Natural Science Report, Ochanomizu University, Vol. 4, No. 2, 1953
お茶の水女子大学 自然科学報告 第4巻 第2号

On the Velocity Distribution over the Surface of a Symmetrical Aerofoil at High Speeds. I[1)]

Saburô Asaka (阿阪三郎)

Department of Physics, Faculty of Science,
Ochanomizu University, Tokyo

§ 1. Introduction

It has become more and more important to investigate the flow past an obstacle at high speeds. Various methods proposed by many authors for this purpose are in general classified into three groups, namely, the M^2-expansion method, the thin-wing-expansion method and the hodograph method. It should be emphasized that Imai has given essential contributions to the development of all these methods. In order to investigate systematically the effects of compressibility for the characteristics of aerofoils of various shape, thickness, and camber, the thin-wing-expansion method is the most suitable among them, because it seems to give very accurate results up to considerably high Mach numbers with rather simple procedures. Moreover we can easily take into account the effect of variations in the shape of profiles by this method, while it is very difficult to estimate this effect directly by the hodograph method. In fact, Kaplan has studied the two-dimensional flow of a compressible fluid on the basis of this method, having obtained the third approximation for the flow past a circular arc aerofoil[(1)] and a cusped wing, now called the Kaplan bump.[(2)] Further, investigating the flow along a sinusoidal wall, he suggested a very interesting conclusion about the limit of the continuous flow.[(3)]

On the other hand, about ten years ago developing his thin-wing-expansion method, Imai gave the analytical formulae for the velocity and pressure distributions, lift and moment of an aerofoil placed in the flow of a compressible fluid up to the second approximation,[(4)] and investigated in detail the characteristics of generalized Joukowski profiles.[(5)] Later, extending this method to the third approximation, Imai and Ôyama studied the flow along sinusoidal walls.[(6,7)] Recently this method has been applied to an elliptic cylinder by Hasimoto[(8)] and to a Kaplan bump by Matunobu.[(9)] Further, in order to test the accuracy of the method, Naruse[(10)] has carried out numerical calculations for the special wing, around which the existence of the continuous transonic flow was demonstrated theoretically by Tomotika and Tamada.[(11)] The superiority of the thin-wing-expansion method has been demonstrated in

1) Contribution from Department of Physics, Faculty of Science, Ochanomizu University, No. 17.

all these cases.

So far, however, the sharp-edged aerofoils have not yet been dealt with in detail by this method. Then we applied the thin-wing-expansion method to a symmetrical biconvex circular arc aerofoil, as representative of these profiles.

In this paper, we shall give the outlines of Imai's thin-wing-expansion method for the sake of convenience. At the same time we shall modify and extend the original method, especially in the procedures for obtaining the complex velocity potentials and the mapping functions. By these improvements it will be made more systematic to proceed to higher order approximations.

In the next paper, a symmetrical circular arc aerofoil will be treated in detail, and the velocity distributions on its surface up to the third approximation will be presented. Then they will be compared with those for an elliptic cylinder and a Kaplan bump. These results are very interesting and important from both theoretical and practical points of view, since they give informations about the compressibility effects dependent on the shape of aerofoils ; and moreover they may be expected to throw some light on the so-called Taylor's problem, i.e., the problem of existence and stability of the continuous and transonic flow around an obstacle.

Here the author wishes to express his sincere thanks to Prof. Imai for his kind guidance and encouragement through this work. He is also indebted to Prof. Tomotika for his kindness in making the references (3) and (11) available to him.

I. The Thin-Wing-Expansion Method

§ 2. The Basic Formulae of the Thin-Wing-Expansion Method

We shall consider a two-dimensional stationary flow of an ideal fluid past a cylinder of arbitrary shape. We denote the magnitude of the fluid velocity, its x- and y-components, the pressure and the density as q, u, v, p and ρ respectively.[2]

If the flow is continuous and irrotational, there exist a velocity potential Φ and a stream function Ψ such that

$$u=\frac{\partial\Phi}{\partial x}, \qquad v=\frac{\partial\Phi}{\partial y}, \tag{2.1}$$

$$u=\frac{\rho_\infty}{\rho}\frac{\partial\Psi}{\partial y}, \qquad v=-\frac{\rho_\infty}{\rho}\frac{\partial\Psi}{\partial x}. \tag{2.2}$$

We assume that the fluid is compressible and its changes of state obey the adiabatic law :

[2] The corresponding quantities in the undisturbed flow at large distances from the cylinder are denoted with the suffix ∞.

$$\frac{p}{p_\infty}=\left(\frac{\rho}{\rho_\infty}\right)^\gamma , \tag{2.3}$$

where γ is the ratio of specific heats. Then, from Bernoulli's equation we obtain the following relation,

$$\frac{\rho}{\rho_\infty}=\left[1-\frac{\gamma-1}{2}M^2\left(\frac{q^2}{q_\infty^2}-1\right)\right]^{1/(\gamma-1)} , \tag{2.4}$$

where M is the Mach number for the undisturbed flow. For simplicity, we take $q_\infty=u_\infty=1$ and $v_\infty=0$. Then we may designate $\Phi=x$ and $\Psi=y$ for the undisturbed flow. The fundamental assumption for the thin-wing-expansion method is that, deviations of the velocity, pressure, and density from those in the undisturbed flow are small, when the thickness of the cylinder is small. Consequently, the deviations of Φ and Ψ from x and y respectively are also small. Then we may assume that Φ and Ψ can be expanded as follows,

$$\Phi=x+\phi_1+\phi_2+\phi_3+\cdots\cdots , \tag{2.5}$$

$$\Psi=y+\psi_1+\psi_2+\psi_3+\cdots\cdots , \tag{2.6}$$

where $\phi_1, \psi_1; \phi_2, \psi_2$ and $\phi_3, \psi_3;\cdots$ are the quantities of the order of magnitude of $\varepsilon, \varepsilon^2$ and $\varepsilon^3,\cdots$ respectively, and ε is a small parameter representing the thickness of the cylinder.

