varphi_i \quad \text{(cf. [W]-2 chap. III, cor. of th. 7)}$$

where φ_i is a function defined on $X(u)$ with values in J, φ_i and φ coincide within an additive constant and φ_i is the canonical function of $X(u)$. Let x_0 be a base Point of $\mathfrak{A}$. Then x_0 is algebraic over k and hence it is rational over $\bar{k}(u)$. Moreover x_0 is a simple Point of $X(u)$ and this shows that one of the φ_i, say φ_1, is defined over $\bar{k}(u)$ (cf. [W]-2 chap. III, cor. th. 7).

Let $|X|$ be a linear system, which is not necessarily complete, on a Variety V and k be a common field of definition for V and $|X|$. Let X be a generic divisor of $|X|$ over k and assume that V and X are Varieties having no singular Subvarieties of dimension $n-2$ and $n-3$ respectively. The induced linear system on X by $|X|$ shall be refered to as *the characteristic linear system* of $|X|$ on X. "The characteristic linear system of $|X|$" will mean the characteristic linear system of $|X|$ on some generic divisior of $|X|$ over k.

There is a maximal algebraic family $\{X\}$ on V such that when a V-divisor Y is algebraically equivalent to zero, there are two divisors X and X' in $\{X\}$ such that $Y\sim X-X'$ (cf. [M]-3, th. 1). We shall call $\{X\}$ as a *total maximal algebraic family*[2]. We shall say that a field k of definition for the Picard Variety $\mathfrak{p}(V)$ of an algebraic Variety V having no singular Subvariety of dimension $n-1$ is *the complete field of definition* for $\mathfrak{p}(V)$ when the following conditions are satisfied:

when X is a V-divisor, algebraically equivalent to zero, such that it is rational over a field K containing k, then the class of X on $\mathfrak{p}(V)$ is rational over K and conversely, when ξ is a Point on $\mathfrak{p}(V)$ rational over a field K containing k, there is a V-divisor X rational over K, algebraically equivalent to zero, such that its class on $\mathfrak{p}(V)$ is ξ. (cf. [M]-3, th. 3 and [M]-4, th. 4).

Denote by L_m the linear system consisting of all the divisors of the form $H_m\cdot V$ where H_m is a hypersurface of order m. Let C_{m_i} be a generic divisor of L_{m_i} $(i=1, 2, \cdots, s)$ over a common field of definition k for V and for every L_{m_i}. Then the intersection-product $C_{m_1}\cdots C_{m_s}$ is defined on V. We shall say that it is a generic $(n-s)$-cycle of order $m_1\cdots m_s$ of V over k. When V has no singular Subvariety of dimension $n-r$ $(r>0)$, its generic $(n-s)$-cycle over k is a Variety and has no singular Subvariety of dimension $n-s-r$ (cf. [N], [Z]-th. 3). In particular, a generic 1-cycle on V over k is a non-singular Curve.

Let U be a Variety, A be an Abelian Variety and f be a function defined on U with values in A. We shall say that U (and f) *generates* A when if $x_1, \cdots, x_n$ are sufficiently many numbers of independent

2) The writer had used the term "regular," and by the advice of Prof. C. Chevalley, the writer will use the term "total".

generic Points of U over a common field of definition k for U, A and f, $\sum f(x_i)$ is a generic Point of A over k. When that is so, we shall also say that A is generated by U (and f).

1. **Proposition.** *Let V be an algebraic Surface free from singular Points in a projective space, $\mathfrak{A}$ be a linear pencil on V and W_0 be a fixed divisor in $\mathfrak{A}$ such that it is a non-singular Curve. Let X be a divisor on V algebraically equivalent to zero such that $X\cdot W_0$ is linearly equivalent to zero on W_0. Then X is linearly equivalent to zero within a linear combination of certain numbers of fixed divisors on V independent of X.* (cf. [W]-3, (*A*)).

Proof. Since the proof may be essentially the same as that of Weil, we shall sketch briefly the outline of it.

