# Some Theorems on Abelian Varieties

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In algebraic theories of Riemann matrices and complex toroids, it is a very important problem to find out, how one can construct an Abelian Variety from a Variety having a normal law of composition, and satisfying a certain condition. A. Weil developped elegant theories on Abelian Varieties (cf. [W-3])[1] and there, he constructed them as abstract Varieties: moreover, he applied the transcendental ground field extensions. But can one construct an Abelian Variety as a projective Variety, without applying any ground field extensions? The latter half was pointed out by A. Weil as an open question in [W-3], and both of the above were solved for Jacobian Varieties of non-singular projective Curves by W. L. Chow in [C-1] and for Picard Varieties of non-singular projective Varieties by the writer in [M-3]. In this paper, we prove the following theorem:

*Let $U$ be a Variety having a normal law of composition, and $k$ a common field of definition for $U$ and for that law. When $U$ is birationally equivalent to an Abelian Variety, it is birationally equivalent over $k$ to an Abelian Variety in a projective space.*

This includes the preceeding results of Chow and the writer, but from the "geometric" standpoint of view, the writer believes, however, that this does not diminish the values of those. The problem was discussed between J. Igusa and the writer and this is the report of the writer's. Igusa's general results in the classical case is not published yet.

A *linear equivalence* of divisors and an *algebraic equivalence* of cycles on a Variety are defined in the usual manner, the former being denoted by $\sim$. We shall say that an algebraic family $\{X\}$ of positive divisors on a normal projective Variety as *maximal*, when $\{X\}$ is a Variety as the totality of divisors and when it is not contained in any algebraic family of the same kind. Let $G_a(V)$ be the group formed by all the divisors on a normal projective Variety $V$ which are algebraically equivalent to zero and $G_l(V)$ be the group formed by all the $V$-divisors which are linearly equivalent to zero. There is an algebraic family of positive $V$-divisors $\{X\}$ such that for any $Y$ in $G_a(V)$, we

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1) The letters and numbers in brackets refer the bibliography at the end. We use terminology and conventions in [W-1], [W-2], [W-3], freely.

can find two divisors $X$ and $X'$ in $\{X\}$ such that

$$Y \sim X - X'.$$

We call such a $\{X\}$ as *regular*. The existence of it is proved in [M-2]-th. 1. The *Picard Variety* $P$ of $V$ is an Abelian Variety satisfying the following conditions:

i) $P$ is isomorphic with $G_a(V)/G_l(V)$ as a group,

ii) there is a common field of definition $K$ for $V$ and $P$ such that when $Y \in G_a(V)$ is rational over a field $K'$ containing $K$, its class on $P$ with respect to linear equivalence is rational over $K'$ and that when a Point $y$ on $P$ is rational over a field $K''$ containing $K$, there is a rational $V$-divisor $Y$ in $G_a(V)$ over $K''$ whose class on $P$ is $y$. For these, see [M-2]-th. 3, [M-3]-§ 2, and for the classical case see [I-2]. The *Albanese Variety* $A$ of $V$ is an Abelian Variety satisfying the following condition:

There is a function $\varphi$ defined on $V$ with values in $A$ such that when $f$ is a function defined on $V$ with values in an Abelian Variety $B$, there is a homomorphism $\lambda$ from $A$ to $B$ and that

$$f = \lambda\varphi + c,$$

where $c$ is a constant. (cf. [M-3]-§ 3, [I-2]).

We say that an Abelian Variety is *generated* by a Variety $V$ when there is a function $f$ defined on $V$ with values in $A$ and a finite number of simple Points $x_1, .., x_s$ on $V$ such that $f(x_1) + \dots + f(x_s)$ is a generic Point of $A$ over a common field of definition for $V$, $A$ and $f$.

Let $W$ be a Variety in a projective space, defined over a field $k$, and $C^1$ be a section of $W$ by the linear Variety defined by a generic set of linear equations over $k$. We say that $C$ is a *generic 1-section* of $W$ over $k$. When the particular reference to the field is not made, we mean by it a generic 1-section of $W$ over a suitable field of definition for $W$.

## § 1

1. **Proposition 1.** *Let $\{W\}$ be an algebraic family of positive cycles of dimension $t$ in a projective space and $F$ be its associated-Variety. Assume that it contains a non-singular Variety. Then there is a frontier $\mathfrak{F}$ on $F$ such that when $W'$ is a cycle of $\{W\}$, such that its Chow-Point is not on $\mathfrak{F}$, $W'$ is a non-singular Variety.*

**Proof.** Let $k$ be a field of definition for $F$ and $W$ be a generic cycle of $\{W\}$ corresponding to a generic Point $w$ of $F$ over $k$. Then $W$ is a Variety and by [M-4]-lemma 3 and [M-1]-prop. 1, $W$ is non-singular. Since $W$ is rational over $k(w)$, there is a Subvariety $X$ of $F \times L^n$, where $L^n$ is the ambient space of $W$, such that $(w \times L).X = w \times X(w) = w \times W$ by [W-1]-th. 12, ch. VII. There is a frontier $\mathfrak{F}_1$ on $F$