Combining (2.1) and (2.2), we have

$$\frac{\rho}{\rho_\infty}\frac{\partial\Phi}{\partial x}-\frac{\partial\Psi}{\partial y}=0 , \qquad \frac{\rho}{\rho_\infty}\frac{\partial\Phi}{\partial y}+\frac{\partial\Psi}{\partial x}=0 . \tag{2.7}$$

Substituting (2.5), (2.6) and (2.4) into (2.7), and arranging various terms in the order of magnitude with respect to ε, we obtain a set of differential equations for ϕ_m and ψ_m.

To solve these equations, we transform the variables by the equations :

$$x=\xi , \qquad y=\eta/\mu ; \quad \mu=\sqrt{1-M^2} , \tag{2.8}$$

$$\phi_m=\phi_m , \quad \psi_m=\mu\chi_m \quad (m=1, 2, 3, \cdots) . \tag{2.9}$$

Further we introduce complex variables,

$$\zeta=\xi+i\eta , \qquad \bar{\zeta}=\xi-i\eta , \tag{2.10}$$

and

$$G_m=\phi_m+i\chi_m . \tag{2.11}$$

Integrations of equations thus transformed give the following expressions for G_m:[4, 6]

$$G_1=f(\zeta) , \tag{2.12}$$

$$G_2=\frac{M^2}{4}\left[\nu\left(\frac{df}{d\zeta}\right)^2\bar{\zeta}+(1+\nu)\left\{2\left(\frac{df}{d\zeta}\right)\bar{f}+\int\left(\frac{d\bar{f}}{d\bar{\zeta}}\right)^2 d\bar{\zeta}\right]+f_2(\zeta) , \tag{2.13}$$

$$G_3=A\left(\frac{df}{d\zeta}\right)^3\bar{\zeta}+B\int\left(\frac{d\bar{f}}{d\bar{\zeta}}\right)^3d\bar{\zeta}+3B\left(\frac{df}{d\zeta}\right)^2\bar{f}+C\frac{df}{d\zeta}\int\left(\frac{d\bar{f}}{d\bar{\zeta}}\right)^2d\bar{\zeta}$$

$$+D\int\left(\frac{df}{d\zeta}+\frac{d\bar{f}}{d\bar{\zeta}}\right)\left(\frac{\partial G_2}{\partial\zeta}+\frac{\partial G_2}{\partial\bar{\zeta}}+\frac{\partial\bar{G}_2}{\partial\zeta}+\frac{\partial\bar{G}_2}{\partial\bar{\zeta}}\right)d\bar{\zeta}$$

$$-\frac{M^2}{2}\frac{df}{d\zeta}\int\left(\frac{\partial G_2}{\partial\zeta}+\frac{\partial\bar{G}_2}{\partial\zeta}\right)d\bar{\zeta}+f_3(\zeta)\ , \tag{2.14}$$

where
$$\nu=\frac{\gamma+1}{4}\frac{M^2}{\mu^2}\ , \tag{2.15}$$

and
$$A=-\frac{(1+\gamma)(3-2\gamma)}{96}\frac{M^6}{\mu^2}\ ,$$

$$B=-\frac{M^4}{96\mu^2}\{6(1-\gamma)-(1-2\gamma)(3-\gamma)M^2\}\ ,$$

$$C=\frac{M^2}{32\mu^2}\{8-8(3-\gamma)M^2+(13-9\gamma+2\gamma^2)M^4\}\ ,$$

$$D=\frac{M^2}{8\mu^2}\{4-(3-\gamma)M^2\}\ .$$

$f(\zeta)$, $f_2(\zeta)$ and $f_3(\zeta)$ are analytic functions of ζ, which must be determined by the following conditions:

(i) $\dfrac{\partial G_m}{\partial\zeta}$ is one-valued and continuous everywhere in the flow,

(ii) as $\zeta\to\infty$, $\dfrac{\partial G_m}{\partial\zeta}\to 0$,

(iii) on the surface of the cylinder, $\Im\left(\dfrac{\zeta}{\mu^2}+G_1+G_2+G_3+\cdots\right)=0.$

When G_m are known, velocity potentials and stream functions are easily obtained successively as follows:

$$\left.\begin{aligned}\phi_m&=\frac{1}{2}\left[G_m(\zeta,\bar{\zeta})+\overline{G_m(\zeta,\bar{\zeta})}\right]\ ,\\ \psi_m&=\frac{\mu}{2i}\left[G_m(\zeta,\bar{\zeta})-\overline{G_m(\zeta,\bar{\zeta})}\right]\ .\end{aligned}\right\} \tag{2.16}$$

§ 3. Determination of $f(\zeta)$, $f_2(\zeta)$ and $f_3(\zeta)$

According to the theory of conformal transformations, there exists always an analytical function, which maps the region outside a given profile of the ζ-plane onto the region outside the unit circle ($Z=e^{i\theta}$) of the Z-plane, and by means of which the points at infinity of both planes correspond with each other. Such a function can be expanded as

$$\zeta(Z)=C_{-1}Z+C_0+\frac{C_1}{Z}+\frac{C_2}{Z^2}+\cdots\ ,\quad \text{for}\quad |Z|\geqq 1\ .$$

Hereafter we designate the chord length of the profile as always covering the interval $-1 \leqq \xi \leqq 1$. Since the thickness of the cylinder is small compared with its chord length, $\zeta(Z)$ may be assumed as follows :

$$\zeta(Z)=\frac{1}{2}\left(Z+\frac{1}{Z}\right)+\zeta_1(Z)+\zeta_2(Z)+\zeta_3(Z)+\cdots , \tag{3.1}$$

where the first term on the right-hand side is the mapping function for the flat plate as the 0-th approximation. $\zeta_m(Z)$ is of the order of magnitude of ε^m and may be expanded :

$$\zeta_m(Z)=c^m_{-1}Z+c^m_0+c^m_1\frac{1}{Z}+c^m_2\frac{1}{Z^2}+\cdots . \tag{3.2}$$

If we know the mapping functions (3.2) for the profile, we can determine the complementary functions $f(\zeta)$, $f_2(\zeta)$, and $f_3(\zeta)$, by considering the conditions (i), (ii) and (iii) prescribed for G_m.

