On the Transformation and Mutual Relations of Adiabatic Charts Convenient to the Height Calculation [1)]

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Introduction and Summary

Though the method of adiabatic charts has been applied to thermodynamical problems in meteorology for a long time, especially in aerology their use has been flourished remarkably during last twenty years, and in this period quick treatments of observational data of aerology and quick calculations of their heights were required, because many aerological observatories were established and at each place aerological observations were carried out several times a day. So far in meteorological circles a well known method by barometric formula has been used as procedures of height calculations for these data. (cf. **1** (i)). About twenty years ago, however, in order to reduce the complication inherent in these procedures, Stüve[(1)] proposed a method to utilize adiabatic charts for this purpose, whose adoption has since prevailed quickly in the aerological circles. Moreover a few special adiabatic charts have appeared, suitable to the above requirement about the quick height calculation. At the same time some theories about height calculations on adiabatic charts and the mutual relations among these charts appeared also as below.

We shall classify the recent development of these theories into the following four classes.

(i) Stüve's[(1)] theory to reduce the height problem of actual atmosphere on Emagram ($T-\log p$) to that of isothermal atmosphere or dry adiabatic one.

(ii) Theories of height calculation on Tephigram by Shaw,[(2)] Brunt[(3)] and Arakawa.[(5)]

(iii) Refsdal's[(4)] theory of Aerogram.

(iv) Yamaoka's theory[(6)] for the transformation of adiabatic charts and his proposition of Taikisendzu (Atmospheric Chart).

As shown in the above classification new adiabatic charts such as Aerogram and Taikisendzu have appeared recently, convenient to the height calculation on these diagrams. According to the author's opinion,[(8)] contrary to the usual considerations[(2,3,4)] about Tephigram, it belongs to

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the same family of adiabatic charts as Aerogram etc. as regards to the convenience of height calculation. Though the general theories for height calculation are described in the above classification (iii) and (iv), Yamaoka's (iv) is, however, considered to be superior to Refsdal's theory (iii). But there remains some questions to be discussed—their assumptions and mutual relations among these diagrams.

In this paper[2] we shall consider mainly these problems, and make good Yamaoka's theory, but in our paper the whole problem is treated quite differently from Yamaoka's. In **1** preliminary notes on fundamental equations and the theory for height calculation so far used are described as fundamentals for the further development of our theory. In **2** conditions to be satisfied for our aim are discussed, as the above coordinates lead to the partial differential equation of Monge-Ampère's type, its general solution being given in **3.** In **4** the outline of Yamaoka's theory is described and its results are compared with ours. In **5** mutual relation among Aerogram, Taikisendzu and Tephigram is derived from our general formula and rough sketch of each case is also delivered.

1. Fundamental equations for the theory of height calculation

We shall commence our discussions by taking up two basic equations for the atmosphere, Laplace's static equation and the equation of state for the gas. As usual the former, Laplace's is

$$\frac{\partial p}{\partial z} = -\rho g\,, \tag{1}$$

where p is the pressure, ρ the density, z the height and g the gravitational acceleration. On the other hand the equation of state is

$$p = \rho RT\,, \tag{2}$$

where R is the gas constant for the atmosphere and T its temperature.

The derivation of the fundamental equation to be used for the discussion of height calculation for the atmosphere is performed by changing Eqs. (1) and (2) as follows

$$\frac{\partial p}{\partial z} = -\frac{g}{R}\frac{p}{T}\,. \tag{3}$$

Now introducing the following variable

$$s = \log\frac{p_0}{p}\,,$$

as a new variable in place of the pressure itself, Eq. (3) becomes

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[2] This paper contains partly an English translation of the author's previous papers[7,8] (in Japanese), and partly new treatment (in **3** and **4**)

$$\frac{\partial s}{\partial z}=\frac{g}{R}\frac{1}{T}. \tag{4}$$

To evaluate the height corresponding to each point of state of the atmosphere, which point is specified by its pressure p and temperature T[3], we used to assume that the pressure and the temperature of each point of state are functions of z only, i.e., the distribution of the pressure and that of the temperature are uniform horizontally. In this case the height of the point is evaluated by the next two sorts of method.