Let W be a generic divisor of $\mathfrak{A}$ over a common field of definition k for V and $\mathfrak{A}$, over which X is rational, then it is a non-singular Curve and it is easy to see that there is a set of finite numbers of Curves $U_1, \cdots, U_s$ all algebraic over k on V having the following properties: let Z be a V-divisor rational over $\bar{k}$ such that $Z\cdot W\sim 0$ on W, then Z is linearly equivalent to a certain linear combination of $U_1, \cdots, U_s$. From this, we can derive the following: let $\sum$ be a one-dimensional algebraic family of positive V-divisors defined over k such that every divisor of $\sum$ induces on W mutually equivalent W-divisors with respect to linear equivalence, then every divisor of $\sum$ is mutually equivalent with respect to linear equivalence.

Let $\{Y\}$ be the total maximal algebraic family of positive V-divisors and U be its associated-Variety, which is defined over $\bar{k}$. Let Y be a generic divisor of $\{Y\}$ over $\bar{k}(w)$ and Y_0 be a rational divisor of it over $\bar{k}$ where w is the Chow-Point of W. Let φ_w be the canonical function of W defined over $k(w)$ and h_w be the function defined on U with values in the Jacobian Variety $\mathfrak{p}(W)$ defined

$$h_w(y)=S[\varphi_w((Y-Y_0)\cdot W)]=\eta_w$$

where y is the Chow-Point of Y. $h_w^{-1}(\eta_w)$ consists of finite numbers of associated-Varieties of complete linear systems and the Locus A_w of η_w over $\bar{k}(w)$ is an Abelian Variety isogeneous to the Picard Variety $\mathfrak{p}(V)$ of V (cf. [M]-2). Let w' be the Chow-Point of a generic divisor W' of $\mathfrak{A}$ over $\bar{k}(w, y)$ and put $h_{w'}(y)=\eta_{w'}$. Then $\eta_{w'}$ is purely inseparable over $\bar{k}(w, w', \eta_w)$ since the group of all the Points of given order on an Abelian Variety is a finite group (cf. [W]-2, cor. 1, th. 33).

Let K be the algebraic closure of $k(w')$. Then, there is a homomorphism λ defined on A_w with values on $A^*_{w'}$ with a field of definition $K(w)$ (cf. [W]-2, th. 27) such that $\lambda(\eta_w)=\eta^*_{w'}$ where * denotes an automorphism of the universal domain defined by

$$u^* = u^{p^t}$$

for a certain non-negative integer t. By using the theorem of complete reducibility for Abelian Varieties (cf. [W]-2, prop. 25, th. 26), we conclude that there is a function $\bar{\psi}$ defined on $\boldsymbol{W}$ with values in $\boldsymbol{A}_w$, such that

$$S[\bar{\psi}((\boldsymbol{Y}-\boldsymbol{Y}_0)\cdot\boldsymbol{W})]=\eta^*_u,$$

$\bar{\psi}$ can be extended to a function ψ defined on $\boldsymbol{V}$ with values in $\boldsymbol{A}_w$, such that $\psi_W=\bar{\psi}$.

We may assume that $\boldsymbol{X}=\boldsymbol{Y}'-\boldsymbol{Y}_0$ where $\boldsymbol{Y}'$ is in $\{\boldsymbol{Y}\}$ and moreover, that $\boldsymbol{Y}'$, $\boldsymbol{Y}_0$ are also non-singular Curves (cf. [M]-5, th. 2). Then applying th. 10 of [W]-2, if $\boldsymbol{X}\cdot\boldsymbol{W}_0\sim 0$, we have

$$0=S[\psi(\boldsymbol{X}\cdot\boldsymbol{W}_0)]=S[\psi(\boldsymbol{X}\cdot\boldsymbol{W})]$$

and from this we conclude that $\boldsymbol{X}\cdot\boldsymbol{W}\sim 0$ on $\boldsymbol{W}$. q.e.d.

As a corollary of the above proposition, we have

Corollary. *Let V^n be a Variety in a projective space, having no singular Subvariety of dimension n-2 and $\mathfrak{A}$ be a linear pencil on V such that $\mathfrak{A}$ contains a Variety W free from singular Subvarieties of dimension n-3. When X is a V-divisor which is algebraically equivalent to zero, such that $W\cdot X$ is defined on V and that it is linearly equivalent to zero on W, X is linearly equivalent to zero within a linear combination of certain numbers of V-divisors which are independent of X.*

This corollary follows immediately from prop. above and from [W]-3, lemma.

