A Remark on the Takahashi-Umezawa-Katayama Theory of Interaction Representation[1)]

(Theory of Quantization of Field. Part I)

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Résumé

The general theory of interaction representation proposed by Takahashi and Umezawa is re-examined by our improved method for a simple case of interaction Lagrange function with the fourth derivative. This example shows that the expansion with respect to the interaction parameter g is impossible for some cases. In the case of quantization by *one* harmonic oscillator (Heisenberg-Pauli's quantization of field), an interaction Hamiltonian H' can be determined, but is proved to be valid only up to its first order of the interaction parameter g, while in the case of quantization of field by *two* harmonic oscillators (Pais-Uhlenbeck's quantization of field), the Hamiltonian is valid up to any order of g. Thus the general proof of the existence of the G-function given by Takahashi and Umezawa becomes doubtful. To quantize the field, our method used in this paper will be more general than that of Takahashi and Umezawa.

Introduction

During the last year a new theory has been developed by Takahashi and Umezawa[(1)] about the interaction representation for the interaction Hamiltonian with higher derivatives, by modifying the method of Yang-Feldman[(2)]. Then, their theory was analyzed by Katayama[(3)] on its meaning as a quantized theory of field. Recently one example with the fourth derivative was discussed between Okabayashi[(4)] and Katayama[(5)]. By their discussions the construction of the T-U theory as the theory of interaction representation seems to be made fairly clear.

In spite of the apparent success of the T-U theory for higher derivative interaction, there remain several points to be discussed before this theory is applied to the theory of non-local interaction. Among these the following two points are discussed. First, though they have introduced a G-function[(6)] in their theory as a subsidiary apparatus so as to make the free field one harmonic oscillator, it seems to us to be unnatural as a method of quantization of field. As this introduction of a G-function is combined with the assumption of the existence of a free

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field, it is doubtful to assume that the G-function can be determined up to any order of the interaction parameter g, because free fields are not necessarily determined for all cases of interaction representation, as shown in this paper. Secondly, the connection between the T-U theory and Pais-Uhlenbeck's theory[7] about the quantization of field (or Ostrogradski's idea[8]) by many harmonic oscillators remains obscure. Though an example for the latter theory was already discussed by Katayama[9], this connection has not been pointed out by him. In our example, which is easily generalized, this relation will be made clear.

About these points we shall publish a general theory in a later paper, in which we shall develop a theory, which is based on a complete construction of all independent solutions of the system specified by a Lagrange function with higher derivative interaction in Heisenberg representation. The solutions of the free field in the interaction representation are constructed so as to satisfy certain conditions, i.e., solutions are to be those of some harmonic oscillators. According to whether these conditions are satisfied or not, the theory of interaction representation becomes possible or not. Consequently, the construction of the G-function becomes possible or not.

In this paper our example, which is the same as the one discussed between Okabayashi and Katayama, i.e., the quantization of a scalar field with an interaction Lagrange function of the fourth order, is re-examined and fully discussed by the procedure of our theory. We believe that the important points mentioned above, i.e., the possibility of the construction of the G-function and the connection between two methods of quantization of field (Heisenberg-Pauli and Pais-Uhlenbeck's methods), will be made clear.

1. Proposition of an Example. Its General Solutions in Heisenberg Representation

To make clear our theory without losing its essential features, we shall take up the case discussed by Okabayashi and Katayama, in which the Lagrange function for a scalar field $\phi(x)$ with the fourth derivative is taken as

$$L=-\tfrac{1}{2}\{\partial_\lambda\phi\partial_\lambda\phi+m^2\phi^2+g\Box\phi\Box\phi\},$$
$$(\text{or } L=-\tfrac{1}{2}\{-\phi\Box\phi+m^2\phi^2+g\phi\Box^2\phi\}), \qquad (1)$$

where $\phi(x)$ represents a field in the Heisenberg representation. Its equation of motion is then

$$(\Box-m^2)\phi(x)=g\Box^2\phi. \qquad (2)$$

The free field $\phi_0(x)$ of a non-perturbing Lagrange function

$$L_0=-\tfrac{1}{2}\{\partial_\lambda\phi_0\partial_\lambda\phi_0+m^2\phi_0^2\}, \qquad (3)$$

satisfies the equation of motion

$$(\Box - m^2)\phi_0(x)=0 . \tag{4}$$

Its Green function is defined by

$$\bar{\Delta}(x-x')=\frac{1+\varepsilon(x, x')}{2}\Delta(x-x') , \tag{5}$$

where

$$\varepsilon(x, x')=\begin{cases} +1 & \text{for} \quad x_0>x_0' \\ -1 & \text{for} \quad x_0<x_0' , \end{cases} \tag{6}$$

