On an Elementary Treatise of Integration[1]

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Since various ways of approaching the integration theory have been studied from all points of view by many authors, nothing essential seems left to be searched for especially in the classical parts of this field.

Still we hope in this short note that there may be something novel in the way of putting special emphasis on the additivity of integrals as set-functions.

It is easily seen from the elements of "calculus" that the whole theoretical parts of integral calculus will be much simplified if the existence of their primitive functions are assumed. This suggests the way of replacing the above primitive functions by some suitable set-functions having a certain relation to the given point-functions, so that the whole theory may, at least to a certain extent, be simplified and clarified.

1. Terminology and Notations. Let S be a fixed set in which is given a completely additive class **B** of subsets $\subset S$ such that $S \in \mathbf{B}$.

Let $m(X)$ be a countably additive measure defined for $X \in \mathbf{B}$ where $m(S) < +\infty$. Further let us write $\sum_{j=1}^{n} X_j = X_1 + \cdots + X_n$ instead of $\cup_{j=1}^{n} X_j$ if the sets X_j are disjoint with each other.

To each finite partition $\varDelta$ of $X \in \mathbf{B}$: $X = X_1 + \cdots + X_n$ $(X_i \in \mathbf{B})$, we may associate two numbers

$$\bar{\sigma}_\varDelta = \textstyle\sum_j \sup X_j^f \, m(X_j) \quad \text{and} \quad \underline{\sigma}_\varDelta = \sum_j \inf X_j^f \, m(X_j)$$

where X_j^f denotes the set of the values taken by f on X_j.

We shall use, without making a special mention, the letters X, X_j, etc., to denote the measurable sets only.

2. Definitions and Theorems. Let us begin with the following

Theorem 1 *To each bounded function f, not necessarily be measurable* **B**, *there exists a countably additive set-function F satisfying the following condition:*

(1) $m(X) \inf X^f \leq F(X) \leq m(X) \sup X^f$ *for every* $X \in \mathbf{B}$.

Among such $F(X)$, there exist the greatest and the smallest ones, which coincide with $\inf_\varDelta \bar{\sigma}_\varDelta$ *and* $\sup_\varDelta \underline{\sigma}_\varDelta$ *respectively, where the* inf *and* sup *are taken over all finite partition $\varDelta$ of X.*

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Proof. Since the proof is easy, only a brief sketch will be given here.

Let us first show that from the finite additivity of F follows the countable one.

We observe that if $X=X_1+X_2+\cdots$, then $m(X)=\lim_{n\to\infty}\sum_{j=1}^{n} m(X_j)$ which shows $m(X-\sum_{j=1}^{n} X_j)=m(X)-\sum_{j=1}^{n} m(X_j)\to 0\ (n\to\infty)$. But, by (1) we have

$$a\, m(X-\sum_{j=1}^{n} X_j)\leqq F(X-\sum_{j=1}^{n} X_j)\leqq b\, m(X-\sum_{j=1}^{n} X_j)$$

where $a\leqq f(x)\leqq b$, whence we have

$$F(X)=\sum_j F(X_j)$$

since

$$F(X-\sum_{j=1}^{n} X_j)=F(X)-\sum_{j=1}^{n} F(X_j)\,.$$

Writing $\overline{F}(X)=\inf_\varDelta \bar{\sigma}_\varDelta$, we find from (1'), $F(X)\leqq\bar{\sigma}_\varDelta$ from which we also have $F(X)\leqq\overline{F}(X)$. Since from the following obvious relation

$$m(X)\ \sup\ X^f\geqq\bar{\sigma}_\varDelta\geqq\underline{\sigma}_\varDelta\geqq m(X)\ \inf\ X^f\,,$$

we see that $\overline{F}$ satisfies (1).

We have only to show that $\overline{F}$ is finitely additive, the proof of which may be carried on in a ready made way, and is omitted here. We may discuss about $\underline{F}$ quite similarly.

Can we take such a $F(X)$ as the definition of the integral of f over X, without assuming further restrictions on f or F? This is answered negatively since, for instance, we can not deduce from the mere condition described above the linearity of the integral. This is perhaps one of the chief reasons why S. Saks in his famous book, giving the definition of integral of positive functions by $\underline{F}$, has to reduce 'linearity' to the case of the integral of simple functions, which gives his construction an appearance of some complexity. In this respect, the book written recently by Halmos is quite satisfactory.

We are now in a position to settle some restrictions on F or f.

If f is measurable on S, then, regarded as a function on $X\in\mathbf{B}$, it becomes measurable also on X, so that, by making 'ε-partition' $\varDelta_\varepsilon$ of X: $X=X_1+\cdots+X_m$, where the diameter of $X_j^f\leqq\varepsilon$ for $j=1,\cdots n$, we find

$$\text{(2)}\qquad \sup\ X_j^f-\inf\ X_j^f\leqq\varepsilon\qquad (j=1,\cdots,n)$$

whence, for an arbitrary $x_j\in X_j$

(3) $$\inf X_j^f m(X_j) \leqq f(X_j) m(X_j) \leqq \sup X_j^f m(X_j).$$

Now, let F be any additive set-function, satisfying all the conditions of Theorem 1.