such that when $w'' \in F - \mathfrak{F}_1$, $X(w'')$ is defined and is a Variety by [M-4]-lemma 5. Passing to representatives, we assume that Points and Varieties under considerations are in affine spaces and we regard representatives of $L^n$ as an affine space $S^n$. Let $Y$ be the projection of $X$ on $S$ and $f_1(X), .., f_m(X)$ be a basis of the prime ideal in $k(w)[X]$ of $W$. We may assume that the coefficients of $f_i(X)$ are in $k[w]$ and hence $f_i(X) = f_i(w, X) \in k[w, X]$. Let $w''$ be a point on $F$ such that $X(w'')$ is defined and is a variety. Then $f_i(w'', X)$ vanishes on $X(w'')$ and is in the prime ideal of $X(w'')$ over $k(w'')$. Let $g_1(w, X), .., g_s(w, X)$ be all the determinants of the matrix $\|\partial f_i(w, X)/\partial X_j\|$ of $n-t$ rows. Since $W$ is non-singular, at least one of them does not vanish at every point of $W$ by [W-1]-§5, ch. IV. The set of equations $g_1(w, X)=0, .., g_s(w, X)=0$ defines on $F \times S$ a bunch $\mathfrak{F}_2'$ and the projection $\mathfrak{F}_2$ of it on $F$ is surely a bunch on $F$ since $w$ is not contained in any component of $\mathfrak{F}_2$. Put $\mathfrak{F} = \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup$(multiple Subvarieties of $F$). When $w'' \in F - \mathfrak{F}_1$, $X(w'')$ is defined, is a variety, and at least one of $g_1(w'', X), .., g_s(w'', X)$ is not zero at every point of it. Therefore it is non-singular. This proves our assertion.

**Proposition 2.** *Let $\{Z\}$ be an algebraic family of cycles on a product of two projective spaces $L_1 \times L_2$ defined over a field $k$ and $Z$ a generic cycle of it over $k$. Assume that $Z$ is a Variety, $U_i$ is the projection of $Z$ on $L_i$ and that the projection of $Z$ on $U_1$ is regular at every Point of $U_1$. Let $W$ be the associated-Variety of $\{Z\}$. There is a bunch $\mathfrak{F}$, normally algebraic over $k$ such that when $z'$ is on $W - \mathfrak{F}$, the corresponding cycle $Z'$ is a Variety and if $U_1'$ is the projection of $Z'$ on $L_1$, the projection of $Z'$ on $U_1'$ is regular at every Point of it.*

**Proof.** Let $K$ be the smallest common field of definition containing $k$ for $Z, U_1, U_2$. Since $Z$ is a Variety, we have $K = k(z)$, where $z$ is the Chow-Point of $Z$. By [W-1]-th. 12, ch. VIII, there is a Variety $X$ on $W \times L_1 \times L_2$ such that

$$(z \times L_1 \times L_2) \cdot X = z \times X(z) = z \times Z.$$

Let $x \times y$ be a generic Point of $X(z)$ over $k(z)$, then $z \times x \times y$ is a generic Point of $X$ over $k$ and $k(z, x) = k(z, x, y)$. Therefore, when $Y$ is the projection of $X$ on $W \times L_1$, the projection of $X$ on it is regular. One can find a bunch $\mathfrak{F}'$ on $Y$ such that when $z' \times x'$ is on $Y - \mathfrak{F}'$, the projection of $X$ on $Y$ is regular at $z' \times x'$ and is not regular otherwise by [W-3]-no. 1, ch. 1. The projection $\mathfrak{F}_1$ of $\mathfrak{F}'$ on $W$ is surely a bunch since $z$ is not contained in any component of $\mathfrak{F}_1$. When we enlarge $\mathfrak{F}_1$ to $\mathfrak{F}$ by adding certain components, if necessary, we may assume that for $z' \in W - \mathfrak{F}$, $X(z')$ is defined and is a Variety.

By the compatibility of specializations with the operation of algebraic projection (cf. [S] or [M-1]), we have

$$pr_{L_1} X(z') = U_1' ,$$

where $U_1'$ is a specialization of $U_1$ over $z \to z'$ with reference to $k$. Now let $x' \times y'$ be a generic Point of $X(z')$ over $k(z')$ and $x'' \times y''$ be a Point on $X(z')$. Since $z' \notin \mathfrak{F}$, $z' \times x'' \notin \mathfrak{F}'$ and hence the projection of $X$ on $Y$ is regular at $z' \times x''$. Therefore every coordinates of a representative of $y$ can be written in the form

$$f(\bar{z}, \bar{x})/g(\bar{z}, \bar{x})$$

with $g(\bar{z}', \bar{x}'') \neq 0$, where we have indicated by $\bar{z}, \bar{x}, \bar{z}', \bar{x}', \bar{x}''$ representatives of $z, x, z', x', x''$ and where $f, g$ are polynomials in $\bar{z}, \bar{x}$. This proves $g(\bar{z}', \bar{x}') \neq 0$ and hence the projection of $X(z')$ on $U_1'$ is regular at $x'$. Our proposition is thereby proved.

The following proposition has been proved in the writer's paper [M-3] on the non-singular projective Variety and one can see immediately that the proof can be extended without essential modifications to a normal projective Variety. Therefore, we write down the proof of it here briefly.

**Proposition 3.** *Let $V$ be a normal projective Variety defined over a field $k$ and $\{X\}$ a regular maximal algebraic family of positive $V$-divisors. When $\{X\}$ contains a rational divisor over $k$, its associated-Variety $U$ is defined over $k$.*

**Proof.** Let $X_0$ be a rational divisor over $k$ in $\{X\}$ and $x_0$ the Chow-Point of it. Let $\sigma$ be an automorphism of $\bar{k}/k$. One can see easily that $U^\sigma = U$, for $U^\sigma$ is the associated-Variety of a regular maximal algebraic family, the divisors of which are algebraically equivalent to a divisor of $\{X\}$, since $x_0$ is also a Point of $U^\sigma$. But this is impossible when $U^\sigma \neq U$. There is a positive integer $h$ such that $k_h = k^{p^{-h}}$ is a field of definition for $U$. Let $x$ be a generic Point of $U$ over $k_h$ corresponding to $X$ and $\tau$ be the isomorphism of $k_h(x)$ and $k(x^{p^h})$ such that for any element $a$ in $k_h(x)$,

$$\tau a = a^{p^h}.$$

$\tau$ maps $k_h$ onto $k$ and hence $k(x^{p^h})$ is a regular extension of $k$. It is clear that $x^{p^h}$ is the Chow-Point of $p^h X$. Let $\{Y\}$ be a maximal algebraic family containing all the specializations of $p^h X$ over $k$. One can see that $\{Y\}$ is a regular maximal algebraic family defined over $\bar{k}$ and $p^h X$ is linearly equivalent to a generic divisor of it over $\bar{k}$ using [M-2]-th. 2, [W-3]-cor. 1, th. 33. Let $U^*$ be the associated-Variety of $\{Y\}$ and $M^*$ the Chow-Point of the associated-Variety $T(p^h X)$ of $|p^h X|$.

