Some Remarks on Primary Lattices[1]

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Three years ago I published a paper on primary lattices in the Journal of the Faculty of Science, Hokkaido University, Series 1, Vol. 11, (1948). In the present note I will give some remarks about this paper, including various definitions of primary lattices, a simplified proof of Theorem 32 and the correction of Theorem 45.

Let a and b be two elements with $a < b$ in a lattice L. The sublattice of all elements x with $a \leqq x \leqq b$ is called an *interval* or a *quotient* in L and denoted with b/a. If an interval is linearly ordered, then it is called to be *linear* or to be a *chain*. A finite-dimensional modular lattice was called *primary* in the previous paper, if every interval is either linear or has no proper neutral element. We shall show that this definition is equivalent with the following. A finite-dimensional modular lattice is called primary, if in every interval b/a, which is not linear, there exist at least three elements, which cover a, and, if b covers at least three elements in b/a. If every interval, which is not linear, has no proper neutral element, then it has the above property, since otherwise it would contain one of intervals with graphs of following types

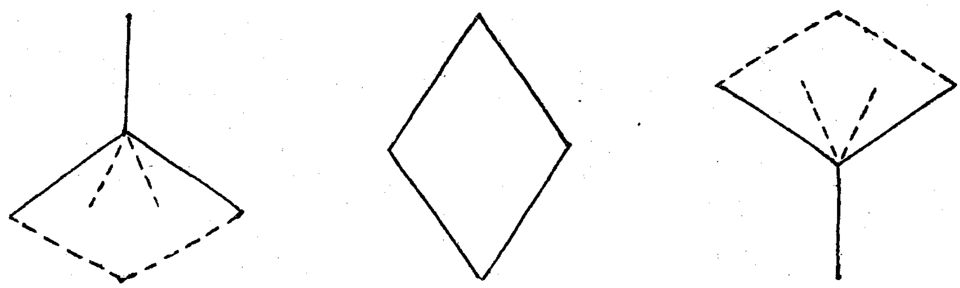

Fig. 1

and these have obviously proper neutral elements. We can prove the converse as follows. By the latter definition the lower closure of an irreducible element in an interval is always linear. For any element c in an interval b/a, where b/a is not linear and $b > c > a$, we choose an irreducible element d in b/a, which does not belong to c/a. Furthermore, if in particular c is irreducible, d can be so chosen that c does not belong to d/a, since b/a is not linear. Then we have $c \frown d < c$ and

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$c \frown d < d$. If e and f cover $c \frown d$ respectively in the intervals $c/c \frown d$ and $d/c \frown d$, then there exists an element g, which covers $c \frown d$ such that $e \smile f = e \smile g = f \smile g$. Then we have $g \frown c = g \frown e = c \frown d$, since otherwise it would follow $g \leqq c$, whence $f \leqq c$ and $f \leqq c \frown d$. Similarly we have $f \frown c = c \frown d$. Now it holds

$$(f \smile g) \frown c = (e \smile f) \frown c \geqq e$$

and
$$(f \frown c) \smile (g \frown c) = \{(g \frown c) \smile f\} \frown c$$
$$= \{(c \frown d) \smile f\} \frown c = c \frown d < e\,.$$

Hence c is not neutral in b/a and the proof is concluded. It is to be remarked that every interval of a primary lattice is primary. Further an indecomposable complemented modular lattice of finite dimension is primary, since its every interval is complemented and indecomposable, any two atoms in it or in its dual being always perspective. In the following by L is meant a primary lattice.

Lemma 1. *If an interval b/a in L is linear, then there exists an irreducible element l, such that $b = a \smile l$ and* dim l *is not less than the dimension of b/a.*

Proof. If it holds $b = b_1 \smile b_2 \smile \cdots \smile b_n$ with irreducible elements b_i, then $a \leqq a \smile b_i \leqq b$ and $a \smile b_j = b$ with some b_j, since b/a is linear.

