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A Remark on my paper "Some Theorems on Abelian Varieties"

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The writer proved in "Some theorems on Abelian Varieties" (vol. 4, no. 1, 1953 of this Report) the following theorem:

Let U be a Variety having the normal law of composition both defined over a field k. When U is birationally equivalent to an Abelian Variety, it is birationally equivalent over k to an Abelian Variety in a projective space defined over k.

But the proof of the above theorem (cf. th. 3) is very complicated and moreover it contains a certain difficulty since the Variety C (cf. p. 34) is not an Abelian Variety in general. The aim of this note is to regulate and correct the proof of th. 3. We use the same notations and conventions as in th. 3.

Let f be a function defined on a Variety U with values on a Variety V, Z be the graph of f and x be a Point of V. Assume that $Z_\frown(U\times x)$ has no component of greater dimension than zero, then we shall say that x is not a fundamental Point for f^{-1}, and when this holds for all Points on V, we shall say that f^{-1} has no fundamental Point on V. When x is not fundamental for f^{-1} we denote by $f^{-1}(x)\times x$ the sum of all the distinct components of

$$Z_\frown(U\times x)\,.$$

In the same way, when $Z_\frown(y\times U)$ has no component of greater dimension than zero, we shall say that y is not a fundamental Point of f and when every Point on U is not fundamental for U, we shall say that f has no fundamental Point on U. We define $f(y)$ in the same way as $f^{-1}(x)$.

Lemma. *Let U^n, W^n, A^n be three Varieties, f and g be functions defined on U and defined on A with values on A and on W respectively having the following properties: (i) U has a normal law of composition defined over a field k of definition for U, and U is birationally equivalent to an Abelian Variety, (ii) f and g are both defined over the algebraic closure $\bar{k}$ of k, (iii) the composite function $g\circ f$ is defined over k, (iv) g is defined everywhere on A and g^{-1} has no fundamental Point on W, (v) A is an Abelian Variety, (vi) W is a projective Variety. When these conditions are satisfied, U is birationally equivalent over k to an Abelian Variety defined over k in a projective space.*

Proof. Let u be a generic Point of U over k and put $f(u)=\xi$,

$g\circ f(u)=x$. Then by (iii), we have $k(x)\subset k(u)$ and by prop. 5, there is a Variety U_1 defined and birationally equivalent to U over k and a function g_1 defined on U_1 with values on W such that g_1 is defined over k, g_1 is defined at every Point of U_1, g_1^{-1} has no fundamental Point on W and that when $\bar{u}$ is the corresponding generic Point of U_1 over k to u, $g_1(\bar{u})=x$. Moreover, we may assume that U_1 is relatively normal with reference to k. There is a function f_1 defined on U_1 with values on A with a field of definition $\bar{k}$ such that $f_1(\bar{u})=\xi$ and hence $g_1=g\circ f_1$.

For simplicity, we omit to write —, and let $u\times v\times w$ be a generic Point of the graph Z of the normal law of composition on U_1 over k. Then, there is a Point α on A rational over $\bar{k}$ such that $f_1-\alpha=h$ is a function defined on U_1 with values on A with a field of definition $\bar{k}$ such that

$$h(u.v)=h(u)+h(v) \quad \text{(cf. [W]-3, III, th.9).}$$

Let T be the Locus of $u\times f_1(u)\times g(f_1(u))=u\times\xi\times x$ over $\bar{k}$. We have $x=g_1(u)$. Let $u'\times\xi'\times x'$ be a Point on T, then since g_1 is defined at every Point on U_1, $g_1(u')=x'$ and since $\xi'\times x'$ is a specialization of $\xi\times x$ over $\bar{k}$, ξ' must be contained in a component of $g^{-1}(x')$ which is a 0-dimensional cycle on A and hence, f_1 has no fundamental Point on U_1. Conversely, since g is defined at every Point of A, we have $x'=g(\xi')$. As $u'\times x'$ is a specialization of $u\times x$ over $\bar{k}$, u' must be a component of $g_1^{-1}(x')$ since g_1^{-1} has no fundamental Point on W. This shows that f_1^{-1} has no fundamental Point on A. Hence h has no fundamental Point on U_1 and h^{-1} has no fundamental Point on A.