Since $p^h X$ is rational over $k(x^{p^h})$ by [C-2], $M^*$ is rational over $k(x^{p^h})$ by [M-2]-lemma 4. Therefore $k(M^*)$ is a regular extension of $k$. Let $y$ be a generic Point of $T(p^h X)$ over $k(M^*)$ corresponding to $Y$. $Y$ is rational over $k(y)$ and hence $M^*$ is rational over it by [M-2]-lemma 4. Therefore, $k(y)=k(y, M^*)$ and $k(y)$ is a regular extension of $k(M^*)$. This proves that $k(y)$ is a regular extension of $k$ and $U^*$ is defined over $k$.

Let $M$ be the Chow-Point of the associated-Variety $T(X)$ of $|X|$. In the same way as above, it is sufficient to prove that $k(M)$ is a regular extension of $k$. $\{Y\}$ contains $p^h X$ and hence it contains also $Y_0=p^h X_0$. Since it is regular, there is a generic divisor $Y$ in it over $\bar{k}$ such that

$$X-X_0\sim Y-Y_0.$$

Let $N$ be the Chow-Point of the associated-Variety $T(Y)$ of $|Y|$ and $y$ the Chow-Point of $Y$. As $Y-Y_0+X_0\sim X$ and as $Y-Y_0+X_0$ is rational over $k(N, y)$, $|X|$ contains a rational divisor over $k(N, y)$ by [W-1]-th. 10, ch. VIII. Hence $M$ is rational over $k(N, y)$ by [M-2]-lemma 4. We may vary freely $y$ on $T(Y)$ and this proves that $k(M)\subset k(N)$. q.e.d.

2. Let $V$ be a normal projective Variety, $A^t$ the Albanese Variety and $P$ the Picard Variety of $V$. Let $k$ be a common field of definition for $V, A, P$ and $C$ a generic 1-section of $V$ over $k$. Since $A$ is generated by $V$ and by a function $f$, $A$ is also generated by $C$ and $f_C$, which is the function induced on $C$ by $f$ (this follows from the writer's remark, "On a generating Curve of an Abelian Variety", Nat. Sci. Rep. Ochanomizu Univ. vol. 3, 1952). Let $Z$ be the graph of $f$.

$$(C\times A)\cdot Z$$

is defined and $pr_A[(C\times A)\cdot Z]=m\Gamma$, where $\Gamma$ is a Curve and $m\neq 0$ since $C$ is non-singular and $A$ is a *minimum model* (cf. [N] and [W-3]-th. 6). We may assume, without loss of generality, that $\Gamma$ contains the Point $o$ on $A$.

Let $z_1, \ldots, z_t$ be $t$ independent generic Points of $\Gamma$ over a common field of definition for $C$ and $f$ containing $k$. $\sum_{i=1}^{t} z_i=x$ is a generic Point of $A$ over $K$. Put $y=\sum_{i=1}^{t-1} z_i$ and let $\Theta$ be the Locus of $y$ over $K$. Let $\{\Theta_a\}$ be the totality of $A$-divisors of the form $\Theta_a$, $a\in A$. Let $b$ be a Point on $A$ and assume that $\Theta_a\sim\Theta_b$. When $c$ is a generic Point of $A$ over $K(a, b)$, we have

$$\Theta'_{c-a}\sim\Theta'_{c-b}$$

since the birational correspondences $a\rightarrow -a$ and $a\rightarrow a+c$ on $A$ are everywhere biregular. Since $c-a$ and $c-b$ are generic Points of $A$ over $K(a, b)$, $\Gamma\cdot\Theta'_{c-a}$ and $\Gamma\cdot\Theta'_{c-a}$ are defined on $A$ by [W-3]-cor., th. 3. Put

$\Gamma\cdot\Theta'_{c-a}=\sum(z_i)$, $\Gamma\cdot\Theta'_{c-b}=\sum(z''_i)$. There is a Point $\bar{z}'_1,\ldots,\bar{z}'_{t-1}$ on $\Gamma$ such that

$$z'_1=(c-a)-(\bar{z}'_1+\ldots+\bar{z}'_{t-1}) \quad \text{i.e.,}\ c-a=\bar{z}'_1+\ldots+\bar{z}'_{t-1}+z'_1.$$

It can be easily seen that $\bar{z}'_1,\ldots,\bar{z}'_{t-1}$, $z'_1$ are independent generic Points of $\Gamma$ over $K$ since $c-a$ is a generic Point of $A$ over $K$. In fact, let $U$ be the graph of the function defined on $\underbrace{\Gamma\times\ldots\times\Gamma}_{t}$ with values on $A$ defined by the relation $\sum_{i=1}^{t}z_i=x$. $U$ is defined over $K$ and *dim* $U=t$.

