お茶の水女子大学 自然科学報告 第4巻 第1号

On the Imbedding Problem of Spaces of Constant Curvature in One Another

Syun-ichi Tachibana (立花俊一)

Department of Mathematics, Faculty of Science,
Ochanomizu University, Tokyo

The problem which imbeds a given n dimensional space S_n of constant curvature K into an another N dimensional space $\tilde{S}_N$ of constant curvature $\tilde{K}$ isometrically was researched by A. Fialkow(1) and A. E. Liber(2). The results obtained by A.E. Liber are as follows:

Theorem. The minimum values of N are given by

(a) $N=n+1,$ *if* $K<0$ *and* $\tilde{K}\geqq 0,$

(b) $N=n+1,$ *if* $K=0$ *and* $\tilde{K}>0,$

(c) $N=2n-1,$ *if* $K>0$ *and* $\tilde{K}\leqq 0,$

(d) $N=2n-1,$ *if* $K=0$ *and* $\tilde{K}<0.$

In this paper we shall prove the following theorem, but I could not deal with (c) of the above one in our method.

Theorem. Let S_n, $\tilde{S}_N$ be spaces of constant curvature with K, $\tilde{K}$ respectively. Then

(A) *if* $\tilde{K}>K$ *and* $N=n+1,$

(B) *if* $\tilde{K}<K=0$ *and* $N=2n-1,$

a given S_n can be imbedded in any $\tilde{S}_N$ isometrically in either case.

A. Fialkow has obtained the results corresponding to (A) in a different manner.

Let S_n be a given n dimensional space of constant curvature, $Pe_1 \dots e_n$ be its frame. Then the infinitesimal displacement of frames is defined with Pfaffian forms ω_i, $\omega_{ij}(i,j=1,\dots,n)$ by the equations

$$dP=\sum_{i=1}^{n}\omega_i e_i,$$

$$De_i=\sum_{j=1}^{n}\omega_{ij}e_j, \quad \omega_{ij}+\omega_{ij}=0,$$

---

(1) A. Fialkow, Einstein spaces in a space of constant curvature. Proc. Nat. Acad. Sci., U.S.A., 24, 30-34.

(2) A.E. Liber, On the immersion of Riemannian spaces of constant curvature in one another. C.R. Doklady Acad. Sci. URSS (N.S) 291-293 (1947), which is not yet accessible to the writer. I knew this result on reading Math. Reviews, 1947, Vol. 8, No. 10.

where D denotes the covariant differentiation. The equations of structure are given by

$$
(1)\qquad
\begin{aligned}
d\omega_i &= \sum_{j=1}^{n} [\omega_j \omega_{ji}], \\
d\omega_{ij} &= \sum_{k=1}^{n} [\omega_{ik}\omega_{kj}] - \frac{1}{2} \sum_{k,h=1}^{n} R_{ijkh}[\omega_k \omega_h], \\
R_{ijkh} &= K(\delta_{ih}\delta_{jk} - \delta_{ik}\delta_{jh}).
\end{aligned}
$$

In the same manner, for a N dimensional space $\tilde{S}_N$ of constant curvature in which S_n is imbedded, the following equations hold good.

$$
(2)\qquad
\begin{aligned}
d\tilde{P} &= \sum_{i=1}^{N} \tilde{\omega}_i \tilde{e}_i \\
D\tilde{e}_i &= \sum_{j=1}^{N} \tilde{\omega}_{ij} \tilde{e}_j, \quad \tilde{\omega}_{ij} + \tilde{\omega}_{ji} = 0, \\
d\tilde{\omega}_i &= \sum_{j=1}^{N} [\tilde{\omega}_j \tilde{\omega}_{ji}], \\
d\tilde{\omega}_{ij} &= \sum_{k=1}^{N} [\tilde{\omega}_{ik}\tilde{\omega}_{kj}] - \frac{1}{2} \sum_{k,h=1}^{N} \tilde{R}_{ijkh}[\tilde{\omega}_k \tilde{\omega}_h], \\
\tilde{R}_{ijkh} &= \tilde{K}(\delta_{ih}\delta_{jk} - \delta_{ik}\delta_{jh}),
\end{aligned}
$$

where i, j, k, h ron over $1, \ldots, N$.

