Markov Processes and Stochastic Equations

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Let $y(t)$, $0 \leq t \leq 1$, be a Markov process whose transition probabilities $F(s, x; t, y) = \Pr.\{y(t) \leq y | y(s) = x\}$ satisfy the following three conditions introduced and studied by Kolmogorov [1] and Feller [2]: When $t \to s$, for an arbitrary $\delta > 0$,

$$(C_1) \qquad \frac{1}{t-s}\int_{|y-x|>\delta} dF(s, x: t, y) \to 0 \ ,$$

$$(C_2) \qquad \frac{1}{t-s}\int_{|y-x|\leq\delta} (y-x)dF(s, x; t, y) \to a(s, x) \ ,$$

$$(C_3) \qquad \frac{1}{t-s}\int_{|y-x|\leq\delta} (y-x)^2 dF(s, x; t, y) \to b^2(s, x) \ .$$

As was shown by Kolmogorov and Feller, $F(s, x; t, y)$ then satisfies a well-known partial differential equation of parabolic type in s, x, and these conditions (C_1)—(C_3) characterise a class of processes of continuous type and distinguish it from others obeying discontinuous changes in time. Under certain regularity conditions imposed on $a(s, x)$, $b(s, x)$ Feller constructed the transition probability $F(s, x; t, y)$ as the fundamental solution of the parabolic equation and proved that it satisfies (C_1)—(C_3) and other naturally postulated conditions. On the basis of the explicit expression of $F(s, x; t, y)$ obtained by Feller continuities of path functions of $y(t)$ were proved by R. Fortet [3] with many results concerning boundary value problems of the parabolic equation. Concerning the same problem S. Bernstein [4] introduced another method making use of approximation by stochastic difference equations. This method has been recently improved by K. Itō [5], making use of his theory of stochastic integrals with respect to the Wiener process $x(t)$, $0 \leq t \leq 1$. He has proved that if $a(t, x)$, $b(t, x)$ are continuous in t, x and satisfy the Lipschitz condition

$$(1) \qquad |a(t, x) - a(t, x')| + |b(t, x) - b(t, x')| < C|x - x'| \ ,$$

then the stochastic integral equation

$$(2) \qquad y(t) = y_0 + \int_0^t a(\tau, y(\tau))\, d\tau + \int_0^t b(\tau, y(\tau)) dx(\tau)$$

has the unique solution $y(t)$, continuous with probability 1, and it is a Markov process satisfying (C_1)—(C_3). By the use of Itō's method we can prove classical results under weaker conditions and clarify intimate connections between these different formulations. As an application we can generalize an "invariance principle" of Erdös and Kac [6] to a form applicable to the heuristic approach by J. L. Doob in the proof of the Kolmogorov-Smirnov limit theorem.

To solve (2) Itō used a method of successive approximation. The following method of difference equations proves to be useful for a later use.

Theorem 1. *Let $a(t, x)$, $b(t, x)$ be two continuous functions satisfying the Lipschitz condition* (1). *Consider a division of* $(0, 1)$, $\Delta=\Delta(t_0, t_1, \cdots, t_n)$, $0=t_0<t_1<t_2<\cdots<t_n=1$, *and define variables* $y_1, y_2, \cdots, y_n$,

$$y_1=y_0+a(t_0, y_0)\Delta t_0+b(t_0, y_0)\Delta x_0 ,$$
$$y_2=y_1+a(t_1, y_1)\Delta t_1+b(t_1, y_1)\Delta x_1 ,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$y_n=y_{n-1}+a(t_{n-1}, y_{n-1})\Delta t_{n-1}+b(t_{n-1}, y_{n-1})\Delta x_{n-1} ,$$

and $y_\Delta(t)$,

$$y_\Delta(t)=y_\mu+a(t_\mu, y_\mu)(t-t_\mu)+b(t_\mu, y_\mu)(x(t)-x(t_\mu)) ,$$

where

$$x_\nu=x(t_\nu),\quad \Delta x_{\nu-1}=x_\nu-x_{\nu-1},\quad \Delta t_{\nu-1}=t_\nu-t_{\nu-1} ,$$
$$t_\mu\le t<t_{\mu+1} .$$

Then for every t, *in the* L^2-*sense, we have*

$$\operatorname*{l.i.m.}_{\rho(\Delta)\to 0} y_\Delta(t)=y(t),\quad \rho(\Delta)=\operatorname*{Max}_{1\le\nu\le n}\Delta t_{\nu-1} ,$$

where $y(t)$ *is the unique solution of* (2).

Theorem 2. *Let* $y^{(n)}(t)$, $0\le t\le 1$, *be a sequence of Markov processes whose transition probabilities satisfy*

$$(C_1')\quad \int_{|y-x|>\sqrt{1+x^2}\delta} d_yF^{(n)}(s, x; s+\Delta s, y)=(\varepsilon_\Delta+\varepsilon_n)\Delta s ,$$

$$(C_2')\quad \int_{|y-x|\le\sqrt{1+x^2}\delta}(y-x)d_yF^{(n)}(s, x; s+\Delta s, y)$$
$$=a(s, x)\Delta s+(\varepsilon_\Delta+\varepsilon_n+\varepsilon_\delta)(1+x^2)^{1/2} ,$$

$$(C_3')\quad \int_{|y-x|\le\overline{1+x^2}\delta}(y-x)^2d_yF^{(n)}(s, x; s+\Delta s, y)$$
$$=b^2(s, x)\Delta s+(\varepsilon_\Delta+\varepsilon_n+\varepsilon_\delta)(1+x^2) ,$$

where $\varepsilon_\Delta\to 0$, $\varepsilon_n\to 0$, $\varepsilon_\delta\to 0$, *respectively when* $\rho(\Delta)\to 0$, $n\to\infty$, $\delta\to 0$, $a(s, x)$, $b(s, x)$ *satisfy* (1), a_{xx}, b_{xx} *exist and are continuous.*