2. **Theorem.** *Let V^n be a Variety free from singular Subvarieties of dimension n-2 in a projective space and B be a linear system on V having the following properties:*

(i) *$\mathfrak{B}$ contains a divisor which is a Variety free from singular Subvarieties of dimension n-3*

(ii) *dim $\mathfrak{B}=2$ and the characteristic linear system of $\mathfrak{B}$ contains a Variety having no singular Subvarieties of dimension n-4.*[3] *Then the Picard Variety of a generic divisor of $\mathfrak{B}$ over a common field of definition for V and $\mathfrak{B}$ is isogeneous to the Picard Variety $\mathfrak{p}(V)$ of V.*

Proof. By the above corollary, it can be easily seen that the Picard Variety $\mathfrak{p}(\boldsymbol{Z})$ of a generic divisor $\boldsymbol{Z}$ of $\mathfrak{B}$ over a common field of definition k for $\boldsymbol{V}$ and $\mathfrak{B}$ contains the Abelian Variety which is isogeneous to the Picard Variety $\mathfrak{p}(\boldsymbol{V})$ of $\boldsymbol{V}$ (cf. [M]-3, prop. 11).

[3] When n-4<0, put 0 instead of n-4.

Let Z and $\bar{Z}$ be two independent generic divisors of $\mathfrak{B}$ over k and f be a function on V such that $(f)_0=Z$, $(f)_\infty=\bar{Z}$. Consider the linear pencil $\mathfrak{A}$ defined by the function 1 and f on V. The base Variety of $\mathfrak{A}$ is $Z\cdot\bar{Z}=C$ and C has no singular Subvariety of dimension n-4 by our assumption (ii). Let K be an algebraically closed field of definition for f, C and for the Picard Variety $\mathfrak{p}(C)$ of C and assume that K is complete as a field of definition for $\mathfrak{p}(C)$. Let v be a generic Point of the associated-Variety of $\mathfrak{A}$ over K and denote by Z_v the corresponding divisor of $\mathfrak{A}$. $\mathfrak{B}$ induces on Z_v a linear pencil $\mathfrak{B}'$ — the characteristic linear system of $\mathfrak{B}$ on Z_v—having C as its divisor. By applying the corollary of our proposition to Z_v, to the linear system $\mathfrak{B}'$ and to a divisor C of B', we see that there is the Abelian Variety isogeneous to $\mathfrak{p}(Z_v)$ in the Picard Variety $\mathfrak{p}(C)$ of C.

Let $\{X\}$ be a maximal total algebraic family of positive Z_v-divisors containing a rational divisor X_0 over $K(v)$ and defined over $K(v)$ (cf. [M]-5, prop. 3) such that $C\cdot X_0$ is defined on Z_v. Let M be the Chow-Point of the associated-Variety $T(X)$ of the complete linear system $|X|$ determined by a generic divisor X of $\{X\}$ over $K(v)$, and x', x'' be two independent generic Points of $T(X)$ over $K(v, M)$ corresponding to X', X'' respectively. Since $C\cdot(X_0-X')$ is rational over $K(v, M, x')$ (cf. [C]-3), its class ξ on $\mathfrak{p}(C)$ is rational over $K(v, M, x')$ and in the same way ξ is rational over $K(v, M, x'')$. This shows that ξ is rational over $K(v, M)$ and its Locus A over $K(v)$ is isogeneous to $\mathfrak{p}(Z_v)$.

