On the Divisor-Problem Generated by $\zeta^{\alpha}(s)$

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We are concerned with a generalization of the Piltz divisor-problem. Let $\zeta(s)$ be the Riemann zeta-function and write, for $\alpha>0$,

$$\zeta^{\alpha}(s)=\sum_{n=1}^{\infty} d_{\alpha}(n)n^{-s} \quad (\sigma=\Re s>1)\ ,$$

where we take that branch of $\zeta^{\alpha}(s)$ which is positive for $s>1$. That this branch can be expanded, as above, into a Dirichlet series absolutely convergent for $\sigma>1$ is evident if we consider, for instance, the Euler product expression of $\zeta(s)$. We are interested in the asymptotic behaviour (as $x\to\infty$) of the summatorial function

$$D_{\alpha}(x)=\sum_{n\leqq x} d_{\alpha}(n) \quad (x>0).$$

For integral values of α our problem reduces to that of Piltz, on which there are numerous investigations. However, it seems to the author that the case of non-integral α has not been treated hitherto. It turns out that in this latter case our problem shows itself much like that of $\pi(x)$, the number of primes $\leqq x$. Now the most remarkable result for $\pi(x)$ that can be proved in a comparatively simple way is the prime-number theorem, and it is the analougue to this theorem that we propose to give an outlined proof in this short note. Our result is the following

Theorem. $$D_{\alpha}(x)\sim\frac{1}{\Gamma(\alpha)}x(\log x)^{\alpha-1} \quad (x\to\infty).$$

This is deduced in a routine way from the following lemma, whose proof is not difficult if we apply the Ikehara-Wiener-Landau method for Tauberian theorems.

Lemma. *Suppose that $g(s)$ is a function defined and continuous on the half-plane $\sigma=\Re s\geqq 1$ and regular for $\sigma>1$, such that $g(1+ti)$ is of bounded variation on the interval $-\tau\leqq t\leqq\tau$ for each $\tau>0$ $(t=\Im s)$. Let A and λ be certain constants, $A>0$, $0<\lambda<1$, and let*

$$f(s)=g(s)+A(s-1)^{-\lambda} \quad (\sigma>1)\ ,$$

wherein $(s-1)^{-\lambda}$ represents the principal branch. Further let $F(u)$ be a non-negative function defined and non-decreasing for $u>0$, such that

$$f(s)=\int_0^{\infty} F(u)e^{-su}du \quad (\sigma>1).$$

Then $$F(u)\sim\frac{A}{\Gamma(\alpha)}e^{u}u^{\lambda-1} \quad (u\to\infty).$$

(*Received Dec. 16, 1953*)