§1. *Proof of* (A). We shall prove that S_n can be imbedded in $\tilde{S}_N$, $(N=n+1)$, when $H \equiv \tilde{K} - K > 0$. Let $\tilde{S}_n$ be the isometric image of S_n in $\tilde{S}_{n+1}$. We attach to each point P of S_n a frame $Pe_1 \ldots e_n$ with an arbitrary analytic law. In $\tilde{S}_{n+1}$ if we take $\tilde{e}_{n+1}$ as the normal to $\tilde{S}_n$,

$$
(3)\qquad \tilde{\omega}_{n+1} = 0
$$

holds good along $\tilde{S}_n$. And as $\tilde{S}_n$ is isometric with S_n, we can take a frame $\tilde{P}\tilde{e}_1 \ldots \tilde{e}_{n+1}$ at each point of $\tilde{S}_n$ such that the equations

$$
(4)\qquad \omega_i = \tilde{\omega}_i, \qquad (i=1, \ldots, n),
$$

hold good. Conversely among the most general frames in $\tilde{S}_{n+1}$, i.e. the frames which is free completely, if we can find a system of frames satisfying (3) and (4), ω_i being given linearly independent Pfaffian forms, our imbedding is possible. Therefore it is sufficient to prove that the differential system (3) and (4) has at least an n dimensional solution. To obtain a closed differential system we form the exterior derivatives of (3) and (4). Forming the exterior derivative of (3), we get, by making use of (4),

$$
(5)\qquad \sum_{i=1}^{n} [\omega_i \tilde{\omega}_{in+1}] = 0.
$$

From (4) by exterior differentiation we obtain

$$
\sum_{i=1}^{n} [\omega_i(\tilde{\omega}_{ij} - \omega_{ij})] = 0, \qquad (j=1, \ldots, n).
$$

Then, by the well known method, from the above equations we get

$$\tilde{\omega}_{ij}=\omega_{ij}, \qquad (i, j=1, \dots, n). \tag{6}$$

Furthermore forming the exterior derivative of (6), the equations

$$[\tilde{\omega}_{in+1}\tilde{\omega}_{jn+1}]=\frac{1}{2}\sum_{k,h=1}^{n} H(\delta_{ik}\delta_{jh}-\delta_{ih}\delta_{jk})[\omega_k\omega_h] \tag{7}$$

are obtained.

Therefore our problem is reduced to whether the closed differential system

$$(\mathrm{I}) \begin{cases} (4) & \tilde{\omega}_i=\omega_i \\ (3) & \tilde{\omega}_{n+1}=0, \qquad\qquad (i, j=1, \dots, n), \\ (6) & \tilde{\omega}_{ij}=\omega_{ij}, \end{cases}$$

$$(\mathrm{II}) \begin{cases} (5) & \sum_{i=1}^{n}[\omega_i\tilde{\omega}_{in+1}]=0, \\ (7) & [\tilde{\omega}_{in+1}\tilde{\omega}_{jn+1}]=\frac{1}{2}\sum_{k,h=1}^{n} H(\delta_{ik}\delta_{jh}-\delta_{ih}\delta_{jk})[\omega_k\omega_h] \end{cases}$$

has at least an n dimensional solution or not.

Let $\tilde{\omega}_{in+1}=\sum_{j=1}^{n}\gamma_{ij}\omega_j$ be the n dimensional integral element[3] of (I), (II). Then from (II), we get

$$\gamma_{ij}=\gamma_{ji} \tag{8}$$

$$\gamma_{ik}\gamma_{jh}-\gamma_{ih}\gamma_{jk}=H(\delta_{ik}\delta_{jh}-\delta_{ih}\delta_{jk}). \tag{9}$$

Now we define γ_{ij} by the equations

$$\gamma_{ij}=\sqrt{H}\,\delta_{ij}, \tag{10}$$

then (10) satisfy (8) and (9), so

$$\tilde{\omega}_{in+1}=\sqrt{H}\,\omega_i, \qquad (i=1, \dots, n), \tag{III}$$

define an n dimensional integral element of the system (I) and (II). As any solution of the linear differential system (I) and (III) is a solution of the system (I) and (II), in place of dealing with the latter, it is sufficint to prove that the former has an n dimensional solution, so it is completely integrable. The equations which are obtained by exterior differentiation of equations of (I) are satisfied by (I) and (II), the latter being satisfied by (III) automatically. So it is sufficient to show that the integrable conditions of (III) hold good. Forming the exterior derivative of the first member of (III) we get, by making use of (2) and (4),