Then $y^{(n)}(t)$ *converges in probability law to the solution* $y(t)$ *of* (2), *i.e., for any* $0\le t_0<t_1<\cdots t_k\le 1$, *and* a_ν, b_ν, $\nu=0, 1, \cdots, k$, *we have*

$$\Pr.\{a_\nu\le y^{(n)}(t_\nu)\le b_\nu,\ \nu=0, 1, \cdots, k\}$$
$$\to\Pr.\{a_\nu\le y(t_\nu)\le b_\nu,\ \nu=0, 1, \cdots, k\},\quad n\to\infty .$$

This proves a result due to Bernstein [4] (c.f. also Khintchine [7], p. 24). To prove Theorem 2 we make use of the following Lemma.

Lemma. *If $a(s,x)$, $b(s,x)$ satisfy the conditions under Theorem 2, and $y(t)$ is the solution of (2), then*

$$\frac{\partial}{\partial y_0}E(\exp[izy(t)])=izE(\exp[\Psi(t)+izy(t)]) ,$$

and

$$\frac{\partial^2}{\partial y_0^2}E(\exp[izy(t)])=E\Big(\exp[izy(t)+\Psi(t)]\Big\{iz\int_0^t\exp[\Psi(\tau)]$$

$$\times((a''-b'b'')d\tau+b''dx(\tau))-z^2\exp[\Psi(t)]\Big\}\Big) ,$$

$$\Psi(t)=\int_0^t\Big(a'-\frac{b'^2}{2}\Big)d\tau+\int_0^t b'dx(\tau),\quad a'=a_y(\tau,y(\tau))\ \textit{etc.},$$

$\partial^2E(\exp[izy(t)])/\partial y_0^2$ is continuous in y_0.

Theorem 3. *Let $y^{(n)}(t)$ be a sequence of Markov processes which satisfy the conditions under Theorem 2 and $y(t)$ the solution of (1), then*

$$\lim_{n\to\infty}\Pr.\{f(t)\le y^{(n)}(t)\le g(t),\ 0\le t\le 1\}$$

$$=\Pr.\{f(t)\le y(t)\le g(t),\quad 0\le t\le 1\} ,$$

where $f(t)$, $g(t)$ are arbitrary continuous functions such that $f(t)\le g(t)$, $0\le t\le 1$, $f(0)<y_0<g(0)$.

The proof follows the same line as in [6]. We can apply this theorem to prove the invariance principle used by J. L. Doob [8] to prove the Kolmogorov-Smirnov limit theorem. We have only to notice that if $F_n(t)$, $0\le t\le 1$, is the empirical distribution function constructed from a sample of size n from a population with uniform distribution over $(0,1)$,

$$y^{(n)}(t)=(F_n(t)-t)\sqrt{n},\quad n=1,2,\cdots,$$

forms a sequence of Markov processes satisfying the conditions of Theorem 2.

Theorem 4. *If the transition probabilities of a Markov process $y(t)$ satisfy*

(C_1) $\quad y(0)=y_0\ ,\quad b(t,y)>0\ ,$

(C_2) $$\int_{|y-x|>\delta\sqrt{1+x^2}}d_yF(s,x;s+\Delta s,y)=\varepsilon_\Delta\Delta s ,$$

(C_3) $$\int_{|y-x|\le\delta\sqrt{1+x^2}}(y-x)d_yF(s,x;s+\Delta s,y)=a(s,x)\Delta s+(\varepsilon_\Delta+\varepsilon_\delta)(1+x^2)^{1/2},$$

(C_4) $$\int_{|y-x|\le\delta\sqrt{1+x^2}}(y-x)^2d_yF(s,x;s+\Delta s,y)$$

$$=b^2(s,x)\Delta s+(\varepsilon_\Delta+\varepsilon_\delta)(1+x^2) ,$$

where $a(s, x)$, $b(s, x)$ satisfy the conditions under Theorem 1, *then there can be constructed from $y(t)$, a Wiener process $x(t)$, with respect to which $y(t)$ is given as the solution of the Itō equation* (2).[1)]

Since the second stochastic integral on the second member of (2) is continuous as shown by Itō, the sample function of $y(t)$ is almost certainly continuous. Moreover, since the first integral is absolutely continuous, continuity of $y(t)$ itself mainly depends on that of the second one, and it will be easy to see that $y(t)$ satisfies the local law of the iterated logarithm

$$\Pr.\left\{\overline{\lim_{h\to 0}}\frac{|y(t+h)-y(t)|}{\sqrt{2h\log\log h^{-1}}}=b(y(t),t)\right\}=1.$$

These results on the continuity of $y(t)$ can be seen as a generalization of corresponding ones due to R. Fortet, who proved them in the case when $F(s, x; t, y)$ is the fundamental solution of a parabolic equation. By the same theorem and Feller's result [2], it can be seen that if $a(s, x)$, $b(s, x)$ are differentiable then $F(s, x; t, y)$ is differentiable with respect to s, x. Details will be published elsewhere.

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1) After I have proved Theorem 4, Itō informed me that according to a letter to Itō, a similar result had been also obtained by J. L. Doob.