As for $G_1=f(\zeta)$, it can contain only terms of negative power of Z for $|Z|\geqq 1$, in order to satisfy the condition (ii). Further, since $\Im(G_1+(\zeta_1/\mu^2))$ must vanish on the profile by the condition (iii), it must also vanish on the unit circle of the Z-plane, where $\bar{Z}=1/Z$. Then we can verify that

$$f(\zeta)=-\frac{1}{\mu^2}\left(\zeta_1(Z)-c^1_{-1}Z-c^1_0-\frac{\overline{c^1_{-1}}}{Z}\right) \tag{3.3}$$ [3)]

is the required function.

By means of this function, we can express G_2 as follows,

$$G_2=f^0_2(Z)+\sum_{j=1}^{3}A^j(g^j_2(Z,\bar{Z})+f^j_2(Z)) , \tag{3.4}$$

where

$$\left.\begin{aligned} &A^1=\frac{M^2}{4}\nu , && g^1_2=\left(\frac{df}{d\zeta}\right)^2\bar{\zeta} , && f^1_2=-\left(\frac{df}{d\zeta}\right)^2\zeta , \\ &A^2=\frac{M^2}{2}(1+\nu) , && g^2_2=\frac{df}{d\zeta}\bar{f} , && f^2_2 , \\ &A^3=\frac{M^2}{4}(1+\nu) , && g^3_2=\int\left(\frac{d\bar{f}}{d\bar{\zeta}}\right)^2 d\bar{\zeta} , && f^3_2=\int\left(\frac{df}{d\zeta}\right)^2 d\zeta . \end{aligned}\right\} \tag{3.5}$$

Here $f^0_2(Z)$ is an analytic function of Z, which has been so determined as to satisfy the conditions (i), (ii) and

(iii′) on the unit circle of the Z-plane

$$\Im\left(f^0_2(Z)+\frac{1}{\mu^2}\zeta_2(Z)\right)=0 ,$$

while $f^j_2(Z)$ are analytic functions of Z, such that $g^j_2+f^j_2$ obeys the conditions (i), (ii) and

(iii″) on the boundary

$$\Im(g^j_2+f^j_2)=0 .$$

3) The constant term in (3.3) can be dropped for simplicity.

Then, $f_2(\zeta)$ in (2.13) is the sum of these functions, namely

$$f_2(\zeta)=f_2^0(Z)+\sum_{j=1}^{3}A^j f_2^j(Z)\,. \tag{3.6}$$

The functions f_2^j can be determined by considerations similar to those for finding f. But in this paper, we shall use the more general and convenient method of conjugate Fourier series, which will be described at the end of this section.

By a treatment similar to that of G_2, we can obtain the third approximation,

$$G_3=f_3^0(Z)+\sum_{j=1}^{24}B^j(g_3^j(Z,\overline{Z})+f_3^j(Z))\ , \tag{3.7}$$

where g_3^j and f_3^j have properties analogous to those of g_2^j and f_2^j. These functions are given in the following table.

$B^1=\dfrac{M^2}{4}\nu$	$g_3^1=\left(\dfrac{df}{d\zeta}\right)^2\bar{\zeta}_1$	f_3^1
$B^2=\dfrac{M^2}{4}\nu$	$g_3^2=\left[\left(\dfrac{df}{d\zeta}\right)^2\right]_3\bar{\zeta}$	$f_3^2=-\left[\left(\dfrac{df}{d\zeta}\right)^2\right]_3\zeta$
$B^3=-\dfrac{M^2}{2}(1+\nu)$	$g_3^3=\left[\dfrac{df}{d\zeta}\bar{f}\right]_3$	f_3^3
$B^4=-\dfrac{M^2}{4}(1+\nu)$	$g_3^4=\left[\int\left(\dfrac{d\bar{f}}{d\bar{\zeta}}\right)^2 d\bar{\zeta}\right]_3$	$f_3^4=\left[\int\left(\dfrac{df}{d\zeta}\right)^2 d\zeta\right]_3$
$B^5=\dfrac{M^2}{2}\nu$	$g_3^5=\dfrac{df}{d\zeta}\dfrac{df_2^0}{d\zeta}\bar{\zeta}$	$f_3^5=-\dfrac{df}{d\zeta}\dfrac{df_2^0}{d\zeta}\zeta$
$B^6=\dfrac{M^2}{2}(1+\nu)$	$g_3^6=\dfrac{df}{d\zeta}\bar{f}_2^0$	f_3^6
$B^7=\dfrac{M^2}{2}(1+\nu)$	$g_3^7=\int\dfrac{d\bar{f}}{d\bar{\zeta}}\dfrac{d\bar{f}_2^0}{d\bar{\zeta}}d\bar{\zeta}$	$f_3^7=\int\dfrac{df}{d\zeta}\dfrac{df_2^0}{d\zeta}d\zeta$
$B^8=\dfrac{M^2}{2}(1+\nu)$	$g_3^8=\dfrac{df_2^0}{d\zeta}\bar{f}$	f_3^8
$B^9=\dfrac{M^4}{12}\nu\{(\gamma+3)+3\nu\}$	$g_3^9=\left(\dfrac{df}{d\zeta}\right)^3\bar{\zeta}$	$f_3^9=-\left(\dfrac{df}{d\zeta}\right)^3\zeta$
$B^{10}=\dfrac{M^4}{16}\{(3\gamma+5)$ $+2(2\gamma+7)\nu+12\nu^2\}$	$g_3^{10}=\left(\dfrac{df}{d\zeta}\right)^2\bar{f}$	f_3^{10}
$B^{11}=\dfrac{M^2}{4}\Big[1+\dfrac{M^2}{2}\{2(\gamma+1)$ $+(2\gamma+9)\nu+7\nu^2\}\Big]$	$g_3^{11}=\dfrac{df}{d\zeta}\int\left(\dfrac{d\bar{f}}{d\bar{\zeta}}\right)^2 d\bar{\zeta}$	f_3^{11}
$B^{12}=\dfrac{M^4}{48}\{3(\gamma+3)$ $+2(2\gamma+11)\nu+16\nu^2\}$	$g_3^{12}=\int\left(\dfrac{d\bar{f}}{d\bar{\zeta}}\right)^3 d\bar{\zeta}$	$f_3^{12}=\int\left(\dfrac{df}{d\zeta}\right)^3 d\zeta$

