On a Generating Curve of an Abelian Variety[1)]

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In his book "Variétés Abéliennes et Courbes Algébriques", A. Weil has proved that every Abelian Variety is generated by a finite number of Curves (cf. prop. 30, § IX of the book). We shall show in this paper that every Abelian Variety is generated by one Curve. This will be one of basic tools for the investigation of Abelian Varieties. Using this, and applying Chow's result on Jacobian Varieties, we shall generalize his celebrated theorem[2)] to arbitrary Abelian Varieties in the forth-coming paper.

§ 1

Let $\boldsymbol{V}^n$ be a complete Variety and $\boldsymbol{X}^{r+1}$ a Subvariety of the product $\boldsymbol{U}^{r\times n}$ of $\boldsymbol{V}$ and another Variety $\boldsymbol{U}^r$. Then $\boldsymbol{X}^{r+1}$ defines on $\boldsymbol{V}$ an algebraic family of Curves $\{\boldsymbol{X}(\boldsymbol{M})\}$ parametrized by $\boldsymbol{U}$. We assume that the following conditions are satisfied by $\{\boldsymbol{X}(\boldsymbol{M})\}$.

(i) $\{\boldsymbol{X}(\boldsymbol{M})\}$ covers $\boldsymbol{V}$, i.e., projection of $\boldsymbol{X}$ on $\boldsymbol{V}$ is $\boldsymbol{V}$,

(ii) there is a bunch $\mathfrak{B}$ on $\boldsymbol{U}$ such that when $\boldsymbol{M}\in\boldsymbol{U}-\mathfrak{B}$, $\boldsymbol{M}$ is simple on $\boldsymbol{U}$, $\boldsymbol{X}(\boldsymbol{M})$ is defined and is a non-singular Curve, and contains no multiple Point of $\boldsymbol{V}$,

(iii) $\{\boldsymbol{X}(\boldsymbol{M})\}$ has a base Point $\boldsymbol{N}$ which is simple on $\boldsymbol{V}$.

Now we shall prove the

Proposition 1. *Let f be a function on $\boldsymbol{V}$ with values in an Abelian Variety $\boldsymbol{A}$. Assume that f induces on $\boldsymbol{X}(\boldsymbol{M}_0)$ $(\boldsymbol{M}_0\in\boldsymbol{U}-\mathfrak{B})$ a constant. Then f is a constant.* (For the definition of the function, see (A)–§ I, n°.1).

Proof. Let $\boldsymbol{Z}$ be the graph of f and K a common field of definition for $\boldsymbol{V}, \boldsymbol{U}, \boldsymbol{X}, \boldsymbol{A}, \boldsymbol{Z}$ and $\boldsymbol{M}_0$ over which $\mathfrak{B}$ is normally algebraic. Let $\boldsymbol{M}\times\boldsymbol{P}$ be a generic Point of $\boldsymbol{X}$ over K and $\boldsymbol{Y}^{r+1}$ the Locus of $\boldsymbol{M}\times\boldsymbol{P}\times f(\boldsymbol{P})$ over K. We have $pr_{U\times V}\boldsymbol{Y}=\boldsymbol{X}$ and the projection of $\boldsymbol{Y}$ on $\boldsymbol{V}\times\boldsymbol{A}$ is $\boldsymbol{Z}$. Consider the intersection

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1) We shall use the the same terminologies and conventions as in Weil's books, "Foundations of Algebraic Geometry" and "Variétés Abéliennes et Courbes Algébriques". We shall denote the former by (F) and the latter by (A).

2) This theorem says that when Γ is a non-singular projective Curve defined over a field k, the Jacobian Variety itself is defined over k, and can be immersed in a projective space. For this, see W. L. Chow: "Algebriac system of positive cycles in an algebraic variety", Amer. Journ. vol. LXXII.

$$(\boldsymbol{M}_0 \times \boldsymbol{V} \times \boldsymbol{A}) \frown \boldsymbol{Y}.$$

It is easy to see that every Point in this intersection is of the form $\boldsymbol{M}_0 \times \boldsymbol{P}' \times \xi'$ where $\boldsymbol{P}' \in \boldsymbol{X}(\boldsymbol{M}_0)$ and $\xi' = f(\boldsymbol{P}')$. Since f induces on $\boldsymbol{X}(\boldsymbol{M}_0)$ a constant, it follows that the only component in the above intersection is $\boldsymbol{M}_0 \times \boldsymbol{X}(\boldsymbol{M}_0) \times \xi'$. Hence

$$(\boldsymbol{M}_0 \times \boldsymbol{V} \times \boldsymbol{A}) \cdot \boldsymbol{Y} = \boldsymbol{M}_0 \times \boldsymbol{X}(\boldsymbol{M}_0) \times \xi'.$$