(i) Method by barometric formula

In this method the relation between the pressure and the height for the point of state is prescribed. After Eq. (3) is modified as follows

$$\frac{dp}{p}=-\frac{g}{R}\frac{1}{T}dz, \tag{5}$$

an integral for the pressure

$$p=p_0 e^{-\int_0^z \frac{g}{R}\frac{dz}{T}} \tag{6}$$

is obtained, which is regarded as the fundamental equation for this method. This method has been widely used from old times in meteorology and is called the method by barometric formula.

(ii) Method of geopotential

From Eq. (4) under the assumption of horizontally uniform distribution we obtain

$$gdz=RTds. \tag{7}$$

As the left-hand side of Eq. (7) is a change of geopotential φ for a differential height element dz, integrating them from z_0 to z_1, we obtain

$$\varphi_1-\varphi_0=\int_{z_0}^{z_1} gdz=R\int_{s_0}^{s_1} Tds. \tag{8}$$

Eq. (8) means that the difference between geopotentials φ_0 and φ_1 is equal to a certain area on the $T-S$ diagram—i.e., Emagram. Following the proposition by Bjerknes we shall make use of this geopotential φ instead of usual height z. Then the problem of height calculation for the state of point of the atmosphere will be solved by evaluating the geopotential-difference on the adiabatic charts based on Eq. (8). But as the form of the right-hand side of Eq. (8) itself is inadequate for the procedure of the practical geopotential calculation on the usual adiabatic charts—Emagram, Aerogram etc.—the problem to transform them into the new form convenient to their uses follows necessarily. To seek their concrete solutions for this problem is the main aim of our study.

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3) The effect of vapor is neglected in our discussions

2. The conditions to be satisfied by the transformation of coordinates in adiabatic charts

In order to satisfy the requirement for adiabatic chart to be used in aerology, that the practical procedure of height calculation on it must be performed very easily, we should express mathematically this requirement in the form that the coordinates of new adiabatic charts were specified by some assumptions.

First, we shall assume that the geopotential of an atmosphere taken as a standard one must be selected as one of its coordinates. In aerology it is very reasonable to take up the geopotential instead of height, as discussed by Bjerknes from the theoretical point of view. Secondly, according to Yamaoka's discussion we shall take up the following parameter τ as another coordinate of a new adiabatic chart. The vertical temperature distribution t of the above standard atmosphere is expressed as a function of s (or the pressure p) and an arbitrary parameter τ as below,

$$t = f(s, \tau).$$

For instance, this parameter may be the temperature of the atmosphere at the standard pressure.

Then as the transformation of one basic adiabatic chart (Emagram) to the new one, we shall have the following one,

$$\begin{cases} u=s=\log\dfrac{p_0}{p} \\ v=RT, \end{cases} \rightleftarrows \begin{cases} x=\tau \\ y=gz. \end{cases} \tag{9}$$

The conditions to be prescribed for this transformation are two; one of them must be Laplace's equation and the other be a condition to specify the transformation properties of area of these charts. Then these mathematical expressions are as below.

(i) The first condition, Laplace's is as shown in Eq. (4),

$$\frac{\partial u}{\partial y} = \frac{1}{v}. \tag{10}$$

(ii) The next condition of equal area transformation, is shown below,

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \text{const.}\ (=k). \tag{11}$$

Eliminating v from two conditions (10) and (11), we have a partial differential equation of the second order with respect to the dependent variable u. This equation belongs to the partial differential equation of the type of Monge and Ampère, its general solution being given in the following section.