$$
\left.\begin{array}{lll}
B^{13}=-\dfrac{M^4}{8}\nu^2 & g_3^{13}=\left(\dfrac{df}{d\zeta}\right)^2\dfrac{d^2f}{d\zeta^2}(\zeta-\bar{\zeta})\bar{\zeta} & f_3^{13}=0 \\
B^{14}=-\dfrac{M^4}{8}\nu^2 & g_3^{14}=\left(\dfrac{df}{d\zeta}\right)^2\dfrac{d^2f}{d\zeta^2}\zeta\bar{\zeta} & f_3^{14}=-\left(\dfrac{df}{d\zeta}\right)^2\dfrac{d^2f}{d\zeta^2}\zeta^2 \\
B^{15}=-\dfrac{M^4}{4}\nu(1+\nu) & g_3^{15}=\dfrac{df}{d\zeta}\dfrac{d^2f}{d\zeta^2}\bar{f}(\zeta-\bar{\zeta}) & f_3^{15}=0 \\
B^{16}=\dfrac{M^4}{8}\nu(1+\nu) & g_3^{16}=\dfrac{df}{d\zeta}\left(\dfrac{d\bar{f}}{d\bar{\zeta}}\right)^2(\zeta-\bar{\zeta}) & f_3^{16}=0 \\
B^{17}=\dfrac{M^4}{12}\nu(1+\nu) & g_3^{17}=\left(\dfrac{d\bar{f}}{d\bar{\zeta}}\right)^3(\zeta-\bar{\zeta}) & f_3^{17}=0 \\
B^{18}=\dfrac{M^4}{4}\nu(1+\nu) & g_3^{18}=\dfrac{df}{d\zeta}\dfrac{df_2^2}{d\zeta}\bar{\zeta} & f_3^{18}=-\dfrac{df}{d\zeta}\dfrac{df_2^2}{d\zeta}\zeta \\
B^{19}=\dfrac{M^4}{4}(1+\nu)^2 & g_3^{19}=f\dfrac{df}{d\zeta}\dfrac{d\bar{f}}{d\bar{\zeta}} & f_3^{19} \\
B^{20}=\dfrac{M^4}{8}(1+\nu)^2 & g_3^{20}=f\left(\dfrac{d\bar{\zeta}}{d\bar{\zeta}}\right)^2 & f_3^{20} \\
B^{21}=\dfrac{M^4}{8}(1+\nu)^2 & g_3^{21}=\dfrac{d^2f}{d\zeta^2}(\bar{f})^2 & f_3^{21} \\
B^{22}=\dfrac{M^4}{4}(1+\nu)^2 & g_3^{22}=\dfrac{df}{d\zeta}\bar{f}_2^2 & f_3^{22} \\
B^{23}=\dfrac{M^4}{4}(1+\nu)^2 & g_3^{23}=\displaystyle\int\dfrac{d\bar{f}}{d\bar{\zeta}}\dfrac{d\bar{f}_2^2}{d\bar{\zeta}}d\bar{\zeta} & f_3^{23}=\displaystyle\int\dfrac{df}{d\zeta}\dfrac{df_2^2}{d\zeta}d\zeta \\
B^{24}=\dfrac{M^4}{4}(1+\nu)^2 & g_3^{24}=\dfrac{df_2^2}{d\zeta}\bar{f} & f_3^{24}
\end{array}\right\} \tag{3.8}
$$

(3.8)[4]

Then

$$f_3(\zeta)=f_3^0(Z)+\sum_{j=1}^{24}B^jf_3^j(Z)\ . \tag{3.9}$$

Determination of f_2^j and f_3^j. We shall denote any one of g_2^j and g_3^j by G, and the corresponding one of f_2^j and f_3^j by F respectively. If we put

$$G(e^{i\theta},\ e^{-i\theta})=P(\theta)+iQ(\theta)\ , \tag{3.10}$$

and

$$F(e^{i\theta})=R(\theta)+iI(\theta)\ , \tag{3.11}$$

then

$$I(\theta)=-Q(\theta)\ , \tag{3.12}$$

by the condition :

$$\Im(G+F)=0\ .$$

---

4) In this table G_3 is the sum of the terms of the magnitude of $\mathcal{E}^3$ contained in G_2 and G_3 of § 2 ((2.13) and (2.14)). $[\ ]_3$ denotes the terms of the magnitude of $\mathcal{E}^3$ in the expression in brackets, and ζ may be considered to stand for $\frac{1}{2}(Z+(1/Z))$.

It is easily seen from its constitution that $G(Z, \bar{Z})$ has no singularity for $|Z|\geqq 1$. Hence $F(Z)$ is regular in the same region and can be expanded in a series:

$$F(Z)=c_0+\frac{c_1}{Z}+\frac{c_2}{Z^2}+\cdots, \quad \text{for} \quad |Z|\geqq 1,$$

$$c_n=a_n+ib_n \qquad (n=0, 1, 2, \cdots).$$

Comparing this expression with (3.11), we have

$$R(\theta)=a_0+\sum_{n=1}^{\infty}(a_n \cos n\theta+b_n \sin n\theta), \tag{3.13}$$

$$I(\theta)=b_0+\sum_{n=1}^{\infty}(b_n \cos n\theta-a_n \sin n\theta). \tag{3.14}$$

The conjugate Fourier series associated with $R(\theta)$, denoted by $R^*(\theta)$, is defined as

$$R^*(\theta)=\sum_{n=1}^{\infty}(a_n \sin n\theta-b_n \cos n\theta). \tag{3.15}$$

Then it is readily verified that

$$R(\theta)=a_0+I^*(\theta), \quad I(\theta)=b_0-R^*(\theta). \tag{3.16}$$

As $Q(\theta)$ is determined from (3.5) and (3.8), we can get $I(\theta)$ and $R(\theta)$ from (3.12) and (3.16) respectively.