$$U\cdot(\Gamma\times\ldots\times\Gamma\times(c-a))$$

is defined and is a prime rational 0-cycle over $K(c-a)$ by [W-1]-th. 1, ch. VI. This proves that $\bar{z}'_1,\ldots,\bar{z}'_{t-1}$, $z'_1$ are independent generic Points of $\Gamma$ over $K$ and every $\bar{z}'_1$ is in $\Gamma\frown\Theta'_{c-a}$. Therefore we have

$$\Gamma\cdot\Theta'_{c-a}=\sum_{i=1}^{s}((z'_{i1})+\ldots+(z'_{it}))\ ,$$
$$\Gamma\cdot\Theta'_{c-b}=\sum_{i=1}^{s}((z''_{i1})+\ldots+(z''_{it}))$$

with $st=[U:A]$ and $\sum_{j=1}^{t}z'_{ij}=c-a$, $\sum_{j=1}^{t}z''_{ij}=c-a$. It follows that $\Gamma\cdot\Theta'_{c-a}$ and $\Gamma\cdot\Theta'_{c-a}$ are $\Gamma-$ divisors and are such that

$$\Gamma\cdot\Theta'_{c-a}\sim\Gamma\cdot\Theta'_{c-b}$$

by [W-1]-cor. 1, th. 4, ch. VIII. This shows that $s\cdot(c-a)=s\cdot(c-b)$, i.e., $s\cdot a=s\cdot b$ by [W-3]-th. 10 and hence, when $\Theta_a$ is given, the divisors $\Theta_b$ such that $\Theta_a\sim\Theta_b$ is in finite numbers and it must hold that $s\cdot(a-b)=o$ by [W-3]-cor. 1, th. 33.

There is a $V\times A$-cycle $X$ such that

$$X\cdot(V\times a)=pr_V[Z\cdot(V\times\Theta_a)]\times a\ ,$$

where $X$ is the transform of $W\times\Theta$ by the everywhere biregular birational correspondence $Q\times x\rightleftarrows Q\times[f(Q)-x]$ between $V\times A$ and itself, whenever the second hand side is defined by [W-3]-prop. 2. It is easily seen that $X$ has the projection $A$ on $A$ and $X\frown(V\times a)$ is non-empty for every $a$ on $A$. Assume that $C$ is contained in a component of $X(a)$. Then $(C\times A)\cdot Z$ is contained in $V\times\Theta_a$ and hence $\Gamma<\Theta_a$. Let $a$ and $b$ be two Points on $A$ such that $\Gamma$ is not contained in $\Theta_a$ and $\Theta_b$, and that $X(a)$ and $X(b)$ are both defined. Assume that $X(a)\sim X(b)$. Consider the intersection

$$C\times A\frown Z\frown V\times\Theta_a\,.$$

This intersect properly on $V\times A$ and $(C\times A)\cdot Z$, $Z\cdot(V\times\Theta_a)$ are both defined. Therefore, we have

$$(C\times A)\cdot[Z\cdot(V\times\Theta_a)]=[(C\times A)\cdot Z]\cdot(V\times\Theta_a),$$

by [W-1]-th. 10, ch. VII. In the same way, it holds

$$(C\times A)\cdot[Z\cdot(V\times\Theta_a)]=[(C\times A)\cdot Z]\cdot(V\times\Theta_a).$$

We have $pr_V\{(C\times A)\cdot[Z\cdot(V\times\Theta_a)]\}=C\cdot X(a)\sim C\cdot X(b)=pr_V\{(C\times A)\cdot[Z\cdot(V$

$\times\Theta_b)]\}$ on $C$ by [W-1]-th. 16, ch. VII. $(C\times A)\cdot Z$ is the graph of the function $f_C$ and it is non-singular, since $C$ is non-singular, by [W-1]-th. 15, ch. IV. It follows that

$$\{(C\cdot X(a)\times A)\cdot[(C\times A)\cdot Z]\}_{C\times A}=[(C\times A)\cdot Z]\cdot(V\times\Theta_a)$$
$$\sim\{(C\cdot X(b)\times A)\cdot[(C\times A)\cdot Z]\}_{C\times A}=[(C\times A)\cdot Z]\cdot(V\times\Theta_b)$$

as $(C\times A)\cdot Z$-divisors by [W-1]-th. 18, ch. VII and [W-1]-cor. 1, th. 4, ch. VIII. Let $L$ be the ambient projective space of $V$. It holds

$$[(C\times A)\cdot Z]\cdot(V\times\Theta_a)=\{[(C\times A)\cdot Z]\cdot(L\times\Theta_a)\}_{L\times A}$$
$$[(C\times A)\cdot Z]\cdot(V\times\Theta_b)=\{[(C\times A)\cdot Z]\cdot(L\times\Theta_b)\}_{L\times A}$$

by [W-1]-cor., th. 18, ch. VII. Therefore we have

$$pr_A\{C\cdot X(a)\times A)\cdot[(C\times A)\cdot Z]\}_{C\times A}=m\Gamma\cdot\Theta_a$$
$$pr_A\{C\cdot X(b)\times A)\cdot[(C\times A)\cdot Z]\}_{C\times A}=m\Gamma\cdot\Theta_b$$

(cf. [W-1]-th. 16, ch. VII). On the other hand, we have $S[f(C\cdot X(a))]=S[f_C(C\cdot X(a))]=S[f_C(C\cdot X(b))]$ by [W-3]-th. 10. This proves that $ms\cdot a=ms\cdot b$ by [W-3]-cor. 1, th. 4. Let $\{X(x)\}$ be the totality of specializations of $X(x)$ over $K$. Then the above arguments and [W-3]-cor., th. 33 show that this algebraic family is parametrized by $A$ and a generic divisor of it over $K$ is isolated with respect to linear equivalence.

$\{X(x)\}$ is the abstract analogue of the family discovered by Poincaré (cf. [P]). Moreover, we can deduce easily from the above results that the Albanese Variety and the Picard Variety of a normal Variety are isogeneous. Hence we have obtained the following theorem.

**Theorem 1.** *Let $P$, $A$ be the Picard Variety and the Albanese Variety of a normal Variety $V$. Then $P$ and $A$ are isogeneous and hence have the same dimensions.*

This theorem proves that the dimension of the Picard Variety is an absolute invariant of the class of $V$. We define this to be the *irregularity* of $V$.