Let Γ be a generic 1-cycle of Z_v of order 1 over $K(v, M)$, then $\mathfrak{p}(Z_v)$ is isogeneously imbedded into the Jacobian Variety $\mathfrak{p}(\Gamma)$ of Γ (cf. [M]-3, prop. 11). We may assume that the degree of X is so large that $deg(\Gamma\cdot X)>2\cdot genus(\Gamma)-2$. Then taking Chow's result on the Jacobian Varieties into account (cf. [C]-2), the same arguments as above show that the class η of $\Gamma\cdot X$ on $\mathfrak{p}(\Gamma)$ is rational over $K(v, M, t)$ where t is the Chow-Point of Γ. The Locus B of η over $K(v, t)$ is an Abelian Variety in $\mathfrak{p}(\Gamma)$ isogeneous to $\mathfrak{p}(Z_v)$ and moreover, $K(v, t, M)$ is a pure inseparable extension of $K(v, t, \eta)$ by Weil's criterion for linear equivalence (cf. [W]-3, (E)). Hence ξ is also purely inseparable over $K(v, t, \eta)$ and when p is the characteristic of our universal domain, there is a positive integer e such that ξ^{p^e} is rational over $K(v, t, \eta)$. Since K is algebraically closed, ξ^{p^e} has also the Locus A' over $K(v^{p^e})\subset K(v)$ and A' is clearly isogeneous to A. By what we have observed above, there is a homomorphism λ defined over $K(v, t)$ from B onto A' and there is a homomorphism μ defined over $K(v)$ from A' onto A, and therefore, there is a homomorphism ν defined over $K(v, t)$ from B onto A, that is, into $\mathfrak{p}(C)$. By [W]-2, chap. VII, prop. 25, there is a homomorphism from $\mathfrak{p}(\Gamma)$ onto B with a field of definition $K(v, t)$ and consequently, there is a symmetric function Ψ defined on the product of sufficiently many factors equal to Γ into $\mathfrak{p}(C)$ such that the image of $\mathfrak{p}(\Gamma)$ is A and that it is defined over $K(v, t)$. Let Γ be cut out on on Z_v by the linear Variety defined by the set of linear equations

$$\sum l_{ij} X_j - s_i X_0 = 0$$

where (l, s) is a set of independent variables over $K(v)$ and put $\overline{K(l)} = K'$. Then Γ has a Point which is rational over $K'(v)$ and so, when we write

$$\Psi = \sum_{i=1}^{m} \bar{\psi}_i ,$$

where $\bar{\psi}_i$ is a function defined on Γ with values in $\mathfrak{p}(C)$, one of it, say $\bar{\psi}_1 = \bar{\psi}$, may be assumed to be defined over $K(v, s)$ (cf. [W]-2, chap. III, cor. th. 7). This function $\bar{\psi}$ can be extended to a function ψ defined on Z_v with values in $\mathfrak{p}(C)$ with a field of definition $K'(v)$ and further, ψ can be extended to a function φ defined on V with values in $\mathfrak{p}(C)$ defined over K' in a natural way such that

$$\varphi_{Z_v} = \psi, \quad \psi_\Gamma = \bar{\psi}.$$

Let $x_1, \cdots, x_m$ be m independent generic Points of V over K' and put $\sum \varphi(x_i) = \zeta$. ζ has the Locus A'' over K' and A'' clearly contains A as a Subvariety. This implies that $A = A''$ and hence V generates the Abelian Variety A isogeneous to $\mathfrak{p}(Z_v)$ and consequently $\mathfrak{p}(V)$ and $\mathfrak{p}(Z_v)$ are isogeneous. q.e.d.

Corollary. *Let V^n be an absolutely locally normal Variety, free from singular Subvariety of dimension n-2 in a projective space and W^{n-1} be its generic (n-1)-cycle over a certain field of definition for V. Then the Picard Variety $\mathfrak{p}(V)$ and $\mathfrak{p}(W)$ of V and W are isomorph.*