3. General solution for our partial differential equation

Our equations to be solved are Eqs. (10) and (11) in a previous section. Eliminating v from Eqs. (10) and (11), we obtain

$$\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y}-\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}=k\left(\frac{\partial u}{\partial y}\right)^2. \tag{12}$$

If we rewrite Eq. (12) by using usual notations[4] (p, q, r, s and t) in the theory of partial differential equations, we have

$$qs-pt=kq^2. \tag{13}$$

This equation, which is a partial differential equation of Monge-Ampère's type with the second order, is solved by the method of Monge's characteristics as shown in the following.

From their definitions of usual notations, there are two relations to be satisfied by r, s and t, in addition to Eq. (13),

$$\begin{cases} dp=rdx+sdy\,, \\ dq=sdx+tdy\,. \end{cases} \tag{14}$$

In order that Eq. (13) may be solved for arbitrary r, s and t, the following two determinants of their coefficients must be zero, i.e.,

$$\begin{vmatrix} dx & dy & \\ & dx & dy \\ & q & -p \end{vmatrix}=0\,, \qquad \begin{vmatrix} dp & dy & \\ dq & dx & dy \\ kq^2 & q & -p \end{vmatrix}=0\,, \tag{15}$$

or

$$\begin{cases} dx(pdx+qdy)=0\,, \\ kq^2(dy)^2-pdpdx-qdpdy+pdydq=0\,. \end{cases} \tag{16}$$

As the former of Eq. (16) split into two conditions, adding them the characteristic condition

$$du=pdx+qdy\,,$$

we have the next two systems of equation (I) and (II) as Monge's system of characteristics

$$(\mathrm{I})\begin{cases} du-pdx-qdy=0 \\ dx=0 \\ kq^2dy-qdp+pdq=0\,, \end{cases} \qquad (\mathrm{II})\begin{cases} du-pdx-qdy=0 \\ pdx+qdy=0 \\ kq^2dy+pdq=0\,. \end{cases} \tag{17}$$

Or we shall rewrite them as follows

4) By usual notations we mean that

$$p=\frac{\partial u}{\partial x},\quad q=\frac{\partial u}{\partial y},\quad r=\frac{\partial^2 u}{\partial x^2},\quad s=\frac{\partial^2 u}{\partial x \partial y}\quad \text{and}\quad t=\frac{\partial^2 u}{\partial y^2}$$

$$(\mathrm{I})\ \begin{cases} dx=0 \\ d\left(ky-\frac{p}{q}\right)=0 \\ du-qdy=0\,, \end{cases} \qquad (\mathrm{II})\ \begin{cases} du=0 \\ d(kx+\log q)=0 \\ pdx-qdy=0\,. \end{cases} \tag{18}$$

According to Ampère, we shall introduce systems of two parameters α and β on the integral surfaces of (13), chosen as the following four cases.

Table I Choice of Parameter

Parameter / Case	α	β
A	x	u
B	x	$kx+\log q$
C	$ky-p/q$	u
D	$ky-p/q$	$kx+\log q$

But as each case has no particular essential features different from others, we shall take the most simple case (A), i.e., $\alpha=x$ and $\beta=u$ in the following discussion. Then as the system of partial differential equation to specify other y, p and q in terms of x and u, we have the following system of four partial equations.

$$\begin{cases} \dfrac{\partial y}{\partial u}=\dfrac{1}{q}, \\ \dfrac{\partial y}{\partial x}=\dfrac{p}{q} \end{cases} \tag{19}$$

$$\begin{cases} \dfrac{\partial}{\partial u}\left(\dfrac{p}{q}\right)=\dfrac{k}{q}, \\ \dfrac{\partial q}{\partial x}=-kq\,. \end{cases} \tag{20}$$

It can be verified for these four equations (19) and (20) to satisfy the conditions of integrability. Their solutions are easily determined as follows,

$$\begin{cases} y(x,u)=e^{kx}\displaystyle\int^{u} e^{\psi(u)}du+f(x)\,, \\ p(x,u)=ke^{-\psi(u)}\displaystyle\int^{u} e^{\psi(u)}du+f'(x)e^{-kx-\psi(u)}\,, \\ q(x,u)=e^{-kx-\psi(u)}\,, \end{cases} \tag{21}$$

where $\psi(u)$ and $f(x)$ are arbitrary functions of u and x respectively. These Eqs. (21) give the general solutions of Eq. (13) or (12).