Lemma 2. *Let $d, c_1, c_2, \cdots, c_n$ be irreducible elements in L with $d \leqq c_1 \smile c_2 \smile \cdots \smile c_n$. Then it holds*

$$\dim d \leqq \operatorname{Max}(\dim c_i)\,.$$

Proof. Putting $\operatorname{Max}(\dim c_i) = \lambda$, we prove by induction on λ. If $\lambda = 1$, then the interval $c_1 \smile c_2 \smile \cdots \smile c_n/O$ is complemented modular and hence $\dim d = 1$. Now we suppose $\lambda > 1$. We denote with c_i' the element, which is covered by c_i, and put $a = c_1' \smile c_2' \smile \cdots \smile c_n'$. If $a \smile d = a$, then $d \leqq c_1' \smile c_2' \smile \cdots \smile c_n''$ and $\dim d \leqq \lambda - 1$ by the induction hypothesis. If $a \smile d > a$, then $a \smile d/a$ is linear and $a \smile d$ covers a, since the interval $c_1 \smile c_2 \smile \cdots \smile c_n/a$ is complemented modular. The element d', which is covered by d, is irreducible and $d' \leqq a$, since $\dim(a \smile d/a) = \dim(d/d \frown a) = 1$. Then it follows $\dim d' \leqq \operatorname{Max}(\dim c_i') = \lambda - 1$ by the induction hypothesis, whence $\dim d \leqq \lambda$.

For an irreducible element a we denote with $a^{(\lambda)}$ the irreducible element of dimension λ in the lower closure a/O.

Lemma 3. *Let a and b be two different irreducible elements in L with a same dimension. We put* $\dim(a \frown b) = \mu$. *If c is an irreducible element of dimension λ with $c \leqq a \smile b$, then it holds $c \leqq a^{(\lambda+\mu)} \smile b^{(\lambda+\mu)}$, where $\lambda \leqq$* dim *$a - \mu$.*

Proof. Putting $\dim a = \nu$, we have $a^{(\lambda+\mu)} \frown b^{(\lambda+\mu)} = a \frown b$ and

$$(1) \qquad \dim(a \smile b) - \dim(a^{(\lambda+\mu)} \smile b^{(\lambda+\mu)}) = 2(\nu - \lambda - \mu)\,.$$

In the interval $a \smile b/c \smile a^{(\lambda+\mu)} \smile b^{(\lambda+\mu)}$ both elements $c \smile a \smile b^{(\lambda+\mu)}$ and

$c \cup b \cup a^{(\lambda+\mu)}$ are irreducible and of dimension $\nu-\lambda-\mu$, since $(c \cup a^{(\lambda+\mu)} \cup b^{(\lambda+\mu)}) \cap a = a^{(\lambda+\mu)}$ by Lemma 2. If we can prove the relation

$$(2) \qquad (c \cup a \cup b^{(\lambda+\mu)}) \cap (c \cup b \cup a^{(\lambda+\mu)}) = c \cup a^{(\lambda+\mu)} \cup b^{(\lambda+\mu)},$$

then we have $\dim(a^{(\lambda+\mu)} \cup b^{(\lambda+\mu)}) = \dim(c \cup a^{(\lambda+\mu)} \cup b^{(\lambda+\mu)})$ by (1), whence $c \leq a^{(\lambda+\mu)} \cup b^{(\lambda+\mu)}$. If (2) were not true, we would have $c \cup b \cup a^{(\lambda+\mu+1)} \leq c \cup b \cup a^{(\lambda+\mu)}$. Since it holds

$$\frac{b \cup a^{(\lambda+\mu+1)}}{b} \simeq \frac{a^{(\lambda+\mu+1)}}{a \cap b},$$

so the interval $b \cup a^{(\lambda+\mu+1)}/b$ is linear of dimension $\lambda+1$ and contained in the interval $c \cup b \cup a^{(\lambda+\mu)}/b$. Furthermore

$$(3) \qquad \frac{c \cup b \cup a^{(\lambda+\mu)}}{b} \simeq \frac{c \cup a^{(\lambda+\mu)}}{b \cap (c \cup a^{(\lambda+\mu)})}$$

and $b \cap a \leq b \cap (c \cup a^{(\lambda+\mu)})$. Since two irreducible elements $c \cup (b \cap a)$ and $a^{(\lambda+\mu)}$ in the interval $c \cup a^{(\lambda+\mu)}/a \cap b$ are linear and their dimensions are not greater than λ, so the interval $c \cup a^{(\lambda+\mu)}/b \cap (c \cup a^{(\lambda+\mu)})$ does not contain a linear interval of dimension $\lambda+1$ by Lemma 2. Hence the interval $c \cup b \cup a^{(\lambda+\mu)}/b$ does not contain a linear interval of dimension $\lambda+1$, which yields a contradiction.

An irreducible element c is called *maximal,* if there does not exist an irreducible element d with $c < d$.