3. Now we shall prove the following theorem

**Theorem 2.** *Let $\{W\}$ be an algebraic family of positive cycles on a projective space, defined over a field $k$. Assume that a generic cycle $W$ of it over $k$ is a Variety and let $A$ be an Abelian Variety in a projective space generated by $W$. Let $\{A\}$ be the algebraic family of positive cycles defined as the totality of specializations of $A$ over $\bar{k}$. Then almost all of the cycles in $\{A\}$ are also Abelian Varieties. Let $(W', A')$ be a specialization of $(W, A)$ over $\bar{k}$. Then for almost all of $(W', A')$, $W'$ is a Variety and $A'$ is an Abelian Variety generated by $W'$.*

**Proof.** Since $A$ is generated by $W$, there is a function $f$ defined on $W$ with values on $A$ and a certain numbers of simple Points $x_1,\dots,x_s$ such that

$$f(x_1)+\dots+f(x_s)$$

is a generic Point of $A$ over a common field of definition containing $k$ for $A$, $W$ and $f$. Hence, there is a symmetric function defined on $\underbrace{W\times\dots\times W}_{s}$ with values on $A$ (cf. [W-3]-ch. 1). Let $Z$ be its graph and $K$ the smallest common field of definition containing $\bar{k}$ for $A$, $W$, $Z$ and for the graph of the law of composition $Y$ on $A$. When $U$ is the projective model of $K$ over $\bar{k}$ and $u$ a generic Point of $U$ over $\bar{k}$, such that $K=k(u)$, there is a Variety $X$ on the product of $U$ and a certain numbers of projective space $L_i$ with the following property:

$$(u\times \Pi L_i)\cdot X = u\times X(u) = u\times W\times A\times Z\times Y,$$

by [W-1]-th. 12, ch. VII. There is a bunch $\mathfrak{F}$ on $U$ such that for $u'\in U-\mathfrak{F}$, $(u'\times \Pi L_i)\cdot X$ is defined and is a Variety by [M-4]-lemma 5. In this case, $X(u')$ is a specialization of $X(u)$ over $u\to u'$ with reference to $\bar{k}$ and hence, when $(W', A', Z', Y')$ is a specialization of $(W, A, Z, Y)$ over $u\to u'$ with reference to $\bar{k}$, it holds

$$X(u') = W'\times A'\times Z'\times Y'$$

since specializations are compatible with the operation of product (cf. [S] or [M-1]). Moreover, as $u'\in U-\mathfrak{F}$, $W'$, $A'$, $Z'$, $Y'$ are Varieties. $Y'$ is a Subvariety of $A'\times A'\times A'$ and the projection of $Y'$ on the product $A'\times A'$ of two factors of $A'\times A'\times A'$ is regular : $Z'$ is a Subvariety of $W'\times\dots\times W'\times A'$ and the projection of it on $W'\times\dots\times W'$ is regular, since specializations are compatible with the operation of algebraic projection (cf. [S], or [M-1]). When we enlarge the bunch $\mathfrak{F}$ to $\mathfrak{F}_1$ if necessary, we may assume that the projection of $Y'$ on the product of any two factors of $A'\times A'\times A'$ is everywhere regular by prop. 2. The law defined on $A'$ by $Y'$ is clearly normal since it is so on $A$ and hence $A'$ is an Abelian Variety. Moreover, the projection of $Z'$ on $A'$ is $A'$. Hence $A'$ is generated by $W'$, since we may assume that the function defined by $Z'$ is symmetric. This completes our proof.

**Corollary.** *Let $\{W\}$ be an algebraic family of positive cycles on a projective space defined over a field $k$ and assume that a generic cycle $W$ of $\{W\}$ over $k$ is a Variety. Then almost all of the cycles in $\{W\}$ are Varieties having irregularities $\geqq$ irregularity of $W$.*

**Proof.** This follows from th. 2, when we know that a certain Abelian Variety, isogeneous to the Albanese Variety of $W$ can be immersed into a projective space. But by th. 1, it is isogeneous to the

Picard Variety of $W$ and the latter can be isogeneously imbedded into the Jacobian Variety of a generic 1-section of $W$ (cf. [W-4] and [M-2]-prop. 11). By the Chow's result, Jacobian Varieties of Curves can be imbedded into projective spaces (cf. [C-1]). q.e.d.

One can show, by replacing $W$ by a (non-singular) Curve $C$ in a projective space, and $A$ by the Jacobian Variety $J$ of it in a projective space in th. 2, that for almost all $(C', J')$, $J'$ is the Jacobian Variety of a Curve $C$. This can be done by applying th. 2 and by noticing to the fact that the Jacobian Variety of a Curve is the Albanese Variety of it (cf. [W-3]-th. 21).

## § 2

4. **Proposition** 4. *Let $V$ be a normal projective Variety defined over a field $k$ and $P$ the Picard Variety of $V$. One can find an Abelian Variety defined over $\bar{k}$ immersed into a projective space, which is isogeneous to $P$.*

**Proof.** By [W-4], [M-2]-prop. 11 and [C-1], there is an Abelian Variety $A$ in a projective space isogeneous to $P$. Let $(V', A')$ be a specialization of $(V, A)$ over $\bar{k}$. Then $V=V'$ and by th. 2, for almost all of $A'$, it is an Abelian Variety generated by $V$. Hence $A'$ is isogeneous to $P$ by th. 1 since $dim\, A = dim\, A' = dim\, P$. Moreover, as this holds for almost all of $(V, W')$, we can choose $W'$ such that it is defined over $\bar{k}$.