Returning to meteorological terms, we shall here remember the definition of the temperature, Eqs. (9) and (10)

$$T=v=\frac{1}{q}=e^{kx+\psi(u)}. \tag{22}$$

Then

$$\begin{cases} x=\dfrac{1}{k}\left\{\log T-\log\varphi\left(\log\dfrac{p_0}{p}\right)\right\}, \\ y=\displaystyle\int^u T(x,u)du+f(x). \end{cases} \tag{23}$$

If we introduce the characteristic temperature Θ(6), the temperature at the pressure ($\psi(s)=0$), we have

$$\begin{cases} x=\dfrac{1}{k}\log\Theta, \\ y=\Theta\displaystyle\int^s e^{\psi(s)}ds+F(\Theta), \end{cases} \tag{24}$$

where $\Theta=T(x,s)e^{-\psi(s)}$ and $F(\Theta)=f\left(\frac{1}{k}\log\Theta\right)$.

In our previous paper(7) the same results were obtained from the first system (I) of the characteristics Eq. (17) only. But in this paper we treated Eq. (18) by Ampère's parameter method, different from the method treated in the former paper. By this derivation the whole scheme of the theory of the height calculation are made clearer than that in the former paper.(7)

4. Outline of Yamaoka's theory and its comparison with ours

In his paper(6) about the transformation of adiabatic charts, Yamaoka discussed it by the following method, which is summarized here from his Japanese paper for the convenience of our discussion about adiabatic charts.

On the Emagram ($T-s$) we shall consider an arbitrary group of curves

$$T=f(s,\tau), \tag{25}$$

where τ is a parameter. Integrating Laplace's equation (7) we have,

$$H=R\int Tds=R\int_0^s f(s,\tau)ds, \tag{26}$$

which expresses the area, enclosed by four curves,— s-axis, a curve $f(s,\tau)$ and two horizontal lines $s=0$ and $s=s$. The curves with equal H-values describe a group of parameter curves. Thus as shown in

Fig. 1, on Emagram we have two kinds of curves H=const. and τ=const. The meaning of these curves is as follows. The curves $T=f(s, \tau)$ mean a group of standard atmospheres, whereas the group of H=const. means that it is a group of equipotential curves.

For the selection of standard atmospheres there are several rules to be prescribed,

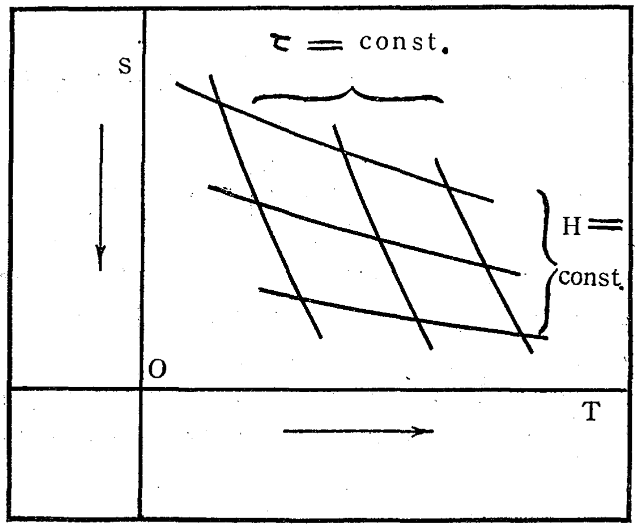

Fig. 1 Emagram

(i) The distributions of the temperature of standard atmosphere and of real observed atmosphere must have a similar form in order to make sure the accuracy of calculation on diagram.