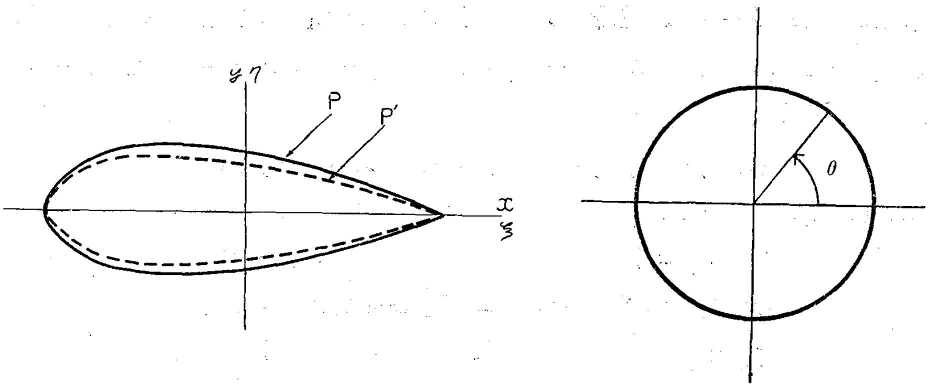
To obtain the conjugate Fourier series associated with a given periodic function $R(\theta)$, we may calculate it from its definition (3.15). It is, however, often more convenient to calculate directly by the following formula:

$$\begin{aligned} R^*(\theta)&=-\frac{1}{2\pi}\int_0^{2\pi} R(\varphi)\cot\frac{\varphi-\theta}{2}d\varphi \\ &=-\frac{1}{2\pi}\int_0^{2\pi}[R(\varphi+\theta)-R(\theta)]\cot\frac{\varphi}{2}d\varphi. \end{aligned} \tag{3.17}$$

When $\zeta_1(Z)$, $\zeta_2(Z)$ and $\zeta_3(Z)$ are given, $f(Z)$, $f_2^0(Z)$ and $f_3^0(Z)$ can be determined by the same procedure.

§ 4. Determination of Mapping Function $\zeta(Z)$

In order to calculate the effect of the compressibility on the velocity distribution over the surface of an aerofoil, we have to determine the mapping functions $z(Z)$ and $\zeta(Z)$ for various Mach numbers. Here $z(Z)$ is the function which maps the region outside the given profile P of the z-plane onto the region outside the unit circle of the Z-plane, while $\zeta(Z)$ is the function corresponding to the profile P' obtained by expanding P, μ times in the direction of the y-axis only (see Fig. 1). The existence of these functions is asserted by the theory of conformal mapping and many practical procedures have been proposed by several



z-and ζ-plane Z-plane

Fig. 1.

authors. Imai has also developed a general method for obtaining $z(Z)$ and $\zeta(Z)$ simultaneously.(4) This method seems to be especially suitable for the present case. In this section we shall extend it to higher order approximations.

As shown in § 3, the mapping function for a thin profile can be expressed in the form:

$$\zeta(Z)=\frac{1}{2}\left(Z+\frac{1}{Z}\right)+\zeta_1(Z)+\zeta_2(Z)+\zeta_3(Z)+\cdots . \tag{4.1}$$

On the unit circle $Z=e^{i\theta}$:

$$\left.\begin{aligned} \xi(\theta)&=\cos\theta+\xi_1(\theta)+\xi_2(\theta)+\xi_3(\theta)+\cdots ,\\ \eta(\theta)&=\qquad\quad\ \eta_1(\theta)+\eta_2(\theta)+\eta_3(\theta)+\cdots . \end{aligned}\right\} \tag{4.2}$$[5]

We shall now seek the relations between $\xi_m(\theta)$ and $\eta_m(\theta)$. Here we confine ourselves to the case in which the chord line of the profile is placed parallel to the undisturbed flow. The mapping function for the profile at some angle of incidence can be obtained from that for zero angle of incidence only by the rotation of the Z-plane.

We have already assumed that the chord length ranges between -1 and $+1$. If the trailing edge and leading edge correspond to $\theta=0$ and θ_l respectively, then

$$\left.\begin{aligned} &\xi_m(0)=0 , \quad \xi_m(\theta_l)=0 ,\\ \text{and}\quad &\xi'_m(0)=0 , \quad \xi'_m(\theta_l)=0 . \end{aligned}\right\} \tag{4.3}$$

The prime denotes differentiation with respect to θ.

The functions $\zeta_m(Z)$ can be expanded as

$$\left.\begin{aligned} &\zeta_m(Z)=c_{-1}Z+c_0+\frac{c_1}{Z}+\frac{c_2}{Z^2}+\cdots ,\\ &c_n=a_n+ib_n \quad (n=-1, 0, 1, \cdots) . \end{aligned}\right\} \tag{4.4}$$

Let us consider the function $\zeta_m(Z)-c_{-1}Z-c_0$. It is regular for $|Z|\geqq 1$ and

5) Since there is no fear of confusion, simple notations $\xi(\theta)$ etc. are used instead of $\xi(e^{i\theta})$ etc.

tends to zero as $Z\to\infty$. Moreover it takes the value on the unit circle :

$$\{\xi_m(\theta)-a_{-1}\cos\theta+b_{-1}\sin\theta-a_0\}+i\{\eta_m(\theta)-b_{-1}\cos\theta-a_{-1}\sin\theta-b_0\}\ .$$

Then, from (3.16),

$$\xi_m(\theta)=\eta^*_m(\theta)+2(a_{-1}\cos\theta-b_{-1}\sin\theta)+a_0\ . \tag{4.5}$$

Owing to (4.3), we have

$$\left.\begin{aligned}
&\eta^*_m(0)+2a_{-1} && +a_0=0\ ,\\
&\eta^*_m(\theta_t)+2a_{-1}\cos\theta_t-2b_{-1}\sin\theta_t+a_0=0\ ,\\
&\eta^{*\prime}_m(0) \qquad -2b_{-1} && =0\ ,\\
&\eta^{*\prime}_m(\theta_t)-2a_{-1}\sin\theta_t-2b_{-1}\cos\theta_t && =0\ .
\end{aligned}\right\} \tag{4.6}$$

