A Note on a Metric Property of Capacity

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1. The object of this short note is to give some complements to the former paper: "On Hausdorff's measure and generalized capacities etc." Jap. Jour. Math. Vol. 19, 1945.

The author was pointed out by Dr. B. Lepson at the Institute of Advanced Study, Princeton, through Prof. K. Kodaira's letter to the author that the proof of Theorem 13 (p. 234) "If E is bounded Borel set of Hausdorff's h-measure positive, then the capacity $C^{\Phi}(E)$ is positive, provided that $-\int_0 h(r)d\Phi(r)<+\infty$" is carried on under the tacit assumption that every Borel set E of Hausdorff's measure m_h infinite contains a closed set F of m_h-measure $\neq 0$. This conclusion, of course, is true if $0<m_h(E)<+\infty$, so that the theorem is true at least for closed sets or F_σ sets. As the author can not yet determine whether the above assumption is true or not, he tried to prove Theorem 13 by another approach, and has obtained a theorem which holds for any set, not necessarily borelian, in terms of outer capacity, but only in case of newtonian or more generally, of potentials of order α. This depends on a theorem concerning modern potential theory by H. Cartan[1] and also a lemma of Ahlfors which is used to prove the well known theorem of Cartan-Ahlfors[2].

2. Let us begin with some notations and definitions. We shall make use of the terminology in Euclidean 3-space, R^3, though the results obtained will easily be generalized to the space of any finite number $\geqq 2$ of dimension.

Let $h(r)$ be an increasing, continuous function defined in the interval $[0, +\infty)$ satisfying $h(0)=0$. The family of all such functions will be denoted by H. For example, $h(0)=\lambda r^{\alpha}\in H$ for $\alpha>0$ and $\lambda>0$.

Given an arbitrary set E in the space and a positive number ε, let us write

$$m_h(E, \varepsilon)=\inf \sum_j h(\operatorname{diam} S_j)$$

where the lower bound is taken over all the sequences of spherical neighbourhoods $\{S_j\}$ such that (i) $\bigcup S_j \supseteqq E$ and (ii) diam S_j, the diameter of S_j, is $<\varepsilon$ for all $j=1, 2, \cdots$.

The limit $\lim\limits_{\varepsilon\to+0} m_h(E, \varepsilon)$ $(=\sup\limits_{\varepsilon} m_h(E, \varepsilon))$ will be denoted by $m_h(E)$. This set function has, as is well known, the property of Carathéodory's outer measure, and is called Hausdorff's measure or h-measure, in the sense of which all the Borel sets are measurable.

Lemma 1. *The necessary and sufficient condition for $m_h(E)>0$ is* $\inf \sum_j h(\operatorname{diam} S_j)>0$, *where the lower bound is taken over all $\{S_j\}$ which cover E*[3].

Let M_E be the family of all the measures μ distributed on a bounded Borel set E, of total measure 1.

Now we shall consider, for each measure $\mu \in M_E$, the potential, $u^\mu(x)$, of order α, defined by the following integral:

$$u^\mu(x)=\int \Phi(\rho(x, y))d\mu(y)$$

where $\rho(x, y)$ denotes the distance between two points x and y and $\Phi(r)$ the 'fundamental function' $r^{-\alpha}(\alpha>0)$. This potential is called newtonian in case $\alpha=1$.

Let us write now for each $\mu \in M_E$

$$V_\mu^\Phi(E)=\sup_{x\in R^3} u^\mu(x), \quad \inf_{\mu\in M_E} V_\mu^\Phi(E)=V^\Phi(E), \quad 1/V^\Phi(E)=C^\Phi(E),$$

where we adopt the usual convention that $C^\Phi(E)=0$, if and only if $V^\Phi(E)=+\infty$.

The set function $C^\Phi(E)$ thus defined is called by O. Frostman Φ-capacity of E[4].

Let E be an arbitrary set. Taking arbitrary compact sets K in E, we shall define the inner capacity, $\underline{C}^\Phi(E)$ of E as follows

$$\sup_{K\subseteq E} C^\Phi(K)=\underline{C}^\Phi(E) .$$

Then, taking arbitrary open sets $G\supseteq E$, we shall define the outer capacity, $\overline{C}^\Phi(E)$, of E as follows

$$\inf_{G\supseteq E} \underline{C}^\Phi(G)=\overline{C}^\Phi(E) .$$

In case that the outer and the inner capacity of E are equal with each other, the common value will be called simply the capacity of E which will be denoted also by $C^\Phi(E)$.

According to this, Frostman's definition of capacity is nothing but the inner capacity, since, to each positive $\varepsilon>0$ and each $\mu \in M_E$, there exists a compact set $K\subseteq E$ such that $\mu(E-K)<\varepsilon$. Using these notations of capacities, Prof. H. Cartan has, in his recent researches on the theory of potentials, extremely generalized Evans' theorem[5] as follows:

Theorem of H. Cartan[6]. *If $\overline{C}^\Phi(E)=0$ where $\Phi(r)=r^{-\alpha}(0<\alpha<3)$, there exists a potential u^μ, of order α, which takes the value $+\infty$ at every point of E.*

For a fixed $\mu \in M_E$, let us write

$$\mu(a, r)=\mu(E\frown U(a, r))$$

where $U(a, r)$ denotes as usual the spherical neighbourhood of a, with radius r.

Then, $\mu(a, r)$ is, considered as a function of r, monotone increasing and satisfies $\mu(a, r)=1$ for sufficiently large r.