Put $h(u)=\xi$, $h(v)=\eta$, $h(w)=\zeta$ and let E be the Locus of $u\times v\times w\times\xi\times\eta\times\zeta$ over $\bar{k}$. Let $u'\times v'\times w'\times\xi'\times\eta'\times\zeta'$ be a Point on E having the projection $u'\times v'$ on $U_1\times U_1$. Then ξ' and η' are components of $h(u')$ and $h(v')$ respectively. Hence the specialization of ζ over $u\times v\to u'\times v'$ with reference to $\bar{k}$ is in finite numbers. Moreover, w is a component of $h^{-1}(\zeta)$ and so specializations of w over $u\times v\to u'\times v'$ with reference to $\bar{k}$ are in finite numbers. This shows that $(u'\times v'\times U_1)\frown Z$ has no component of greater dimension than zero. The same holds for $(u'\times U_1\times w')\frown Z$ and from this we conclude that U_1 is an Abelian Variety by prop. 6. q.e.d.

Proof of th. 3. Since U is birationally equivalent to an Abelian Variety, we may assume that there is an Abelian Variety A defined over $\bar{k}$ in a projective space such that there is a function f defined on U with values on A with a field of definition $\bar{k}$ (cf. prop. 4).

Put $U=U$, $W=A$, $A=A$, $f=f$, $g=\delta A$, $k=\bar{k}$. Then our lemma shows that there is an Abelian Variety in a projective space defined and birationally equivalent to U over $\bar{k}$. Assume that A is already such an Abelian Variety. Let T be a birational correspondence between U and A and $(T_1=T, \ldots, T_m)$ be the set of complete conjugates of T over k. Let u be a generic Point of U over $\bar{k}$, L be the ambient space of A and put

$$(u \times L) \cdot \sum T_i = u \times \sum \xi_i \,.$$

$\sum \xi_i$ is rational over $k(u)$ and hence its Chow-Point x is rational over $k(u)$. Hence x has the Locus W over k. ξ_i is a Point of $A_i = pr_L T_i$ which is an Abelian Variety and (ξ_i, ξ_j) is a generic Point of the birational correspondence T_{ji} between A_i and A_j over $\bar{k}$ since it holds that

$$\bar{k}(u) = \bar{k}(\xi_i) = \bar{k}(\xi_j) (\supset k(x)).$$

Put $\xi_1 = \xi$. Let f, g be functions such that $f(u) = \xi$, $g(\xi) = x$. The conditions (i), (ii) of our lemma are clearly satisfied. The composite function $g \circ f$ is defined over k since x is rational over $k(u)$.

Let Z be the Locus of $\xi_1 \times \ldots \times \xi_m$ over $\bar{k}$, then $pr_{ij} Y = T_{ji}$. Let ξ_1' be a Point on A and extend the specialization $\xi_i \to \xi_1$ to a specialization $\xi_1 \times \cdots \times \xi_m \to \xi_{i_1}' \times \cdots \times \xi_{i_m}'$ over $\bar{k}$ when ξ_1' is a specialization of ξ_i over $\bar{k}$. Then this is uniquely determined since T_{ji} is a birational transformation between A_i and A_j (cf. [W]-3, II, th. 6). This proves that when G is the graph of g, $G \frown (W \times \xi_1')$ has no component of greater dimension than zero and since A has no singular Point, g is defined at ξ_1' by Zariski's main theorem on birational transformations (cf. [Z]). Moreover, it is easy to see that g^{-1} has also no fundamental Point on W and the condition (iv) is also satisfied. Conditions (v), (vi) are satisfied in our case and hence our theorem is proved.

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