$$[\phi_0(x), \ \phi_0(x')]=i\Delta(x-x') . \tag{7}$$

Then the solutions of Eq. (2) are expressed in the form of an integral equation as

$$\phi_i(x)=\phi^{in}(x)+g\int_{-\infty}^{+\infty}K^{(i)}(x, x')\phi_i(x')dx' , \tag{8}$$

where its integral kernels $K^{(i)}(x, x')$ are given as follows,

$$\left.\begin{aligned} K^{(1)}(x, x')&=\frac{1+\varepsilon(x, x')}{2}\Delta(x-x')\Box'^2 , \\ K^{(2)}(x, x')&=S_1(\partial)\frac{1+\varepsilon(x, x')}{2}\Delta(x-x')S_2(\partial')\Box' , \\ K^{(3)}(x, x')&=\Box\frac{1+\varepsilon(x, x')}{2}\Delta(x-x')\Box' , \\ K^{(4)}(x, x')&=\Box S_1(\partial)\frac{1+\varepsilon(x, x')}{2}\Delta(x-x')S_2(\partial') , \\ K^{(5)}(x, x')&=\Box^2\frac{1+\varepsilon(x, x')}{2}\Delta(x-x') , \end{aligned}\right\} \tag{9}$$

where $S_1(\partial)$ and $S_2(\partial)$ are some linear combinations of the first derivative.

Among these integral kernels $K^{(i)}$, we shall make use of $K^{(1)}$, $K^{(3)}$ and $K^{(5)}$ only, in this paper, by virtue of the special form of our Lagrange function. Furthermore, from the theory of linear partial differential equations of the fourth order, Eq. (2) has four independent particular solutions for the initial value problem. Thus the above five integral kernels $K^{(i)}(x, x')$ are not independent of each other, and $K^{(5)}$, say, will be a linear combination of $K^{(1)}$ and $K^{(3)}$. Our solution (8) coincides with Okabayashi's Eq. (3′) when $i=5$, and with Katayama's Eq. (3′′) when $i=3$.

2. Discussion of Particular Solutions on a σ-Surface (Discussion of boundary conditions)

Next, in order to discuss the properties of particular solutions for the above-mentioned three integral kernels $K^{(1)}$, $K^{(3)}$ and $K^{(5)}$, we shall introduce solutions $\phi_{0i}(x, \sigma)$ in the interaction representation

$$\phi_{0i}(x, \sigma)=\phi^{in}(x)+g\int_{-\infty}^{+\infty}\frac{1+\varepsilon(\sigma, x')}{2}\Delta(x-x')\Box'^{2}\boldsymbol{\phi}_i(x')dx', \tag{10}$$

where

$$\varepsilon(\sigma, x')=\begin{cases} +1 & \text{when } \sigma \text{ is later than } x', \\ -1 & \text{when } x' \text{ is later than } \sigma. \end{cases}$$

When the point x is not on the σ-surface, from its properties of Eqs. (4) and (7) $\phi_{0i}(x, \sigma)$ satisfies the equation of free field,

$$(\Box-m^2)\phi_{0i}(x, \sigma)=0. \tag{11}$$

When x is on the σ-surface, as noticed by Okabayashi, by performing the procedure x/σ of putting x on σ after the differentiation, the solutions satisfy again equations of motion (11). Taking the difference of $\boldsymbol{\phi}_i(x)$ and $\phi_{0i}(x)$, we have

$$\left.\begin{aligned}
\boldsymbol{\phi}_1(x)-\phi_{01}(x/\sigma)&= g\int_{-\infty}^{\infty}\frac{\varepsilon(x, x')-\varepsilon(\sigma, x')}{2}\Delta(x-x')\Box'^{2}\boldsymbol{\phi}_1(x')dx', \\
\boldsymbol{\phi}_3(x)-\phi_{03}(x/\sigma)&= g\int_{-\infty}^{\infty}\frac{\Box\varepsilon(x, x')-\varepsilon(\sigma, x')\Box'}{2}\Delta(x-x')\Box'\boldsymbol{\phi}_3(x')dx', \\
\boldsymbol{\phi}_5(x)-\phi_{05}(x/\sigma)&= g\int_{-\infty}^{\infty}\frac{\Box^2\varepsilon(x, x')-\varepsilon(\sigma, x')\Box'^2}{2}\Delta(x-x')\boldsymbol{\phi}_5(x')dx'.
\end{aligned}\right\} \tag{12}$$