From these relations θ_t, a_{-1}, b_{-1}, and a_0 are determined, namely,

$$\left.\begin{aligned}
\tan\frac{\theta_t}{2}&=-\frac{\eta^{*\prime}_m(0)+\eta^{*\prime}_m(\theta_t)}{\eta^*_m(0)-\eta^*_m(\theta_t)}\ ,\\
a_{-1}&=-\frac{1}{2(1-\cos\theta_t)}\{\eta^*_m(0)-\eta^*_m(\theta_t)+\eta^{*\prime}_m(0)\sin\theta_t\}\ ,\\
b_{-1}&=\frac{1}{2}\eta^{*\prime}_m(0)\ ,\\
a_0&=-\frac{1}{1-\cos\theta_t}\{\eta^*_m(0)\cos\theta_t-\eta^*_m(\theta_t)+\eta^{*\prime}_m(0)\sin\theta_t\}\ .
\end{aligned}\right\} \tag{4.7}$$

When $\eta(\theta)$ is known, θ_t is given by the first equation and thereby a_{-1}, b_{-1}, and a_0 are obtained. Then $\xi_m(\theta)$ is determined by substituting these values into (4.5). We shall denote this procedure of deriving $\xi_m(\theta)$ from $\eta_m(\theta)$ by the symbol $\dagger$, that is

$$\xi_m(\theta)=\eta^\dagger_m(\theta)\ . \tag{4.8}$$

When the profile is symmetric about its chord, $\theta_t=\pi$ and

$$\xi_m(\theta)=\eta^\dagger_m(\theta)=\eta^*_m(\theta)-\frac{1}{2}\{\eta^*_m(0)-\eta^*_m(\pi)\}\cos\theta-\eta^{*\prime}_m(0)\sin\theta$$
$$-\frac{1}{2}\{\eta^*_m(0)+\eta^*_m(\pi)\}\ . \tag{4.9}$$

Now we shall determine the mapping function by means of these relations between $\xi_m(\theta)$ and $\eta_m(\theta)$. Let a profile in the physical plane be given by the equation

$$y=y(x) \qquad (-1\leqq x\leqq 1)\ .$$

We introduce a parameter ϑ such that

$$\left.\begin{aligned}
&x=\cos\vartheta\ ,\\
&y=g(\vartheta)=g_1(\vartheta)+g_2(\vartheta)+g_3(\vartheta)+\cdots\ .
\end{aligned}\right\} \tag{4.10}$$

Points at $\vartheta=0$ and $\vartheta=\pi$ correspond to the trailing and leading edge of the profile respectively. Since the thickness is small, $g(\vartheta)$ is a small quantity and $g_1(\vartheta)$, $g_2(\vartheta)$ and $g_3(\vartheta)$, $\cdots$ are of the order of magnitude of ε, ε^2 and ε^3, $\cdots$ respectively.

Remembering $x=\xi$ and $y=\eta/\mu$, and comparing (4.2) with (4.10), we can infer that ϑ is nearly equal to θ, and we may put

$$\vartheta=\theta+\vartheta_1(\theta)+\vartheta_2(\theta)+O(\varepsilon^3)\ . \tag{4.11}$$

Here ϑ_1 and ϑ_2 are functions of θ and of the order of magnitude of ε and ε^2 respectively.

When we substitute (4.11) into (4.10) and compare the first equation with that of (4.2), we get

$$\left.\begin{aligned}\vartheta_1(\theta)&=-\frac{\xi_1(\theta)}{\sin\theta}\ ,\\ \vartheta_2(\theta)&=-\frac{\xi_2(\theta)}{\sin\theta}-\frac{1}{2}\cot\theta\left(\frac{\xi_1(\theta)}{\sin\theta}\right)^2 .\end{aligned}\right\} \tag{4.12}$$

By comparing the second equation of (4.2) with that of (4.10) and making use of (4.12), we can obtain the following expressions of $\eta_m(\theta)$ and $\xi_m(\theta)$ in terms of $g_m(\theta)$:

$$\left.\begin{aligned}
\eta_1(\theta)&=\mu g_1(\theta)\ ,\\
\xi_1(\theta)&=\eta_1^\dagger(\theta)=\mu g_1^\dagger(\theta)\ ,\\
\eta_2(\theta)&=\mu g_2(\theta)-\mu^2 g_1'(\theta)\frac{g_1^\dagger(\theta)}{\sin\theta}\ ,\\
\xi_2(\theta)&=\eta_2^\dagger(\theta)=\mu g_2^\dagger(\theta)-\mu^2\left[g_1'(\theta)\frac{g_1^\dagger(\theta)}{\sin\theta}\right]^\dagger ,\\
\eta_3(\theta)&=\mu g_3(\theta)-\mu^2 g_2'(\theta)\frac{g_1^\dagger(\theta)}{\sin\theta}-\mu^2 g_1'(\theta)\frac{g_2^\dagger(\theta)}{\sin\theta}+\mu^3\frac{g_1'(\theta)}{\sin\theta}\left[g_1'(\theta)\frac{g_1(\theta)}{\sin\theta}\right]^\dagger\\
&\quad-\frac{\mu^3}{2}\cot\theta\cdot g_1'(\theta)\left(\frac{g_1^\dagger(\theta)}{\sin\theta}\right)^2+\frac{\mu^3}{2}g_1''(\theta)\left(\frac{g_1^\dagger(\theta)}{\sin\theta}\right)^2 ,\\
\xi_3(\theta)&=\eta_3^\dagger(\theta)\ .
\end{aligned}\right\} \tag{4.13}$$

The symbol $\dagger$ has the meaning already defined in (4.5) and (4.7) or (4.9). Now the mapping function has been determined completely up to the third order.