(ii) The easiness for the interpolation of height curves must be retained.

(iii) The group of curves for standard ones as scale of height must be expressed by displacement of one curve instead of real plottings.

To the last requirement we shall attain by transforming the diagram into the form, in which equi-altitude curves are straight and parallel with abscissa and have constant mutual intervals. Now to perform the transformation satisfying the above requirements, we take up two successive transformation of equal area

$$(T, s) \rightarrow (\tau, H) \rightarrow (t, H), \tag{27}$$

where τ and t have the same meanings as in **2** and **3**.

Then the above condition of the conservation of the area is expressed as below, by taking the coordinates (τ, H) as standard,

$$\frac{\partial(T, s)}{\partial(\tau, H)} = \frac{\partial(t, H)}{\partial(\tau, H)}. \tag{28}$$

Calculating the left-hand side of Eq. (28) by using Eqs. (25) and (26), we obtain

$$\frac{\partial(T, s)}{\partial(\tau, H)} = \frac{1}{R}\frac{\partial}{\partial\tau}\log f\,,$$
$$\frac{\partial(t, H)}{\partial(\tau, H)} = \frac{\partial t}{\partial\tau}\,, \qquad (29)$$

or

$$\frac{1}{R}\frac{\partial}{\partial\tau}\log f = \frac{\partial t}{\partial\tau}\,, \qquad (30)$$

which is integrated as follows after expressing the left-hand side of Eq. (30) by τ and H,

$$t(\tau, H) = \frac{1}{R}\int\left[\frac{\partial}{\partial\tau}\log f\right]_s d\tau + \varphi(H)\,, \qquad (31)$$

where $\varphi(H)$ is an arbitrary function of H and $[\quad]_s$ means that $f(s, \tau)$ is partially differentiated with respect to τ, by taking s=const. and then we eliminate s by using H.

The next condition to be satisfied by his transformation is that the standard atmosphere becomes parallel to the ordinate axis, it being expressed as

$$\int\left[\frac{\partial}{\partial\tau}\log f\right]_s d\tau = \text{a function of } \tau \text{ only.} \qquad (32)$$

Then the integrand of the left-hand side of Eq. (32) must be the function of τ only, because it is expressed as

$$\psi\{s(H, \tau), \tau\}$$

by eliminating s by using Eq. (26). Then

$$\left[\frac{\partial}{\partial\tau}\log f\right]_s = \psi(\tau)\,, \qquad (33)$$

or

$$\log f(s, \tau) = \int\psi(\tau)d\tau + \chi(s)\,,$$

or

$$f(s, \tau) = \Psi(\tau)X(s)\,, \qquad (34)$$

where $\Psi(\tau)$ and $X(s)$ are respectively an arbitrary function of τ or s only. The above function is expressed as

$$f(s, \tau) = \Theta g(s)\,, \qquad (35)$$

by taking the characteristic temperature at standard pressure s_0, which is expressed as

$$\Theta = \Psi(\tau)X(s_0)\,.$$

The abscissa of new adiabatic chart is obtained by Eq. (31) and (35)

$$t(\tau, H)=\frac{1}{R}\log\Theta+\varphi(H)\,. \tag{36}$$

Finally we shall compare Yamaoka's results, Eqs. (26) and (36), with our results obtained in a previous section by changing slightly the notation for the convenience of comparison.

(i) Yamaoka's results

$$\begin{cases} x=\dfrac{1}{R}\log\Theta+\varphi(H)\,, & (31)\\[2ex] y=\displaystyle\int^{s} f(s,\tau)ds=\Theta\int^{s} e^{\psi(s)}ds. & (26)\end{cases}$$

(ii) Our results

$$\begin{cases} x=\dfrac{1}{k}\log\Theta\,,\\[2ex] y=\Theta\displaystyle\int^{s} e^{\psi(s)}+F(\Theta)\,. \end{cases} \tag{24}$$

The only difference is how to add an arbitrary function $\varphi(H)$ or $F(\Theta)$ to either coordinate. If we require that the transformation (9) must be an equal area transformation, our results (24) must be prefered instead of Yamaoka's, for Eqs. (26) and (36) do not satisfy Eq. (11). The rightness of this justification is clear from the fact that in Eq. (18) x becomes parameter, but not $du-qdy$. This disagreement of Yamaoka's results with ours must originate in the complication of his transformation (27) or in the early assumption of the form of Eq. (26).