Then after some calculations, we have for x on the σ-surface

$$\left.\begin{aligned}
\phi_{01}(x/\sigma)&=\boldsymbol{\phi}_1(x), \\
\phi_{03}(x/\sigma)&=(1-g\Box)\boldsymbol{\phi}_3(x), \\
\phi_{05}(x/\sigma)&=\{1-g(\Box+m^2)\}\boldsymbol{\phi}_5(x).
\end{aligned}\right\} \tag{13}$$

By Eq. (13), which represents the relation between $\boldsymbol{\phi}_i(x)$ and $\phi_{0i}(x/\sigma)$ on the σ-surface, we have made clear the properties of the boundary conditions of the particular solutions obtained in Eq. (8).

3. Mutual Relation between the General Solution in Heisenberg Representation and the Free Field Solution in the Interaction Representation

If we replace the kernels $K^{(i)}$ in Eq. (8) by a linear combination of the three integral kernels

$$aK^{(1)}+bK^{(3)}+cK^{(5)}, \tag{14}$$

we have

$$\boldsymbol{\phi}(x)=\phi^{in}(x)+g\int_{-\infty}^{\infty}(aK^{(1)}+bK^{(3)}+cK^{(5)})\boldsymbol{\phi}(x')dx', \tag{15}$$

as a general solution of Eq. (2) in Heisenberg representation. From the condition that Eq. (15) should satisfy Eq. (2), we have

$$a+b+c=1\,. \tag{16}$$

As a relation which transforms the wave function (15) in Heisenberg representation into the one in interaction representation we have

$$\phi_0(x,\sigma)=\phi^{in}(x)+g\int_{-\infty}^{\infty}\frac{1+\varepsilon(\sigma,x')}{2}\Delta(x-x')\Box'^2\phi(x')dx\,. \tag{17}$$

The difference between $\phi(x)$ and $\phi_0(x,\sigma)$ is

$$\phi(x)-\phi_0(x,\sigma)=g\int_{-\infty}^{\infty}\Big\{aK^{(1)}+bK^{(3)}+cK^{(5)}-\frac{1+\varepsilon(\sigma,x')}{2}\Delta(x-x')\Box'^2\Big\}\phi(x')dx'\,. \tag{18}$$

When x is on the σ-surface, we have

$$\phi(x)-\phi_0(x/\sigma)=\{bg\Box+cg(\Box+m^2)\}\phi(x)\,, \tag{19}$$

or

$$\phi(x)=\{1-(b+c)g\Box-cgm^2\}^{-1}\phi_0(x/\sigma)\,. \tag{19'}$$

4. Approximation (or Substitution) of Lagrange Function by one (or two) Harmonic Oscillator

Takahashi and Umezawa have emphasized that the essence of the theory of quantization of field by making use of the interaction representation exists in the reduction of the form of Lagrange function into that for several independent harmonic oscillators (or free fields). Adopting this point of view, we shall examine how the indefinite factors a, b and c, introduced above, are determined for the several examples appearing in the theories of Takahashi-Umezawa and Katayama.

According to our theory, for the Lagrange function, Eq. (1) we have the following relation between $\phi_0(x/\sigma)$ and $\phi(x)$

$$\phi_0(x/\sigma)=\{1-gcm^2-g(b+c)\Box\}\phi(x)\,. \tag{20}$$

Because of the special structure of this Lagrange function it will be sufficient to examine the substitution of one or two harmonic oscillators in place of it. As shown immediately below, the approximation for one harmonic oscillator is permissible only up to the first order of the interaction parameter g, while the substitution of two harmonic oscillators can be done completely.

(i) Approximation by one harmonic oscillator.

For the treatment of one harmonic oscillator the function $\phi_0(x/\sigma)$ satisfies the following free field equation:

$$(\Box - m'^2)\phi_0(x/\sigma) = 0, \tag{21}$$

where m' is its rest mass. Then the Lagrange function for $\phi_0(x/\sigma)$ must be

$$L = \tfrac{1}{2}\phi_0(x/\sigma)(\Box - m'^2)\phi_0(x/\sigma). \tag{22}$$

We shall examine how to identify Eq. (22) with Eq. (1) under the relation (20). Substituting the relation (20) into Eq. (22) we have

$$L = \tfrac{1}{2}\phi(x)[g^2(b+c)^2\Box^3 - \{2g(b+c)(1-gcm^2) + m'^2g^2(b+c)^2\}\Box^2 + \{(1-gcm^2)^2 + 2m'^2g(1-gcm^2)(b+c)\}\Box - (1-gcm^2)^2m'^2]\phi(x). \tag{23}$$