If the mapping function $z(Z)$ for the profile in the physical plane is knowm from the beginning, $\zeta(Z)$ is derived from it by means of the above formulae. If we put

$$\left.\begin{aligned}z(e^{i\theta})&=x(\theta)+iy(\theta)\ ,\\ x(\theta)&=\cos\theta+x_1(\theta)+x_2(\theta)+x_3(\theta)+\cdots\ ,\\ y(\theta)&=\qquad\quad y_1(\theta)+y_2(\theta)+y_3(\theta)+\cdots\ ,\end{aligned}\right\} \tag{4.14}$$

we obtain

$$\begin{aligned}\eta_1(\theta)&=\mu y_1(\theta)\ ,\\ \xi_1(\theta)&=\mu x_1(\theta)\ ,\end{aligned}$$

$$\left.\begin{aligned}
&\eta_2(\theta)=\mu y_2(\theta)+\mu(1-\mu)y_1'(\theta)\frac{x_1(\theta)}{\sin\theta}\,,\\
&\xi_2(\theta)=\mu x_2(\theta)+\mu(1-\mu)\left[y_1'(\theta)\frac{x_1(\theta)}{\sin\theta}\right]^{\dagger},\\
&\eta_3(\theta)=\mu y_3(\theta)+\mu(1-\mu)y_2'(\theta)\frac{x_1(\theta)}{\sin\theta}\\
&\quad+\mu(1-\mu)y_1'(\theta)\left\{\frac{x_2(\theta)}{\sin\theta}+\frac{x_1(\theta)}{\sin\theta}\frac{x_1'(\theta)}{\sin\theta}-\frac{1-\mu}{2}\cot\theta\left(\frac{x_1(\theta)}{\sin\theta}\right)^2\right.\\
&\quad\left.-\mu\frac{1}{\sin\theta}\left[y_1'(\theta)\frac{x_1(\theta)}{\sin\theta}\right]^{\dagger}\right\}+\frac{\mu(1-\mu)^2}{2}y_1''(\theta)\left(\frac{x_1(\theta)}{\sin\theta}\right)^2,\\
&\xi_3(\theta)=\eta_3^{\dagger}(\theta)\,.
\end{aligned}\right\}\qquad(4.15)$$

§ 5. Velocity Distribution

Now we are ready to calculate the velocity distribution over the surface of the cylinder. If the shape of the cylinder is given by (4.10), the mapping function is determined from (4.13); thereby G_m is obtained by the formulae (3.3), (3.4), (3.5), (3.7) and (3.8), the sum of their real parts being velocity potentials. It is recognized at once by inspection of (3.5) and (3.8), that the following pairs of g_m^j and f_m^j contribute nothing to the value of Φ on the surface:

$$g_2^1+f_2^1\,;$$
$$g_3^2+f_3^2\,,\quad g_3^5+f_3^5\,,\quad g_3^9+f_3^9\,,\quad g_3^{13}+f_3^{13}\sim g_3^{18}+f_3^{18}\,.$$

Owing to this fact, a great deal of calculation is saved.

When Φ is obtained, the magnitude of the velocity q is determined by the formula:

$$q=\frac{d\Phi}{ds}=\frac{d\Phi}{d\theta}\Big/\frac{ds}{d\theta}\,.\qquad(5.1)$$

Here s is the distance measured along the surface of the profile and

$$\left.\begin{aligned}
&\Phi(\theta)=\xi(\theta)+\phi_1(\theta)+\phi_2(\theta)+\phi_3(\theta)+\cdots\,.\\
&\frac{ds}{d\theta}=\pm\sqrt{\left(\frac{d\xi(\theta)}{d\theta}\right)^2+\left(\frac{1}{\mu}\frac{d\eta(\theta)}{d\theta}\right)^2}\,,\ {}^{6)}\\
&\phi_m(\theta)=\Re G_m(e^{i\theta},e^{-i\theta})\,,\quad(m=1,2.3,\cdots)\,.
\end{aligned}\right\}\qquad(5.2)$$

The velocity q is expressed alternatively:

$$q=q_0(\theta)+q_1(\theta)+q_2(\theta)+q_3(\theta)+\cdots\,,$$

where

$$\left.\begin{aligned}
&q_0(\theta)=\cos\theta\Big/\frac{ds}{d\theta}\,,\\
&q_m(\theta)=(\xi_m'(\theta)+\phi_m'(\theta))\Big/\frac{ds}{d\theta}\,.
\end{aligned}\right\}\qquad(5.3)$$

[6] The sign of the right hand side shall be decided, according as s increases in the same direction as θ or not.

q_0 is nearly equal to the velocity distribution for the flow of incompressible fluid, if the thickness of the cylinder is sufficiently small.

For the range $\sin\theta \gg \varepsilon$, $d\theta/ds$ can be expanded in a series such as

$$\frac{d\theta}{ds} = \pm\frac{1}{\sin\theta}\Big[1+\frac{\xi_1'(\theta)}{\sin\theta}+\Big\{\frac{\xi_2'(\theta)}{\sin\theta}+\Big(\frac{\xi_1'(\theta)}{\sin\theta}\Big)^2-\frac{1}{2}\Big(\frac{1}{\mu}\frac{\eta_1'(\theta)}{\sin\theta}\Big)^2\Big\}$$
$$+\Big\{\frac{\xi_3'(\theta)}{\sin\theta}+2\frac{\xi_2'(\theta)}{\sin\theta}\frac{\xi_1'(\theta)}{\sin\theta}-\frac{\eta_2'(\theta)}{\mu\sin\theta}\frac{\eta_1'(\theta)}{\mu\sin\theta}+\Big(\frac{\xi_1'(\theta)}{\sin\theta}\Big)^3$$
$$-\frac{3}{2}\frac{\xi_1'(\theta)}{\sin\theta}\Big(\frac{1}{\mu}\frac{\eta_1'(\theta)}{\sin\theta}\Big)^2\Big\}\Big]$$
$$+O(\varepsilon^4)\,. \tag{5.4}$$
[7]