Let $\boldsymbol{M}$ be a generic Point of $\boldsymbol{U}$ over K and put

$$(\boldsymbol{M} \times \boldsymbol{V} \times \boldsymbol{A}) \cdot \boldsymbol{Y} = \boldsymbol{M} \times \boldsymbol{Y}(\boldsymbol{M}).$$

Let $U \times V \times A$ be a representative of $\boldsymbol{U} \times \boldsymbol{V} \times \boldsymbol{A}$ on which $\boldsymbol{M}_0$, $\boldsymbol{Y}$, $\boldsymbol{X}(\boldsymbol{M}_0)$, have representatives M_0, Y, $X(M_0)$, $\bar{\xi}'$. Then $\boldsymbol{Y}(\boldsymbol{M})$ has a representative $Y(M)$ on $V \times A$. Let $M \times P \times \bar{\xi}$ be a generic Point of $M \times Y(M)$ over $K(\boldsymbol{M})$. Then, it is a generic Point of Y over K by (F)–VI, th. 11 and so $(M_0, \bar{\xi}')$ is a specialization of $(M, \bar{\xi})$ over K. Let C be the projection of $Y(M)$ on A. Denote by $\boldsymbol{C}^*$ the uniquely determined projective cycle determined by C. Then the above argument shows that there is a specialization $\boldsymbol{C}^{*\prime}$ over $(M, \bar{\xi}) \to (M_0, \bar{\xi}')$ with reference to K such that a certain component $\boldsymbol{C}_1^{*\prime}$ of $\boldsymbol{C}^{*\prime}$ has a representative C_1' on the same affine space as C and that C_1' constains $\bar{\xi}'$.[3]

Let ξ'' be a Point of C_1'. Then ξ'' is a specialization of ξ (the Point whose representative is $\bar{\xi}$) over $\boldsymbol{M} \to \boldsymbol{M}_0$ with reference to K. Extend this specialization to a specialization $\boldsymbol{P} \to \boldsymbol{P}''$ over K. Then $\boldsymbol{M}_0 \times \boldsymbol{P}''$ is on $\boldsymbol{X}(\boldsymbol{M}_0)$ and so $f(\boldsymbol{P}'') = \xi''$ since $\boldsymbol{P}''$ is simple on $\boldsymbol{V}$ by our assumption (cf. (A)–§ II, th. 6). But as f is constant along $\boldsymbol{X}(\boldsymbol{M}_0)$ and as $\boldsymbol{P}''$ is simple on $\boldsymbol{X}(\boldsymbol{M}_0)$, we must have

$$f(\boldsymbol{P}'') = f(\boldsymbol{P}') = \xi'.$$

This proves that $\boldsymbol{C}_1' = \xi'$ and so dim $C = 0$. Therefore it holds

$$(\boldsymbol{M} \times \boldsymbol{V} \times \boldsymbol{A}) \cdot \boldsymbol{Y} = \boldsymbol{M} \times \boldsymbol{X}(\boldsymbol{M}) \times \xi.$$

Thus f must be constant along $\boldsymbol{X}(\boldsymbol{M})$.

Now, let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be two independent generic Points of $\boldsymbol{V}$ over K, then we can find two generic Points $\boldsymbol{M}$ and $\boldsymbol{R}$ of $\boldsymbol{U}$ over K such that $X(\boldsymbol{M})$ and $X(\boldsymbol{R})$ contains $\boldsymbol{P}$ and $\boldsymbol{Q}$ respectively. Since $\boldsymbol{N}$ is a base Point of $\{\boldsymbol{X}(\boldsymbol{M})\}$, simple on $\boldsymbol{V}$, on $\boldsymbol{X}(\boldsymbol{M})$ and on $\boldsymbol{X}(\boldsymbol{R})$, it follows that

$$f(\boldsymbol{P}) = f(\boldsymbol{N}) = f(\boldsymbol{R}),$$

by (A)–§ II, th. 6 by what we have proved above. This completes our proof.

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3) For the definition of the specializations, see P. Samuel's "Thésè", Paris, 1951, or T. Matsusaka's "Specialization of cycles on a projective Model", Mem. Col. Sci., Kyoto Univ., Ser. A, 1951. The dimension of the cycle is not altered by specializations.

Proposition 2. *There is always an algebraic family of Curves on V satisfying conditions (i), (ii), (iii) when V is a normal projective Variety. Moreover, when V is defined over k, we can find such a family that U and every component of $\mathfrak{B}$ are defined over $\bar{k}$.*