$$d\tilde{\omega}_{in+1}=\sum_{k=1}^{n+1}[\tilde{\omega}_{ik}\tilde{\omega}_{kn+1}]-\frac{1}{2}\sum_{k,h=1}^{n+1}\tilde{R}_{in+1kh}[\tilde{\omega}_k\tilde{\omega}_h] \tag{11}$$

$$=\sum_{k=1}^{n}[\omega_{ik}\tilde{\omega}_{kn+1}]=\sqrt{H}\sum_{k=1}^{n}[\omega_{ik}\omega_k].$$

(3) E. Cartan. Les systemes differentiels exterieurs et leurs applications geometriques.

On the other hand the exterior derivative of the second member of (III) gives us the equations

$$d(\sqrt{H}\,\omega_i)=\sqrt{H}\,d\omega_i=\sqrt{H}\sum_{k=1}^{n}[\omega_k\omega_{ki}]=\sqrt{H}\sum_{k=1}^{n}[\omega_{ik}\omega_k],$$

which equal to (11). Hence the system (I) and (III) is completely integrable, so the system (I) and (II) has an n dimentsional solution, Hence (A) is proved.

§2. Proof of (B). In the next place we shall prove that S_n with $K=0$ can be imbedded in $\tilde{S}_N(N=2n-1)$, when $\tilde{K}<0$. By the analogous reason with that the proof of (A) was done, our imbedding problem is reduced to whether the differential system

$$(12)\qquad \tilde{\omega}_i=\omega_i,\qquad (i=1,\dots,n),$$

$$(13)\qquad \tilde{\omega}_\alpha=0,\qquad (\alpha=n+1,\dots,2n-1),$$

has at least an n dimensional solution or not. As the exterior derivative (12) we get

$$\sum_{j=1}^{n}[\omega_i(\tilde{\omega}_{ij}-\omega_{ij})]=0.$$

Hence we have

$$(14)\qquad \tilde{\omega}_{ij}=\omega_{ij},\qquad (i,j=1,\dots,n).$$

By the same process we get from (13)

$$(15)\qquad \sum_{i=1}^{n}[\omega_i\tilde{\omega}_{i\alpha}]=0.$$

Furthermore to obtain the closed system, forming the exterior derivative of (14),

$$(16)\qquad \sum_{\alpha=n+1}^{2n-1}[\tilde{\omega}_{i\alpha}\tilde{\omega}_{j\alpha}]=\frac{1}{2}\tilde{K}\sum_{k,h=1}^{n}(\delta_{ik}\delta_{jh}-\delta_{ih}\delta_{jk})[\omega_k\omega_h],$$

after some computation by making use of (2), (12) and (14). Hence the equations with which we shall deal are the following closed differential system:

$$(\mathrm{IV})\begin{cases}(12) & \tilde{\omega}_i=\omega_i, & (i=1,\dots,n),\\ (13) & \tilde{\omega}_\alpha=0, & (\alpha=n+1,\dots,2n-1),\\ (14) & \tilde{\omega}_{ij}=\omega_{ij}, & (i,j=1,\dots,n),\end{cases}$$

$$(\mathrm{V})\begin{cases}(15) & \sum_{i=1}^{n}[\omega_i\tilde{\omega}_{i\alpha}]=0,\\ (16) & \sum_{\alpha=n+1}^{2n-1}[\tilde{\omega}_{i\alpha}\tilde{\omega}_{j\alpha}]=\frac{1}{2}\tilde{K}\sum_{k,h=1}^{n}(\delta_{ik}\delta_{jh}-\delta_{ih}\delta_{jk})[\omega_k\omega_h].\end{cases}$$

Let $\tilde{\omega}_{i\alpha}=\sum_{k=1}^{n}\pi_{\alpha ik}\omega_k$ be the n dimensional integral element of (IV) and (V). Then by the fact that they must satisfy (V), the equations,

$$(17)\qquad \pi_{\alpha ik}=\pi_{\alpha ki},$$

$$(18)\qquad \sum_{\alpha=n+1}^{2n-1}(\pi_{\alpha ik}\pi_{\alpha jh}-\pi_{\alpha ih}\pi_{\alpha jk})=\tilde{K}(\delta_{ik}\delta_{jh}-\delta_{ih}\delta_{jk}).$$

are obtained.