The term containing $\Box^3$ in this expression is of the second order with respect to the parameter g. So we cannot determine the constants a, b and c so as to identify Eq. (23) and Eq. (1) above the first order of g. On the other hand, if we retain the approximation up to the first order of g, the identification can be done and the constants can be determined as follows by using the relation (16)

$$a = c = \tfrac{1}{2}, \qquad b = 0.$$

Then the rest mass of free field $\phi_0(x/\sigma)$ can also be determined as follows

$$m'^2 = \frac{m^2}{(1-gcm^2)^2} = m^2 + gm^4 + \cdots. \tag{24}$$

Thus according to our method the approximation of the interaction field in Heisenberg representation through the free field of one harmonic oscillator can be done only up to the first order of g. On the other hand, according to the theory of Takahashi and Umezawa this approximation could be done to any higher order of g by properly determining the G-function. Their result contradicts our present result. Furthermore, Katayama has shown that he obtained the following solution as a perfect approximation by one harmonic oscillator

$$\phi_0(x/\sigma) = \{1 - g(\Box + \mu_1^2)\}^{1/2}\phi(x). \tag{25}$$

Expanding this function into the infinite series with respect to $\Box$, we have

$$\phi_0(x/\sigma) = \{1 - \tfrac{1}{2}g(\Box + \mu_1^2) - \tfrac{1}{8}g^2(\Box + \mu_1^2)^2 + \cdots\}\phi(x). \tag{26}$$

In order to satisfy the free equation for ϕ_0, the function ϕ must satisfy the equation of motion with the infinite order of $\Box$, which is obtained by operating $\Box - \mu_1^2$ to the right-hand side of Eq. (26)

$$(\Box - \mu_1^2)\{1 - \tfrac{1}{2}g(\Box + \mu_1^2) - \tfrac{1}{8}g^2(\Box + \mu_1^2)^2 + \cdots\}\phi(x) = 0. \tag{27}$$

On the other hand, $\phi(x)$ has to satisfy the equation of motion (2)

$$(m^2 - \Box + g\Box^2)\phi(x) = 0. \tag{28}$$

The equations (27) and (28) contradict each other, since one is of finite order in $\Box$ while the other of infinite order. This contradiction arises from the fact that the particular solution obtained by Katayama does not satisfy the equation of motion for a free field (2) derived from the Lagrange function (1) in Heisenberg representation. This mistake would arise from his adoption of factorization of $(1-g(\Box+\mu_1^2))$ into the products of two square root factors $(1-2g(\Box+\mu_1^2))^{1/2}$, because the square root of the differentiation operator has not a correct meaning, and its expansion in powers of g does not necessarily converge.

(ii) Substitution of two harmonic oscillators.

We shall extend our solution $\phi_0(x/\sigma)$ obtained above to the approximation for two harmonic oscillators, in which two sets of parameters (a_1, b_1, c_1) and (a_2, b_2, c_2) are introduced analogously to the case of Eq. (15). Then, when x is on the σ-surface, $\phi_{01}(x/\sigma)$ and $\phi_{02}(x/\sigma)$ are defined by the following relations

$$\left.\begin{aligned}\phi_{01}(x/\sigma)&=\{1-gc_1m^2-g(b_1+c_1)\Box\}\boldsymbol{\phi}(x)\,,\\ \phi_{02}(x/\sigma)&=\{1-gc_2m^2-g(b_2+c_2)\Box\}\boldsymbol{\phi}(x)\,,\end{aligned}\right\}\tag{29}$$

where $a_1+b_1+c_1=1$ and $a_2+b_2+c_2=1$. ϕ_{01} and ϕ_{02} obey the equation of motion of free field analogous to Eq. (4). Then the Lagrange function for two harmonic oscillators (or free fields) must be of the following form

$$L=\tfrac{1}{2}\{\phi_{01}(\Box-\mu_1^2)\phi_{01}+\alpha\phi_{02}(\Box-\mu_2^2)\phi_{02}\}. \tag{30}$$

Substituting Eq. (29) into Eq. (30), and comparing the coefficients of $\Box^i\boldsymbol{\phi}(x)$ $(i=0,\ 1,\ 2,\ 3)$ we have six equations of condition for the seven undetermined constants a_1, b_1, c_1, a_2, b_2, c_2 and α. Therefore these seven constants are not completely determined, but one of them remains undetermined. So this substitution of two harmonic oscillators can always be carried out and in infinitely many ways.