Proof. Let k be a field of definition for V and L^N the ambient projective space of V. Let L^{N-n-1} be a generic linear Variety over K and put

$$C = V \cdot L^{N-n-1}$$

Then C is a non-singular Curve, and every Point of it is simple on V.[4] Let $\sum u_{ij}X_j - v_i X_0 = 0$ $(1 \leqq i \leqq n-1)$ be the set of defining equations for $L^{N-n-1} = L_{(u,v)}$. Then for non-special values (c) of (u), $L_{(c,v)} \cdot V$ is defined, is non-singular and every Point of it is simple on V. As $\bar{k}$ contains infinitely many elements, we can take (c) from $\bar{k}$. After applying a projective transformation, if necessary, we may assume that

$$L_{(c,o)} \cdot V$$

is defined, is non-singular and that every Point of it is simple on V. Further, if H is the hyperplane defined $X_o = 0$, we may assume that $V \cdot H$ is defined. Let L^{n-1} be the $(n-1)$-dimensional projective space. Considering $(1, v_1, \cdots, v_{n-1})$ as a representative of a generic Point M of L^{n-1} over $\bar{k}$, we can find a Subvariety X^n of $L^{n-1} \times V$ such that

$$pr_V[X \cdot (M \times V)] = X(M) = V \cdot L_{(c,v)} \quad \text{(cf. (F)–VIII, th. 12).}$$

Then the conditions (i) and (ii) are satisfied by the algebraic family $\{X(M)\}$ defined by X^n. Let N be a Point of $V \cdot L_{(c,o)} \frown H$. Then N is contained in every member of $\{X(M)\}$. Moreover, by our assumption on $L_{(c,o)}$, N is a simple Point of V. Thus, (iii) is also satisfied by $\{X(M)\}$. The rest of our assertion follows immediately from our observation made above.

§2

We shall say that an Abelian Variety A is generated by a Variety V when there is a function f defined on V with values in A and a set of a finite number of simple Points $(P_1, \ldots, P_s)$ such that

$$f(P_1) + \cdots + f(P_s) = \xi$$

is a generic Point of A over a common field of definition K for A, V, and f. If that is so, we shall also say that f generates A.

Theorem. *Let A^m be an Abelian Variety generated by a Variety V, and k be a field of definition for V. Then A is generated by a*

[4] Cf. Y. Nakai: "On the section of an algebraic variety by the generic hyperplane". Mem. Col. Sci., Kyoto Univ., Ser. A, 1951.

Curve $\boldsymbol{Y}$ on $\boldsymbol{V}$ and we can choose $\boldsymbol{Y}$ so that it is defined over the algebraic closure of k.

Proof. After applying a normalization, if necessary, we may assume the existence of the algebraic family of Curves $\{\boldsymbol{X}(\boldsymbol{M})\}$ on $\boldsymbol{V}$ satisfying the conditions (i), (ii) and (iii) by prop. 2. Let f be a function on $\boldsymbol{V}$ with values in $\boldsymbol{A}$ generating $\boldsymbol{A}$. Let $\boldsymbol{M}'$ be a Point of $\boldsymbol{U}-\mathfrak{B}$ such that f induces a function $f_{X(M')}$. We may assume that there is a Point $\boldsymbol{Q}$ on $\boldsymbol{X}(\boldsymbol{M}')$ with $f_{X(M')}(\boldsymbol{Q})=f(\boldsymbol{Q})=0$, where 0 is the unit element of $\boldsymbol{A}$.

Let $\boldsymbol{B}^r$ be the Abelian Subvariety of $\boldsymbol{A}^m$ generated by $\boldsymbol{X}(\boldsymbol{M}')$ and f, that is, generated by $\boldsymbol{X}(\boldsymbol{M}')$ and $f_{X(M')}$. There is the Abelian Variety $\boldsymbol{C}^{m-r}$ on $\boldsymbol{A}$ such that $\boldsymbol{B}\frown\boldsymbol{C}$ is a finite subgroup of $\boldsymbol{A}$ by (A)-§ VII, th. 26. By (A)-§ VII, prop. 25, there is a homomorphism γ of $\boldsymbol{A}$ on $\boldsymbol{C}$ and a homomorphism β of $\boldsymbol{A}$ on $\boldsymbol{B}$ such that $\beta\cdot\gamma=\gamma\cdot\beta=0$. It is clear that $\boldsymbol{C}$ is generated by $\boldsymbol{V}$ and $\gamma\cdot f=g$, which is a function on $\boldsymbol{V}$ with values in $\boldsymbol{C}$. Let $\boldsymbol{P}$ be a Point of $\boldsymbol{X}(\boldsymbol{M}')$. Then $f(\boldsymbol{P})$ is a Point of $\boldsymbol{B}$ and hence can be written in the form $f(\boldsymbol{P})=\beta\cdot\xi$, where ξ is a Point of $\boldsymbol{A}$. It follows that

$$g(\boldsymbol{P})=\gamma\cdot f(\boldsymbol{P})=\gamma\cdot(\beta\cdot\xi)=(\gamma\cdot\beta)\cdot\xi=0.$$

Hence g induces on $\boldsymbol{X}(\boldsymbol{M}')$ the constant O. By prop. 1, g is therefore a constant function on $\boldsymbol{V}$ and so $\boldsymbol{B}$ must be $\boldsymbol{A}$. The rest of the proof follows immediately from prop. 2.