**Proposition** 5. *Let $W$ be a projective Variety defined over a field $k$ and $U$ an abstract Variety defined over $k$, having the same dimension as $W$. Assume that there is a function $g$ defined on $U$ with values on $W$ defined over $k$. There is, then, a birational transformation defined over $k$, transforming $U$ to a projective Variety $U_1$ and $g$ to a function $g_1$ defined on $U_1$ with values on $W$ defined over $k$ such that, when $Z_1$ is the graph of $g_1$, (i) the projection of $Z_1$ on $U_1$ is regular at every Point of $U_1$, (ii), $(x' \times U_1) \frown Z_1$ has no component of greater dimension than zero for every $x'$ on $W$.*

**Proof.** Let $(x_1, \ldots, x_N)$ be a generic Point of the representative cone of $V$ over $k$, then $(x/x_0) = (1, x_1/x_0, \ldots, x_N/x_0)$ is a generic point of a representative $V$ of $V$ over $k$ and one can find a generic Point $Q$ of $U$ over $k$ such that $k(Q) \supset k(x/x_0)$. Since $k(Q)$ is an algebraic extension of $k(x/x_0)$, one can find a module basis $(y_1, \ldots, y_t)$ of $k(Q)$ over $k(x/x_0)$ consisting of integral elements over $k[x/x_0]$. $y_i$ satisfies then the equation of the form

$$y_i^{r_i}+f_{i1}^*(x/x_0)y_i^{r_i-1}+\dots+f_{ir_i}^*(x/x_0)=0 \quad (1\leq i\leq t)$$

with $f_{ij}^*(x/x_0)$ in $k[x/x_0]$. One can find a positive integer $s_i$ such that

$$x_0^{s_i}y_i^{r_i}+f_{i1}(x)x_0^{\alpha_i}\cdot y_i^{r_i-1}+\dots+f_{ir_i}(x)x_0^{\beta_i}=0$$

where $\alpha_i\geqq 0, \dots, \beta_i\geqq 0$ and $f_{ij}(x)\in k[x]$. Let $m_i$ be a sufficiently large integer and put

$$s_i+\gamma_i=m_i\cdot r_i,$$

then we may assume that $\gamma_i\geqq r_i$, $\gamma_i\geqq m_i(r_i-1)$. Multiplying $x_0^{\gamma_i}$ we get

$$(x_0^{m_i}y_i)^{r_i}+f_{i1}(x)x_1^{\alpha'_i}(x_0^{m_i}y_i)^{r_i-1}+\dots+f_{ir_i}(x)x_0^{\beta'_i}=0$$

This proves that $x_0^{m_i}y_i$ is integral over $k[x]$. Putting $\sum m_i=m$, we conclude that $x_0^m y_i$ is integral over $k[x]$. Let $(\xi)$ be the totality of monomials $x_0^{e_0}x_1^{e_1}\dots\dots x_N^{e_N}$ with $\sum e_i=m$ arranged in a suitable order. Since $k[x]$ is integral over $k[\xi]$, $x_0^m y_i$ is integral over $k[\xi]$. The Variety $\overline{W}$ whose homogeneous coordinate of a generic Point over $k$ is $(\xi)$ is in everywhere biregular birational correspondence with $W$ over $k$. The Variety $U_1$ defined over $k$ whose homogeneous coordinate of a generic Point over $k$ is $((\xi), x_0^m y_1, \dots, x_0^m y_t)$ is in birational correspondence with $U$ over $k$. The Variety $\overline{Z}$ in a doubly projective space defined over $k$ defined by the pair $(\xi\,;\xi, x_0^m y_1, \dots, x_0^m y_t)$ is the transform of the graph of $g$ by the birational transformations defined above. $\overline{Z}$ clearly satisfies all our requirements when we replace $W$ by $\overline{W}$. Now our proposition follows immediately from this.

**Remark.** In the above proposition, we may assume that $U_1$ is normal with reference to $k$.

The following proposition has been used and has played an important rôle in my previous paper [M-3]. But for completeness, we sketch the proof briefly.

**Proposition 6.** *Let $U$ be a Variety in a projective space, having a normal law of composition. Let $Z$ be the graph of that law, $k$ a common field of definition for $U$ and $Z$ and assume that*

$$(x'\times y'\times U)\frown Z,\ (x'\times U\times y')\frown Z,\ (U\times x'\times y')\frown Z$$

*have on component having the greater dimension than zero for every $x', y'$ on $U$. Then $U$ is an Abelian Variety defined over $k$ when $U$ is relatively normal with reference to $k$.*

**Proof.** In the proof, Zariski's "main theorem on birational transformations" (cf. [Z]) plays an essential rôle. First we show that $U\times U$ is relatively normal with reference to $k$. Then it proves that the projection of $\boldsymbol{Z}$ on the product of any two factors of $\boldsymbol{U}\times\boldsymbol{U}\times\boldsymbol{U}$ is regular everywhere by Zariski's theorem mentioned above. Hence the law is defined everywhere and $\boldsymbol{U}$ is an Abelian Variety defined over $k$. q.e.d.

**Proposition 7.** *Let $\mathfrak{a}=\sum a_i\boldsymbol{Q}_i$ be the reduced expression for a cycle in a projective space $\boldsymbol{L}^n$ and assume that $a_i\not\equiv 0$ (mod. $p$), where $p$ is the characteristic of the universal domain of our algebraic geometry. Let $x$ be the Chow-Point of $\mathfrak{a}$ and $k$ a field of definition for $\boldsymbol{L}^n$. Then $k(x)$ is the smallest field containing $k$ over which $\mathfrak{a}$ is rational.*

**Proof.** First we assume that every $\boldsymbol{Q}_i$ has a representative $Q_i$ on one and the same representative $L$ of $\boldsymbol{L}$. Let $Q_i=(x_0^{(i)},\ldots,x_n^{(i)})$ be the coordinate of $Q_i$. We may assume that $x_0^{(i)}=1$, without loss of generality. Put $\mathfrak{b}=\sum_i Q_i$ and $s=deg(\mathfrak{b})$. Let $u_j^{(\alpha)}(0\leq j\leq n,\ 1\leq\alpha\leq n)$ be $(n+1)n$ independent variables over $k(Q_1,\ldots,Q_s)=k(Q)$ and consider the polynomials in $(u_0^{(\alpha)},\ldots,u_n^{(\alpha)})$ defined by

$$f(u^{(\alpha)})=\Pi\,[u_0^{(\alpha)}-(u_1^{(\alpha)}x_1^{(i)}+\ldots+u_n^{(\alpha)}x_n^{(i)})]^{a_i},\quad 1\leq\alpha\leq n.$$