5. Mutual relations among Aerogram, Taikisendzu and Tephigram and their details

In this section we shall show that there are intimate relations among many adiabatic charts such as Aerogram, Taikisendzu and Tephigram, convenient to height calculation, which are derived from our general solutions in a previous section, by specifying particular forms for arbitrary functions. By taking the special case of a simple assumption $F(\Theta)=0$ and $\psi(s)=\alpha s$, in Eq. (24), for the coordinates of these charts we have an unified formula as follows,

$$\begin{cases} x=\log\Theta\,,\\[1ex] y=\Theta\displaystyle\int_{s_0}^{s} e^{\alpha s}ds\,, \end{cases} \tag{37}$$

where $$\Theta=Te^{-\alpha s} \quad\text{and}\quad s=\log\frac{p_0}{p}, \tag{38}$$

the constants being taken as $R=1$, $g=1$ and $k=1$.

The above three adiabatic charts are classified as follows, according to the special choice of the constants α and s_0 in Eq. (37).

Table II Classification of Constants used in Adiabatic Charts

Diagram \ Constant	α	s_0 (or p_0)
Aerogram	0	0 (const.)
Taikisendzu	const.	0 (const.)
Gen. Tephigram	const.	$-\infty$ (0)

(i) Aerogram

This chart is the case, $\alpha=0$ and $s_0=0$ in Eq. (37), where isothermal atmosphere is taken as a standard atmosphere. The height of this isothermal atmosphere is measured from the pressure $p=p_0$. For this

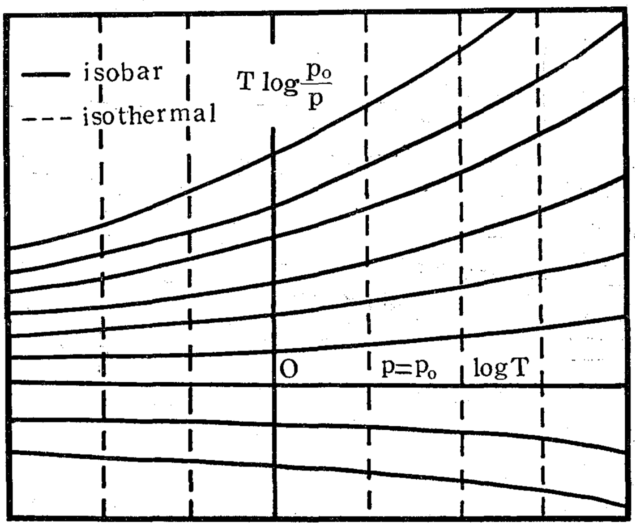

Fig. 2 Outline of Aeogram

case, taking the above values for α and s_0, the coordinates x and y are taken as follows

$$\begin{cases} x=\log T, \\ y=T\int^{s} ds=T\log\dfrac{p_0}{p}. \end{cases} \tag{39}$$

The rough sketch of this diagram is shown in Fig. 2, where the isothermals (dotted lines) are parallel with the ordinate and the isobars (full lines) are exponential lines, whose asymptotes are x-axis.