As a special case of these the following example is shown by Katayama.

$$\left.\begin{aligned}\phi_{01}(x/\sigma)&=-(1-4gm^2)^{-1/4}(\Box-\mu_1^2)\boldsymbol{\phi}(x)\,,\\ \phi_{02}(x/\sigma)&=-(1-4gm^2)^{-1/4}(\Box-\mu_2^2)\boldsymbol{\phi}(x)\,,\end{aligned}\right\}\tag{31}$$

where μ_1^2 and μ_2^2 are two roots of

$$\left.\begin{aligned}&gt^2-t+m^2=0\\ &t=\mu_1^2,\ \mu_2^2=\frac{1\pm\sqrt{1-4\,g\,m^2}}{2g}\end{aligned}\right\}\tag{32}$$

and the following relation is used

$$(\Box-\mu_1^2)^2(\Box-\mu_2^2)-(\Box-\mu_2^2)^2(\Box-\mu_1^2)=\sqrt{1-4gm^2}/g(g\Box^2-\Box+m^2). \tag{33}$$

For this case our indefinite constants are taken as

$$\begin{cases} a_1=1-\dfrac{1}{K} \\ b_1=\dfrac{g(m^2+\mu_1^2)-K}{gm^2K} \\ c_1=\dfrac{K-g\mu_1^2}{gm^2K} \end{cases} \qquad \begin{cases} a_2=1-\dfrac{1}{K} \\ b_2=\dfrac{g(m^2+\mu_2^2)-K}{gm^2K} \\ c_2=\dfrac{K-g\mu_2^2}{gm^2K}\,, \end{cases}$$

$$\alpha=-1\,,$$

where $$K=(1-4gm^2)^{-1/4}.$$

5. Quantization of Field

For the Lagrange function of Eq. (1) after the approximation has been determined for some harmonic oscillators, the quantization of its field can be performed easily as follows.

(i) Case of approximation using one harmonic oscillator

For this case the quantization is performed as usual. As the Lagrange function is

$$L=\tfrac{1}{2}\phi_0(\Box-m'^2)\phi_0\approx-\tfrac{1}{2}(\partial_\lambda\phi_0\partial_\lambda\phi_0+m'^2\phi_0^2)\,, \tag{34}$$

its Hamiltonian becomes

$$H=\tfrac{1}{2}(m'^2\phi_0^2+\phi_0\Delta\phi_0+P_0^2)\,, \tag{35}$$

where $$P_0=\partial L/\partial\dot{\phi}_0\,.$$

Using the value of m'^2 given by Eq. (24), the Hamiltonian can be divided into two parts

$$H=H_0+H_1 \qquad \text{where} \qquad H_1=\tfrac{1}{2}gm^4\phi_0^2. \tag{36}$$

Its commutation relation is

$$[P_0(x),\,\phi_0(x')]_{t=t'}=(\hbar/i)\delta(x-x'). \tag{37}$$

This case coincides with the quantization by means of the method of Heisenberg-Pauli, the interaction Hamiltonian being Eq. (36). But we must remark that for this case the approximation can be done correctly only up to the first order of g in its expansion.

(ii) Quantization by two harmonic oscillators

As the Lagrange function for this case is

$$L=\tfrac{1}{2}\{\phi_{01}(\Box-\mu_1^2)\phi_{01}-\phi_{02}(\Box-\mu_2^2)\phi_{02}\}, \tag{38}$$

its Hamiltonian is

$$H=\tfrac{1}{2}\{P_1^2+\phi_{01}\Delta\phi_{01}+\mu_1^2\phi_{01}^2-(P_2^2+\phi_{02}\Delta\phi_{02}+\mu_2^2\phi_{02}^2)\}, \tag{39}$$

where P_1 and P_2 are variables canonically conjugate to ϕ_{01} and ϕ_{02}. After the commutation relations

$$\left.\begin{aligned}[P_1(x), \phi_{01}(x')]=(\hbar/i)\delta(x-x'),\\ [P_2(x), \phi_{02}(x')]=(\hbar/i)\delta(x-x'),\end{aligned}\right\} \quad (40)$$

are introduced, the fields ϕ_{01} and ϕ_{02} can be quantized. This method of quantization of field coincides with the method of Pais-Uhlenbeck, which is an extension of Ostrogradski's method for particle mechanics.

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