According to the lemma found by L. Ahlfors cited in (2), to each $h \in H$ such that $h(+\infty)>1$, the complementary set of points x satisfying

(1) $$\mu(x, r) \leqq h(r) \text{ for all } r \in [0, +\infty)$$

can be covered by a sequence of compact spheres $\{S_j\}$ with 'radii' $\{r_j\}$ such that

(2) $$\sum_j h(r_j) \leqq k$$

where the number k is independent of $\mu \in M_E$, $h \in H$ and E.

Since we can cover each S_j by a fixed finite number l, independent of each S_j, of congruent spherical neighbourhoods with diameters r_j, we may replace, in the above statement, 'radii' and k by 'diameters' and $kl=\sigma$ respectively. Thus:

Lemma 2. *To each $h \in H$ such that $h(+\infty)>1$, the set of points x which do not satisfy* (1) *can be covered by a sequence of spherical neighbourhoods $\{U_j\}$ with diameters $d_j = diam\ U_j$ such that*

$$\sum_j h(d_j) \leqq \sigma$$

where the number σ is independent of $\mu \in M_E$, $h \in H$, and the set E.

3. We are now in the position to state and prove our theorems.

Theorem 1. *To each $\mu \in M_E$, $h \in H$ and positive integer n, let us write*

$$A_n = \{x \mid \mu(x, r) > n \cdot h(r) \text{ for some } r \geqq 0\} .$$

Then, the set

$$\bigcap_{n \geqq 1} A_n = A$$

is of m_h-measure 0.

Proof. Since m_h has connection with the behaviour of h only for small r, we may suppose without loss of generality $h(+\infty)>1$.

Let $$n \cdot h(r) = h_n(r) \quad \text{for } n=1, 2, \cdots.$$

From $h_n \in H$ and $h_n(+\infty)>1$, we can find, by Lemma 2, a sequence of spherical neighbourhoods $\{U_j\}$ such that $\sum_j h^n(\text{diam}\, U_j) \leqq \sigma$ and that the set of points x at which

(3) $$\mu(x, r) > h_n(r) \quad \text{for some } r \geqq 0$$

is coverd by $\{U_j\}$. This shows $\sigma \geqq \sum_j h_n(\text{diam}\, U_j) = n \cdot \sum h(\text{diam}\, U_j)$, whence we have

(4) $$\sum_j h(\text{diam}\, U_j) \leqq \sigma / n .$$

From the fact $A_n \supseteq A = \bigcap A_n$, we see that $\{U_j\}$ covers also A, whence we have $m_h(A)=0$ by Lemma 1 and (4) of which the right hand side may be made smaller than any assigned positive number by taking n large.

Theorem 2. *Let E be an arbitrary set of outer capacity* 0 *of order α. Then, E is of m_h-measure* 0 *for any $h \in H$ such that* $\int_0 h(r) r^{-1-\alpha} dr < +\infty$.

Proof. We may suppose without loss of generality that E is bounded, since every set may be expressed as a sum of at most countable bounded sets.

By H. Cartan's theorem already mentioned, there exists a potential, of order α, $u^\mu(x) = \int^r \Phi(\rho(x, y)) d\mu(y)$ $(\Phi(r) = r^{-\alpha})$ such that

$$u^\mu(x) = +\infty \text{ at every } x \in E. \tag{5}$$

Then, for a sufficiently large r, we have

$$\begin{aligned} u^\mu(x) &= \int_0^{r_0} \Phi(r) d\mu(x, r) \\ &= \Phi(r_0)\mu(x, r_0) - \lim \Phi(\varepsilon)\mu(x, \varepsilon) - \int_0^{r_0} \mu(x, r) d\Phi(r) \\ &\leqq \Phi(r_0)\mu(x, r_0) - \int_0^{r_0} \mu(x, r) d\Phi(r) \end{aligned}$$

from which we find that if $u^\mu(x) = +\infty$, then $-\int_0^{r_0} \mu(x, r) d\Phi(r) = +\infty$.

But if

$$+\infty = u^\mu(x) = -\int_0^{r_0} \mu(x, r) d\Phi(r) = \alpha \int_0^{r_0} \mu(x, r) r^{-1-\alpha} dr, \tag{6}$$

then x must belong to the set A stated in Theorem 1.

For, if $x \notin A$, then x must belong to the complementary set A^c, of A, whence $x \in \bigcup_n A_n^c$ and we would find a number n such that $x \in A_n^c$, namely

$$\mu(x, r) \leqq n \cdot h(r) \quad \text{for all } r \geqq 0. \tag{7}$$

Combining (7) with the right hand-side of (6), we would have

$$\alpha \int_0^{r_0} \mu(x, r) r^{-1-\alpha} dr \leqq n \int_0^{r_0} h(r) r^{-1-\alpha} dr < +\infty$$

which would contradict (6).

By this consideration, we can conclude that

$$A \supseteq E$$

which shows, by Theorem 1, $m_h(E) = 0$.

Notes

1) H. Cartan: Théorie du potentiel newtonian. Bull. Soc. Math. France, T. 73 (1945).
2) L. Ahlfors: Ein Satz von H. Cartan and seine Anwendung auf die Theorie der meromorphen Funktionen, Sec. sci. fenn. Comment. Phy.—Math. 5 (1931).
3) S. Kametani: Lemma 2 in the paper quoted in 1.
4) O. Frostman: Potentiel d'Equilibre et capacité des Ensembles. (1935) Lund.
5) G. C. Evans: Potentials and positively infinite singularlity of harmonic functions. Monatshefte 46 (1936).
6) Ibid. Théorème 3 bis with additional remarks.

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