Now if we define $\pi_{\alpha ik}$ by the equations

$$(19)\qquad \pi_{\alpha ik}\begin{cases} =0, & \text{if } i\neq k, \\ =\pi_{\alpha i}\neq 0, & \text{if } i=k, \end{cases}$$

then (17) are satisfied automatically and (18) reduce to the equations

$$(20)\qquad \sum_{\alpha=n+1}^{2n-1}\pi_{\alpha i}\pi_{\alpha j}=\tilde{K}, \qquad (i\neq j).$$

The quantities $\pi_{\alpha i}$ satisfying (20) exist actually, for, in (n-1) dimensional space E_{n-1}, (20) show that the inner products by twos of n vectors $\overrightarrow{\pi_i}=\{\pi_{\alpha i}\}$ are $\tilde{K}$. Such vectors, for example, can be obtained by taking the n vectors which go to each vertex from the center of gravity of a regular $(n-1)$ dimensional simplex. Then the equations

$$(\mathrm{VI})\qquad \tilde{\omega}_{i\alpha}=\pi_{\alpha i}\omega_i, \qquad (i=1, \ldots, n\,; \alpha=n+1, \ldots, 2n-1),$$

with (20) define an n dimensional integral element of (IV) and (V). So in place of dealing with (IV), (V), it is sufficient to show that the linear differential system (IV) and (VI) has an n dimensional solution. As the system (IV) and (VI) is not closed, we shall do so. The exterior derivatives of (IV) are contained in (IV) and (V), and the latter is satisfied by (VI) automatically. Forming the exterior derivative of (VI). From the first member of (VI) we have, by making use of (2), (14) and (VI),

$$\begin{aligned} d\tilde{\omega}_{i\alpha}&=\sum_{k=1}^{n}[\tilde{\omega}_{ik}\tilde{\omega}_{k\alpha}]+\sum_{\beta=n+1}^{2n-1}[\tilde{\omega}_{i\beta}\tilde{\omega}_{\beta\alpha}]-\frac{1}{2}\sum_{k,h=1}^{2n-1}\tilde{R}_{i\alpha kh}[\tilde{\omega}_k\tilde{\omega}_h] \\ &=\sum_{k=1}^{n}\pi_{\alpha k}[\omega_{ik}\omega_k]-\sum_{\beta=n+1}^{2n-1}\pi_{\beta i}[\omega_i\tilde{\omega}_{\alpha\beta}]. \end{aligned}$$

From the second member of (VI), the equations

$$\begin{aligned} d(\pi_{\alpha i}\omega_i)&=[d\pi_{\alpha i}\omega_i]+\pi_{\alpha i}\sum_{k=1}^{n}[\omega_k\omega_{ki}] \\ &=-[\omega_i d\pi_{\alpha i}]+\pi_{\alpha i}\sum_{k=1}^{n}[\omega_{ik}\omega_k] \end{aligned}$$

are obtained. Hence the exterior derivatives of (VI) are following equations.

$$(21)\qquad \sum_{k=1}^{n}[(\pi_{\alpha k}-\pi_{\alpha i})\omega_{ik}\omega_k]=[\omega_i, \sum_{\beta=n+1}^{2n-1}\tilde{\omega}_{\alpha\beta}\pi_{\beta i}-d\pi_{\alpha i}].$$

In this place we shall prove a lemma.

Lemma. In S_n with $K\geqq 0$, there exists a system of frames satisfying the equations

$$\omega_{ij}=a(\omega_i-\omega_j), \quad na^2=K, \qquad (i, j=1, \ldots, n).$$

Proof. We shall show by computation that the integrability conditions are satisfied.