By definition, all the set of coefficients of $f(u^{(\alpha)})$ is a representative of $x$. Put $k(\{u_0^{(1)},\ldots,n_n^{(n)}\}-\{u_0^{(1)},\ldots,u_0^{(n)}\})=k(u)$. $x$ is clearly rational over $k(Q)$ and $k(u)$ is linearly disjoint over $k$ with reference to $k(Q)$. Hence $k(Q)$ and $k(x,u)$ are linearly disjoint over $k(x)$. This shows that the extension $k(Q)$ of $k(x)$ which is clearly algebraic, is separable or inseparable according as $k(Q,u)$ is separable or inseparable over $k(x,u)$. Consider $f(u^{(\alpha)})$ as a polynomial in $u_0^{(\alpha)}$. Since $a_i\not\equiv 0$ (mod. $p$), it is a separable polynomial in $n_0^{(\alpha)}$ and hence the extension

$$k(u)(\textstyle\sum_j u_j^{(\alpha)}x_j^{(i)},\ \forall\alpha,\ i)$$

is separable, and is contained in $k(u,Q)$. Since the determinant $\begin{vmatrix} u_1^{(1)}\ldots u_n^{(1)} \\ \cdots\cdots \\ u_1^{(n)}\ldots u_n^{(n)}\end{vmatrix}\neq 0$, $x_j^{(i)}$ is in that field and hence $k(u,\ \sum u_j^{(\alpha)}x_j^{(i)},\ \forall\alpha,\ i)=k(u,\ Q)$. This proves that $k(u,Q)$ is separable over $k(u,x)$, i.e., $k(Q)$ is separable over $k(x)$. Now $\sum a_iQ_i$ has no conjugate than itself over $k(x)$, and hence it is rational over $k(x)$. General case can be proved easily from this. q.e.d.

**5.** Now we shall prove the main theorem of this paper.

**Theorem 3.** *Let $\boldsymbol{U}$ be an abstract Variety, having the normal law of composition, both defined over a field $k$. When $\boldsymbol{U}$ is birationally equivalent to an Abelian Variety, it is birationally equivalent to an Abelian Variety in a projective space defined over $k$.*

**Proof.** Since $U$ is birationally equivalent to an Abelian Variety $A$, $A$ is the Albanese Variety of $U$. By prop. 4, there is an Abelian Variety $A_1$ defined over $\bar{k}$ in a projective space, isogeneous to $A$. There is a function $g$ defined on $U$ with values on $A_1$ by the definition of isogenity and we may assume that it is also defined over $\bar{k}$. Let $u$ be a generic Point of $U$ over $k$, $k_1$ the smallest common field of definition for $U, A_1, g$ containing $k$ and put

$$g(u)=x.$$

It holds $k_1(u)\supset k_1(x)$ and the former is an algebraic extension of the latter. One can find a projective Variety $U_1$ defined over $k_1$, birationally equivalent to $U$, and a function $g_1$ defined on $U_1$ with values on $A_1$ with a field of definition $k_1$ such that when $Z_1$ is the graph of $g_1$, every component of

$$(x'\times U_1)\frown Z_1$$

is of dimension zero by prop. 5. By the remark of prop. 5, we may assume that $U_1$ is relatively normal with reference to $k_1$. $U_1$ has a normal law of composition. Let $\Gamma_1$ be the graph of that law. Let $u'$, $u''$ be arbitrary two Points on $U_1$ and consider the intersection

$$(u'\times u''\times U_1)\frown\Gamma_1$$

and $u'\times u''\times v'$ be a Point of a component of the above intersection. Then by prop. 5, we must have $g_1(u')+g_1(u'')=g_1(v')$. Hence $g_1(v')\times v'$ is a Point of

$$(g_1(v')\times U_1)\frown Z_1.$$

This proves that every component of $(u'\times u''\times U_1)\frown\Gamma_1$ is of dimension zero. The same holds for $(u'\times U_1\times u'')\frown\Gamma_1$ and for $(U_1\times u'\times u'')\frown\Gamma_1$. Therefore $U_1$ is an Abelian Variety by prop. 6.

Now let $k_2$ be the largest separable extension of $k$ in $k_1$, then $k_1$ is a purely inseparable extension of $k_2$ and there is a positive integer $h$ such that $k_1^{p^h}\subset k_2$. Let $u$ and $u_1$ be the corresponding generic Points of $U$ and $U_1$ over $k_1$ by the birational correspondence between them. If we denote by $u_1^{p^h}$ the transform of $u_1$ by the automorphism of $\overline{k_1(u)}$ defined by $a\to a^{p^h}$, we have

$$k_2(u_1^{p^h})\subset k_2(u).$$

Moreover, $u_1^{p^h}$ has the Locus over $k_1^{p^h}\subset k_2$ which is clearly an Abelian Variety. Since the situation is the same as above, one can prove, in the same way as above, the existence of an Abelian Variety $B$ defined over $k_2$ birationally equivalent to $U$ over $k_2$. Let $T$ be a birational correspondence between $U$ and $B$ defined over $k_2$ and $T_1=T, \ldots, T_m$ be all the distinct conjugates of $T$ with respect to $k$. Let $u\times x_1$ be a generic Point of $T_1$ over $k_2$ and $K$ the smallest normal extension of $k$ containing $k_2$. Put $(u\times L)\cdot(\sum_{i=1}^m T_i)=u\times(\sum_{i=1}^m T_i(u))=u\times\sum x_i$ where $L$ is the ambient space of $\boldsymbol{B}$. We have $K(u)=K(x_1)=\cdots=K(x_m)$. When $\boldsymbol{B}_i$ is the projection of $\boldsymbol{T}_i$ on $\boldsymbol{L}$, this shows that the Locus $\boldsymbol{T}_{ji}$ of $x_i\times x_j$ over $K$ is a birational correspondence between $\boldsymbol{B}_i$ and $\boldsymbol{B}_j$. Since $\boldsymbol{B}_i$'s are Abelian Varieties, $\boldsymbol{T}'_{ji}$ are everywhere biregular by [W-3]-th. 6. Moreover, $x_i\neq x_j$ when $i\neq j$. Let $\xi$ be the Chow-Point of $\sum x_i$. Since $\sum\boldsymbol{T}_i$ is rational over $k$, $\sum x_i$ is rational over $k(u)$ and hence $\xi$ is rational over it. Therefore $\xi$ has the Locus $\boldsymbol{C}$ over $k$.