By means of this expansion, we get another expression of q:

$$q=1+q_1(\theta)+q_2(\theta)+q_3(\theta)+O(\varepsilon^4)\,,$$

where

$$\left.\begin{aligned}
-q_1(\theta)&=\frac{\phi_1'(\theta)}{\sin\theta}\,,\\
-q_2(\theta)&=\frac{\phi_2'(\theta)}{\sin\theta}+\frac{\phi_1'(\theta)}{\sin\theta}\frac{\xi_1'(\theta)}{\sin\theta}+\frac{1}{2}\Big(\frac{1}{\mu}\frac{\eta_1'(\theta)}{\sin\theta}\Big)^2,\\
-q_3(\theta)&=\frac{\phi_3'(\theta)}{\sin\theta}+\frac{\phi_2'(\theta)}{\sin\theta}\frac{\xi_1'(\theta)}{\sin\theta}+\frac{\phi_1'(\theta)}{\sin\theta}\Big\{\frac{\xi_2'(\theta)}{\sin\theta}+\Big(\frac{\xi_1'(\theta)}{\sin\theta}\Big)^2\\
&\quad-\frac{1}{2}\Big(\frac{\eta_1'(\theta)}{\mu\sin\theta}\Big)^2\Big\}+\Big\{\frac{\xi_1'(\theta)}{\sin\theta}\Big(\frac{\eta_1'(\theta)}{\mu\sin\theta}\Big)^2+\frac{\eta_2'(\theta)}{\mu\sin\theta}\frac{\eta_1'(\theta)}{\mu\sin\theta}\Big\}.
\end{aligned}\right\} \tag{5.5}$$

Here q is reckoned positive when the flow is directed from the nose to the tail on both the upper and lower surfaces. In this formula, $q_0=1$ represents the velocity distribution on a flat plate for an incompressible flow and q_1 is essentially the same as that given by Prandtl and Glauert's linear theory.

Expressions in terms of ϑ. Although we have determined ϑ as the function of θ in § 4, we may treat them conversely; that is

$$\theta=\vartheta+\theta_1(\vartheta)+\theta_2(\vartheta)+O(\varepsilon^3)\,.$$

where

$$\left.\begin{aligned}
\theta_1&=\frac{\xi_1(\vartheta)}{\sin\vartheta}\,,\\
\theta_2&=\frac{\xi_2(\vartheta)}{\sin\vartheta}+\frac{\xi_1'(\vartheta)}{\sin\vartheta}\frac{\xi_1(\vartheta)}{\sin\vartheta}-\frac{1}{2}\cot\vartheta\Big(\frac{\xi_1(\vartheta)}{\sin\vartheta}\Big)^2.
\end{aligned}\right\} \tag{5.6}$$

Making use of these relations, the various formulae for q obtained above are expressed in terms of ϑ. For instance,

$$q=\frac{d\Phi(\vartheta)}{d\vartheta}\Big/\frac{ds}{d\vartheta}\,,$$

[7] See foot note 6).

where

$$\left.\begin{aligned} &\Phi(\vartheta)=\cos\vartheta+\phi_{\rm I}(\vartheta)+\phi_{\rm II}(\vartheta)+\phi_{\rm III}(\vartheta)+O(\varepsilon^4)\ , \\ &\phi_{\rm I}(\vartheta)=\phi_1(\vartheta)\ , \\ &\phi_{\rm II}(\vartheta)=\phi_2(\vartheta)+\phi_1'(\vartheta)\theta_1(\vartheta)\ , \\ &\phi_{\rm III}(\vartheta)=\phi_3(\vartheta)+\phi_2'(\vartheta)\theta_1(\vartheta)+\phi_1'(\vartheta)\theta_2(\vartheta)+\frac{1}{2}\phi_1''(\vartheta)\theta_1^2(\vartheta)\ , \\ &\frac{ds}{d\vartheta}=\pm\sqrt{\sin^2\vartheta+\left(\frac{dg(\vartheta)}{d\vartheta}\right)^2}\ . \end{aligned}\right\} \tag{5.7}$$

The functions ϕ_1, ϕ_2 and ϕ_3 are the same as those given in (5.2), and the prime denotes differentiation with respect to ϑ.

Corresponding to (5.5) we have also,

$$q=1+q_{\rm I}(\vartheta)+q_{\rm II}(\vartheta)+q_{\rm III}(\vartheta)+O(\varepsilon^4)\ ,$$

here

$$\left.\begin{aligned} &q_{\rm I}(\vartheta)=q_1(\vartheta)\ , \\ &q_{\rm II}(\vartheta)=q_2(\vartheta)+q_1'(\vartheta)\theta_1(\vartheta)\ , \\ &q_{\rm III}(\vartheta)=q_3(\vartheta)+q_2'(\vartheta)\theta_1(\vartheta)+q_1'(\vartheta)\theta_2(\vartheta)+\frac{1}{2}q_1''(\vartheta)\theta_1^2(\vartheta)\ , \end{aligned}\right\} \tag{5.8}$$

and q_1, q_2, q_3 are given in (5.5).

When the chord of the cylinder is not placed parallel to the undisturbed flow, but with the angle of incidence α, Z must be multiplied by the factor $e^{-i\alpha}$ and terms due to circulation, if necessary, must be taken into account. Further, since we have determined the velocity distribution, we can obtain the pressure distribution on the surface, whence lift and moment acting on the cylinder are calculated. The procedures for obtaining these quantities up to the order of magnitude of ε^2 have been formulated also by Imai.(4) These methods can be readily extended to obtain terms of the order of ε^3. In this regard his original paper should be referred to.

References

(1) Kaplan, C. N.A.C.A. Tech. Rep. No. 794 (1949).
(2) 〃 ibid. No. 768 (1943).
(3) 〃 read at 8th Int. Cong. Theor. Appl. Mech., Istanbul (1952).
(4) Imai, I. Rep. Aeron. Res. Inst., Tokyo Imp. Univ. No. 294 (1944).
(5) 〃 Kagaku (Science) **14** (1944) 44.
(6) Imai, I. and Ôyama, S. Rep. Inst. Sci. Tech., Univ. Tokyo, **2** (1948) 5.
(7) 〃 ibid **2** (1948) 7.
(8) Hasimoto, H. Journ. Phys. Soc. Japan **7** (1952) 322.
(9) Matunobu, Y. unpublished.
(10) Naruse, H. unpublished.
(11) Tomotika, S. and Tamada, K. Quart. Appl. Math. **9** (1951) 129.

(*Received Nov. 14, 1953*)