Lemma 4. *Let a and b be two different irreducible elements in L with a same dimension. There exists a maximal irreducible element c in $a \cup b/O$ such that $a \cap c = O$, $b \cap c = O$ and $a \cup c = b \cup c = a \cup b$. Furthermore it is possible to choose this c in $a \cup b/O$ such that $c \geq d$ for a given irreducible element d with $d \cap a = O$ and $d \cap b = O$.*

Proof. Since the interval $a \cup b/O$ is not linear, there exists certainly an atom, which belongs neither to a/O nor to b/O and belongs to $a \cup b/O$. The maximal irreducible element c in $a \cup b/O$, whose lower closure contains this atom or d, satisfies the relations $a \cap c = b \cap c = O$ and $a \cup c \leq a \cup b$. In order to have $a \cup c = a \cup b = b \cup c$, it suffices to prove $\dim(a \cup c) = \dim(a \cup b)$. If it were not the case, then $\dim c < \dim a - \dim(a \cap b)$, since $a \cap c = O$ and $\dim a = \dim b$. Putting $\dim c = \lambda$, $\dim(a \cap b) = \mu$ and $\dim a = \nu$, we have $\lambda < \nu - \mu$. By Lemma 3 we have $c \leq a^{(\lambda+\mu)} \cup b^{(\lambda+\mu)}$ and this implies $c \cup a^{(\lambda+\mu)} = a^{(\lambda+\mu)} \cup b^{(\lambda+\mu)}$. The interval $a \cup b/a^{(\lambda+\mu)} \cup b^{(\lambda+\mu)}$ is not linear, since $\lambda+\mu < \nu$. Consequently the interval $a \cup b/c$ is not linear. Now $a^{(\lambda+\mu)} \cup b^{(\lambda+\mu)}$ is irreducible and of dimension $\lambda+\mu$ in the interval $a \cup b/c$. We can choose an element f in $a \cup b/c$ such that f covers c and $f \cap (a^{(\lambda+\mu)} \cup b^{(\lambda+\mu)}) = c$. There exists an irreducible element l such that $f = c \cup l$ by Lemma 1. Then we have $\dim l > \lambda$, since otherwise we would have $l \leq a^{(\lambda+\mu)} \cup b^{(\lambda+\mu)}$ by Lemma 3, contrary to the choice of f. Then $\dim f = \lambda+1$ yields $\dim l = \lambda+1$ and

$c < l$, which yields a contradiction, since c is maximal irreducible in $a \cup b/O$, q. e. d.

Now we can correct Theorem 45 in my previous paper as follows.

Theorem 1. *A finite-dimensional modular lattice is primary, if and only if it holds in any interval following two conditions:*
(A) *The lower closure of every irreducible element is linear.*
(B) *Any two different irreducible elements of a same dimension are perspective.*

Proof. That a and b are perspective, means that there exists c such that $a \cup c = b \cup c$ and $a \cap c = b \cap c = O$. Lemma 4 asserts that a primary lattice satisfies (A) and (B). Conversely, if (A) and (B) hold, and if an interval b/a is not linear, then there exist two different irreducible elements of a same dimension in b/a. Hence by (B) there exist two elements, which cover a. Since these are perspective, there exists another element, which covers a. Further, b being not irreducible, there exist two elements d and e, which are covered by b. Since d and e are perspective in the interval $b/d \cap e$, there exists another element, which is covered by b, q. e. d. It is to be remarked that Theorem 1 yields another definition of primary lattice.

Lemma 5. *Let $a_1, a_2, \cdots, a_n$ be n independent irreducible elements in L. For every maximal irreducible element b with $b \leqq a_1 \cup a_2 \cup \cdots \cup a_n$, it holds* $\dim b \geqq \mathrm{Min}\,(\dim a_i)$.

Proof. We put $\mathrm{Min}\,(\dim a_i) = \lambda$. In the case, where $n=2$ and $a_1 \cap b = O$, $a_2 \cap b = O$, we have $a_1^{(\lambda)} \cup b = a_2^{(\lambda)} \cup b \geqq a_1^{(\lambda)} \cup a_2^{(\lambda)}$ by Lemma 4, whence $\dim b \geqq \lambda$. In the case, where $n=2$ and $a_1 \cap b > O$, $a_2 \cap b = O$, there exists an irreducible element c by Lemma 4 such that $a_1^{(\lambda)} \cup c = a_2^{(\lambda)} \cup c = a_1^{(\lambda)} \cup a_2^{(\lambda)}$ and $a_1 \cap c = a_2 \cap c = O$, whence $\dim c = \lambda$. If $\dim b \leqq \lambda$, we have $b \leqq a_2^{(\lambda)} \cup c$ by Lemma 2. Since $b \cap a_2^{(\lambda)} = O$ and $b \cap c = O$, this case can be reduced to the preceding case and we have $\dim b = \lambda$. The case, where $a_2 \cap b > O$ and $a_1 \cap b = O$, can be similarly treated. Now we proceed by induction on n. Since b is independent of some element a_i, we can assume that $b \cap a_n = O$. We put $a_1 \cup a_2 \cup \cdots \cup a_{n-1} = A$ and $(b \cup a_n) \cap A = d$. Then it holds