(ii) Taikisendzu

This chart is the case in which we take α=const. and $s_0=0$ in Eq. (37), the coordinates being

$$\begin{cases} x=\log T/p^{\alpha}, \\ y=T/\alpha\left\{1-\left(\dfrac{p}{p_0}\right)^{\alpha}\right\}. \end{cases} \tag{40}$$

In the special case $a=R/c_pJ$, the atmosphere taken as a standard for the geopotential is a dry adiabatic atmosphere. The special adiabatic chart with $a=0.164$, Taikisendzu, has been used since 1939 in Japanese military meteorological fields, which was constructed by T. Yamaoka and the author etc., in that chart the lapse rate of temperature being taken as 0.56°C/100m by the request of the artillery. This Taikisendzu resembles to Tephigram in some points, but differs from the latter in the point that in the latter geopotentials are meassured from the isobar $p=0$, while in the former they are meassured from the isobar $p=p_0$. (cf. Table II) Isothermals (full lines) and isobars (dotted lines) are shown in Fig. 3. Resembling to Aerogram isobars are exponential lines

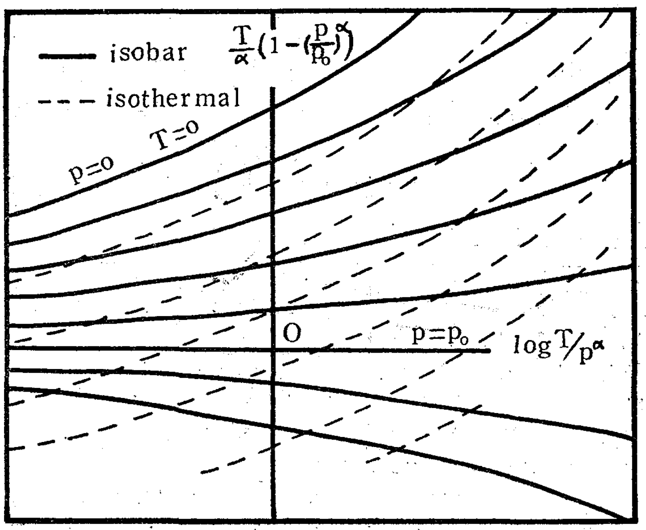

Fig. 3 Outline of Taikisendzu

asymptoting the x-axis (the isobar $p=p_0$), but isothermals are constructed by parallel displacement of the isobar $p=0$ (this isobar coincides with the isothermal $T=0$) in the direction of the ordinate.

(iii) Generalized Tephigram

This chart is the case $a=$const. and $s_0=-\infty$,

$$\begin{cases} x=\log T/p^a, \\ y=T/a. \end{cases} \tag{41}$$

This case is regarded as a generalized one of Tephigram, whose ordinate is taken as the geopotential of polytropic atmosphere at the pressure p meassured from the isobar $p=0$. In the special case $a=R/c_pJ$ the chart is nothing but the ordinary Tephigram, in which the ordinate is the geopotential meassured from the isobar $p=0$ for the dry atmosphere. The rough sketch of this diagram is shown in Fig. 4. In this chart isobars are exponential curves asymptoting the x-axis and isothermals are parallel lines to x-axis.

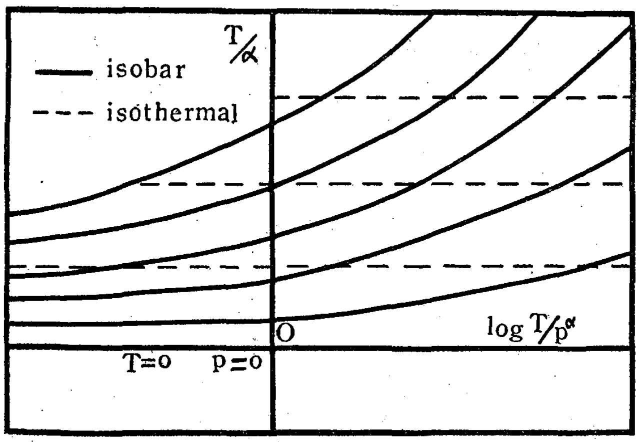

Fig. 4 Outline of Tephigram

The author wishes his sincere thanks to Mr. Yamaoka for his suggestion of this problem and many advices in the course of his study.

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