$$d\omega_{ij}=\sum_{k=1}^{n}[\omega_{ik}\omega_{kj}]-\frac{1}{2}K\sum_{k,h=1}^{n}(\delta_{ih}\delta_{jk}-\delta_{ik}\delta_{jh})$$
$$=a^2\sum_{k=1}^{n}[(\omega_i-\omega_k)(\omega_k-\omega_j)]+K[\omega_i\omega_j]$$
$$=a^2\sum_{k=1}^{n}([\omega_i\omega_k]+[\omega_k\omega_j])-na^2[\omega_i\omega_j]+K[\omega_i\omega_j]$$
$$=a^2\sum_{k=1}^{n}([\omega_i\omega_k]+[\omega_k\omega_j]),$$
$$d(a(\omega_i-\omega_j))=ad(\omega_i-\omega_j)=a(\sum_{k=1}^{n}[\omega_k\omega_{ki}]-\sum_{k=1}^{n}[\omega_k\omega_{kj}])$$
$$=a^2\sum_{k=1}^{n}([\omega_k(\omega_k-\omega_i)]-[\omega_k(\omega_k-\omega_j)])$$
$$=a^2\sum_{k=1}([\omega_i\omega_k]+[\omega_k\omega_j]). \qquad q.e.d.$$

As the system of the frames which we chose in S_n is completely arbitrary, it is allowed that from the begining we take the one whose existence is proved by the lemma. Then with respect of the system of frames, (21) become the following equations

$$0=a\sum_{k=1}^{n}(\pi_{\alpha k}-\pi_{\alpha i})[\omega_i\omega_k]=[\omega_i(\sum_{\beta=n+1}^{2n-1}\tilde{\omega}_{\alpha\beta}\pi_{\beta i}-d\pi_{\alpha i})]. \tag{22}$$

Apparently the above equations are satisfied by the equations

$$\sum_{\beta=n+1}^{2n-1}\tilde{\omega}_{\alpha\beta}\pi_{\beta i}-d\pi_{\alpha i}=0\ . \tag{VII}$$

Hence to solve our problem, it is sufficient to prove that linear system (IV), (VI) and (VII) is completely integrable. The integrability conditions of (IV) and (VI) are satisfied automatically by (IV), (VI) and (VII). Hence we shall show that the integrability conditions of (VII) are satisfied by (IV), (VI) and (VII). It is performed by computation as follows: Forming the exterior derivative of the first member of (VII), we get

$$d(\sum_{\beta=n+1}^{2n-1}\tilde{\omega}_{\alpha\beta}\pi_{\beta i}-d\pi_{\alpha i})=\sum_{\beta=n+1}^{2n-1}(d\tilde{\omega}_{\alpha\beta}\pi_{\beta i}-[\tilde{\omega}_{\alpha\beta}d\pi_{\beta i}])$$
$$=\sum_{\beta=n+1}^{2n-1}\{(\sum_{k=1}^{n}[\tilde{\omega}_{\alpha k}\tilde{\omega}_{k\beta}]+\sum_{\gamma=n+1}^{2n-1}[\tilde{\omega}_{\alpha\gamma}\tilde{\omega}_{\gamma\beta}]-\frac{1}{2}\sum_{k,h=1}^{2n-1}\tilde{R}_{\alpha\beta kh}[\tilde{\omega}_k\tilde{\omega}_h])\pi_{\beta i}-[\tilde{\omega}_{\alpha\beta}d\pi_{\beta i}]\}$$
$$=\sum_{\beta=n+1}^{2n-1}[\tilde{\omega}_{\alpha\beta}(\sum_{\gamma=n+1}^{2n-1}\tilde{\omega}_{\beta\gamma}\pi_{\gamma i}-d\pi_{\beta i})]=0$$

by making use of (2), (VI) and (VII). It should be noticed that the equations obtained from exterior differentiation of (20) are satisfied by (VII). Because it holds that

$$\sum_{\alpha=n+1}^{2n-1}(\pi_{\alpha i}d\pi_{\alpha j}+\pi_{\alpha j}d\pi_{\alpha i})=0\ . \tag{23}$$

Hence the linear differential system (IV), (VI) and (VII) with (20) is completely integrable, so our imbedding is done. *q.e.d.*

In the case that $K \geqq 0$ and $\tilde{K}-K \equiv H < 0$, it follows from (22)

$$(24) \qquad \sum_{\beta=n+1}^{2n-1} \tilde{\omega}_{\alpha\beta}\pi_{\beta i} - d\pi_{\alpha i} = a \sum_{k=1}^{n} (\pi_{\alpha k} - \pi_{\alpha i})\omega_k + \rho_{\alpha i}\omega_i \,.$$

It seems to be impossible that functions $\rho_{\alpha i}$ are chosen such as (24) satisfy (23).

(*Received Feb. 2, 1953*)