Proof. By Weil's criterion for linear equivalence (cf. [W]-3, (E)), when V-divisor X is algebraically equivalent to zero, $X \cdot W \sim 0$ on W and $X \sim 0$ are equivalent (cf. also [W]-1, chap. VIII, th. 4). Moreover, when we apply our theorem to this case, $\mathfrak{p}(V)$ and $\mathfrak{p}(W)$ are isomorph as abstract groups. Let $\{X\}$ be a maximal total algebraic family of positive V-divisors and K be a common field of definition for V, W, $\{X\}$, $\mathfrak{p}(V)$, $\mathfrak{p}(W)$ over which a certain divisor X_0 in $\{X\}$ is rational and assume that K is a complete field of definition for $\mathfrak{p}(V)$ and $\mathfrak{p}(W)$. Let M and M^* be the Chow-Points of the associated-Varieties of the complete linear system $|X|$ and $|W \cdot X|$ where X is a generic divisor of $\{X\}$ over K. Then the remark made at the top of this proof implies that $K(M) \supset K(M^*)$ and moreover, M is purely inseparable over $K(M^*)$ (cf. [M]-3, § 4). Hence it is sufficient to prove that when we extend the field of reference K to the algebraic closure $\bar{K}$ of it, $\bar{K}(M)$ is a separable extension of $\bar{K}(M^*)$. It is easy to see that $|X|$ and $|W\cdot X|$ are both defined respectively over $\bar{K}(M)$ and over $\bar{K}(M^*)$, that is, the base of the modules $L(X)$ and $L(W\cdot X)$ have the basis consisting of functions defined over $\bar{K}(M)$ and over $\bar{K}(M^*)$ respectively (cf. [W]-1, chap. VIII, th. 10). Since V^n is absolutely locally normal, and free from singular Subvarieties of dimension n-2, W is also absolutely locally normal and free from singular Subvarieties of dimension n-3 (cf. [N], [Z]-th. 3). Then in view of the Castelnuovo's lemma[4] (cf. [M]-6, p. 126), we may assume that $|X|$ induces on W the complete linear system $|X\cdot W|$ (cf. also [M]-5, th. 2). Let X^* be a rational divisor of $|X\cdot W|$ over $\bar{K}(M^*)$. There is a divisor X' in $|X|$ such that

$$X^*=X'\cdot W$$

Let x' and x^* be the Chow-Points of X' and X^* respectively. Then we have $\bar{K}(x^*)=\bar{K}(M^*)$ and $\bar{K}(x')\supset\bar{K}(M)$ (cf. [M]-3 lemma 4). We may assume that X^* and X' are both Varieties. Then $\bar{K}(x')$ is a separable extension of $\bar{K}(x^*)$ by [M]-5, prop. 7 and this completes our proof.

Bibliography

W. L. Chow, Algebraic system of positive cycles in an algebraic variety, Am. J. Math. vol. 72, 1950...[C]-1.

W. L. Chow, The Jacobian Varieties of algebraic varieties, to appear soon...[C]-2.

W. L. Chow, On the defining field of a divisor in an algebraic variety, Proc. Am. Math. Soc. vol. 1, 1950...[C]-3.

T. Matsusaka, The theorem of Bertini on linear systems, Mem. Col. Sci. Kyoto Univ. Ser. A, 26, 1951...[M]-1.

T. Matsusaka, On the algebraic construction of the Picard Variety I Jap. J. Math., vol. 21, 1951...[M]-2.

T. Matsusaka, On the algebraic construction of the Picard Variety II Jap. J. Math. vol. 22, 1952...[M]-3.

T. Matsusaka, Some theorems on Abelian Varieties, Nat. Rep., Ochanomizu Univ. vol. 4, 1953...[M]-4.

T. Matsusaka, On algebraic families of positive divisors and their associated-Varieties on a projective Variety, J. Math. Soc. Jap. vol. 5, 1953...[M]-5.

A. Weil, Foundations of Algebraic Geometry, Am. Math. Soc. Colloq., vol. 29, 1946... [W]-1.

A. Weil, Variétès Abéliennes et Courbes algébriques, Act. Sc. et Ind., no 1046, 1948 ...[W]-2.

A. Weil, Criteria for linear equivalence, Proc. Nat. Acad. Sci. U.S.A., vol. 38, no. 3, 1952...[W]-3.

4) The Castelnuovo's lemma we need here is the following: let V^n be an absolutely locally normal Variety in a projective space, having no singular Subvariety of dimension n-2, defined over a field k. Let C be a generic divisor of L_1 over k. When X is a V-divisor, then the complete linear system $|X+hC|$ induces on C a complete linear system if h is large.

O. Zariski, Complete linear systems on normal varieties and a generalization of a lemma of Enriques-Severi, Ann. Math. vol. 55, 1952...[Z].

Y. Nakai, On the section of an algebraic variety by the generic hyperplane, Mem. Col. Sci. Kyoto Univ. Ser. A. 26, 1951...[N].

(*Received November 28, 1953*)