We shall show that $\boldsymbol{C}$ has a normal law of composition defined over $K$. Let $\xi$, $\eta$ be independent generic Points of $\boldsymbol{C}$ over $K$. $\eta$ is the Chow-Point of the cycle $\sum y_i$ defined by

$$(v\times\boldsymbol{L})\cdot(\sum\boldsymbol{T}_i)=v\times\sum y_i\ ,$$

where $v$ is a generic Point of $\boldsymbol{U}$ over $K(u)$. Let $\Gamma$ be the Locus of $x_1\times\ldots\times x_m$ over $K$, which is an Abelian Variety. Put $z_1\times\ldots\times z_m=x_1\times\ldots\times x_m+y_1\times\ldots\times y_m$ and $\zeta$ the Chow-Point of $\sum z_i$. By [W-3]-th.1, and by our prop.7 we have

$$K(\xi,\eta)=K(\xi,\zeta)=K(\eta,\zeta)\,.$$

Hence $\xi\cdot\eta=\zeta$ defines on $\boldsymbol{C}$ the law of composition. It is associative since the law on $\Gamma$ is associative and the above law is normal. Let $x_1'\times\ldots\times x_m'$ be a Point on $\Gamma$, then to this, there corresponds in the unique way, the Point on $\boldsymbol{C}$ — the Chow-Point of $\sum x_i'$ — and to a Point $\xi'$ on $\boldsymbol{C}$, there corresponds only a finite numbers of Points on $\Gamma$. Let $\boldsymbol{D}$ be a projective Variety defined and normal over $k$, birationally equivalent to $\boldsymbol{U}$ over $k$, having the property enunciated in prop.5, relative to $\boldsymbol{U},\boldsymbol{C}$ and the function $g$ defined by $g(u)=\xi$. Let $u,\bar{u}$ be the corresponding generic Points of $\boldsymbol{U}$ and $\boldsymbol{D}$ over $k$. Then by prop.5, there is a function $f$ defined on $\boldsymbol{D}$ with values on $\boldsymbol{C}$ such that $f(\bar{u})=\xi$ and that when $\boldsymbol{Z}$ is the graph of $f$,

$$(\boldsymbol{D}\times\xi')\frown\boldsymbol{Z}$$

has no component having the greater dimension than zero. Moreover, $f$ is defined at every Point of $\boldsymbol{D}$. Let $w$ be a generic Point of $\boldsymbol{U}$ over $k$ such that $(w\times\boldsymbol{L})\cdot(\sum\boldsymbol{T}_i)=w\times\sum z_i$ and $v,\bar{v}$, and $w,\bar{w}$ be corresponding generic Points of $\boldsymbol{U}$ and $\boldsymbol{D}$ over $k$. $\bar{u}\times\bar{v}\times\bar{w}$ has the Locus $\boldsymbol{X}$ over $k$ since we may assume that $k(u,v)=k(u,w)=k(v,w)$. The law defined on $\boldsymbol{D}$ by $\boldsymbol{X}$ is clearly normal. Let $\bar{u}'\times\bar{v}'$ be arbitrary Points on $\boldsymbol{D}$ and consider the intersection $(\bar{u}'\times\bar{v}'\times\boldsymbol{D})\frown\boldsymbol{X}$. Let $\bar{u}'\times\bar{v}'\times\bar{w}'$ be a Point of that intersection. As $f(\bar{u})=\xi$, $f(\bar{v})=\eta$, $f(\bar{w})=\zeta$ and as $\xi\cdot\eta=\zeta$, it holds $f(\bar{u}')\cdot f(\bar{v})=f(\bar{w}')$. Hence $\bar{w}'\times f(\bar{w}')$ must be contained in $(\boldsymbol{D}\times f(\bar{w}'))\frown\boldsymbol{Z}$ and this intersection has no component of greater dimension than zero. This proves that the intersection $(\bar{u}'\times\bar{v}'\times\boldsymbol{D})\frown\boldsymbol{X}$ has no component of greater dimension than zero. The same holds for $(\bar{u}'\times\boldsymbol{D}\times\bar{v}')\frown\boldsymbol{X}$, $(\boldsymbol{D}\times\bar{u}'\times\bar{v}')\frown\boldsymbol{X}$ and from these we conclude that $\boldsymbol{D}$ is an Abelian Variety by prop. 6. q.e.d.

Let $V$ be a normal Variety defined over a field $k$ in a projective space and $\{X\}$ a regular maximal algebraic family of positive $V$-divisors. We may assume that $\{X\}$ has a rational divisor over $k$. Then by prop. 3, the associated-Variety $W$ of it is defined over $k$. Let $X$ be a generic divisor of $\{X\}$ over $k$ and $M$ the Chow-Point of $T(X)$ of the associated-Variety of $|X|$. Then $M$ has the Locus $U$ over $k$ and moreover, it has a normal law of composition defined over $k$ (cf. [M-2]-lemma 4 and prop. 8). Hence, as $U$ is birationally equivalent to the Picard Variety of $V$ (cf. [M-2]-prop. 10, th. 2 and [M-3]-§ 2), the Picard Variety of $V$ can be constructed in such a way that it is defined over $k$ and can be immersed into a projective space.

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