$$\frac{d}{O} = \frac{d}{d \cap a_n} \simeq \frac{a_n \cup d}{a_n} = \frac{b \cup a_n}{a_n} \simeq \frac{b}{O}.$$

Hence d is irreducible and $\dim d = \dim b$. Since $d \leqq a_1 \cup \cdots \cup a_{n-1}$, there exists a maximal irreducible element d' such that $d \leqq d'$ and $d' \leqq a_1 \cup \cdots \cup a_{n-1}$. By induction hypothesis we have $\dim d' \geqq \lambda$ and $d \cup a_n = b \cup a_n$ implies $b \leqq d' \cup a_n$. It follows $d' \cap a_n = O$ from $d \cap a_n = O$ and we obtain $\dim b \geqq \lambda$ by the result for the case $n=2$.

By this lemma we can prove Theorem 32 in the previous paper as follows.

Theorem 2. *In a primary lattice every element is a join of independent irreducible elements.*

Proof. For a given element a we choose irreducible elements a_i with $a=a_1 \cup a_2 \cup \cdots \cup a_n$ such that $\sum \dim a_i$ is least. We can assume that $\dim a_1 \geqq \dim a_2 \geqq \cdots \geqq \dim a_n$. We shall prove that $a_1, a_2, \cdots, a_n$, are independent. If it were not the case, then we can suppose that $a_1, \cdots, a_{i-1}$ are independent and $a_i \cap (a_1 \cup \cdots \cup a_{i-1}) > O$. If we put $b=a_i \cap (a_i \cup \cdots \cup a_{i-1})$, then $a_i > b > O$, since we can remove a_i, if $a_i \leqq a_1 \cup \cdots \cup a_{i-1}$. Let c be a maximal irreducible element with $c \leqq a_1 \cup \cdots \cup a_{i-1}$ and $c \geqq b$. Then we have $\dim c \geqq \dim a_i > \dim b$ by Lemma 5. Putting $\dim a_i=\lambda$, two irreducible elements $c^{(\lambda)}$ and a_i are different. By Lemma 4 there exists an irreducible element d such that $c^{(\lambda)} \cap d=O$ and $c^{(\lambda)} \cup d=a_i \cup d=a_i \cup c^{(\lambda)}$. Hence we have

$$a_1 \cup \cdots \cup a_{i-1} \cup a_i=a_1 \cup \cdots \cup a_{i-1} \cup d,$$
$$\dim (c^{(\lambda)} \cup d)=\dim (a_i \cup c^{(\lambda)})=2\lambda-\dim b.$$

Then $c^{(\lambda)} \cap d=O$ implies $\dim d=\lambda-\dim b < \dim a_i$. Hence we can replace $a=a_1 \cup a_2 \cup \cdots \cup a_n$ by $a=a_1 \cup \cdots \cup a_{i-1} \cup d \cup a_{i+1} \cup \cdots \cup a_n$ with $\dim d < \dim a_i$, contrary to the assumption.

Theorem 3. *A finite-dimensional modular lattice is primary, if and only if the following conditions are satisfied*:

(A) *In any interval the lower closure of an irreducible element is linear.*

(B) *In any interval every element is representable as a join of independent irreducible elements.*

(C) *Every interval is indecomposable.*

Proof. If an interval b/a is not linear, from (A) and (B) it follows that at least two elements c and d cover a. Then there exists another element, which covers a, since $c \cup d/a$ is indecomposable. Further, b being not irreducible, there exist at least two elements, which are covered by b. Then it follows by the same reasoning as above that b covers at least three elements.

Finally I must remark that the lattices with the above properties were already studied by many authors, such as Baer, Iwasawa, Jones, Ore, in connection with the structure of finite groups, but my research was purely lattice-theoretical.