By (1), (2), (3), we find

$$|F(X)-\sum_j f(X_j) m(X_j)| \leqq \sum_j (\sup X_j^f - \inf X_j^f) m(X_j) \leqq \varepsilon m(X)$$

in which, making $\varepsilon \to 0$, the sum $\sum f(x_j)\ m(X_j)$ tends to $F(X)$ which shows also, that the function F is uniquely determined by f.

Thus we have the following

Theorem 2 *For a measurable f, the set-function F of Theorem 1 is uniquely determined and*

$$\inf_\Delta \bar{\sigma}_\Delta = \sup_\Delta \underline{\sigma}_\Delta .$$

Let Δ_ε: $X=X_1+\cdots+X_m$ be an arbitrary ε-partition of X. Forming the sum $\sum_{j=1}^{n} f(x_j)\ m(X_j)$ for arbitrarily chosen $x_j \in X_j$ $(j=1,\cdots,n)$, then

(4) $$\lim_{\varepsilon\to 0} \sum f(X_j) m(X_j) = F(X).$$

In view of this theorem, we can give the definition of integral of bounded f over X_0 as follows:

we say f is integrable over X_0 if and only if F of Theorem 1, *for all $X \subset X_0$, is uniquely determined.*

The value $F(X)$ is called the integral of f over X and denoted as usual by

$$F(X) = \int_X f(x)\ dm(X) = \int_X f.$$

Thus we see that, by Theorem 2, every bounded measurable function is integrable and the integral given by (4) is also equal to $\inf_\Delta \bar{\sigma} = \sup_\Delta \underline{\sigma}_\Delta$, which is identical with the ordinary one.

If we write

$$\sup_\Delta \underline{\sigma}_\Delta = \underline{\int}_X f, \qquad \inf_\Delta \bar{\sigma}_\Delta = \overline{\int}_X f$$

and call these the lower and the upper integrals of f over X, we find easily

$$\underline{\int}_X f \leqq \overline{\int}_X f, \qquad f \leqq g \rightarrow \overline{\int}_X f \leqq \overline{\int}_X g, \qquad \underline{\int}_X f \leqq \underline{\int}_X g.$$

According to our definition, the necessary and sufficient condition that f be integrable over X_0 is that, for every $X \subset X_0$,

$$\underline{\int}_X f = \overline{\int}_X f$$

should hold.

This apparently restricted condition may be relaxed as follows:

$$\underline{\int}_{X_0} f = \overline{\int}_{X_0} f$$

implies the integrability of f over X_0.

According to the additivity of the upper and the lower integrals, we have

$$\left(\overline{\int}_X f - \underline{\int}_X f\right) + \left(\overline{\int}_{X_0-X} f - \underline{\int}_{X_0-X} f\right) = \overline{\int}_{X_0} f - \underline{\int}_{X_0} f = 0$$

in which two expressions enclosed in parentheses are $\geqq 0$ and consequently must be 0.

Also by the additivity, if f is integrable over X and Y $(X \frown Y = 0)$, then it is also integrable over $X + Y$.

We shall denote the family of all the integrable functions over S by $\mathbf{I}_S$ or $\mathbf{I}$.

In particular, $\mathbf{I}_S$ contains the class of all simple functions and the characteristic function of every $X \subset S$, of which the integral is $m(X)$.

We shall derive here some of the fundamental properties directly obtainable from our very definition. By (1), we see first

$$\left|\int_X f\right| \leqq \left\|f\right\|_X m(X) \qquad \text{(boundedness)}$$

where $\|f\|_X = \sup_{x \in X} |f(x)|$ and

$$\int_X f \geqq 0 \qquad \text{for } f \geqq 0 \qquad \text{(monotonity).}$$

Next to show

$$f,\ g \in I \Rightarrow f + g \in I, \qquad cf \in I \qquad \text{for any constant } c$$

and

$$\int_X (f+g) = \int_X f + \int_X g \quad \text{and} \quad \int_X cf = c\int_X f \qquad \text{(linearity)}$$

we observe

$$\underline{\int}_X (f+g) \geqq \underline{\int}_X f + \underline{\int}_X g = \int_X f + \int_X g$$

$$\overline{\int}_X (f+g) \leqq \overline{\int}_X f + \overline{\int}_X g = \int_X f + \int_X g$$

and

$$\int_X f + \int_X g \leqq \underline{\int}_X (f+g) \leqq \overline{\int}_X (f+g) \leqq \int_X f + \int_X g,$$

which shows that, in the above inequalities, only the equal signs can occur.

It is also easy to see that

$$\int_X cf = c\int_X f \qquad \text{for any constant } c.$$

Thus our integral is a bounded, linear functional defined over $\mathbf{I}_S$ and, regarded as an indefinite integral, a m-absolutely continuous additive set-function.

3. Equivalence of our definition with the ordinary one. Since, on these lines described above, we shall be able to derive all the important properties to develop the integration theory, we may ask ourselves if we have defined the integral for the class of functions wider than that of measuable ones.

This will be answered negatively, namely from the uniqueness of F will follow the measurability of f.

Consequently, by developing our theory not only directly from our definition but also from the properties of measurable functions, for instance, from (4) of Theorem 2, etc., we may simplify the whole theory fairly well.

To show that $\mathbf{I}_S$ coincides with the class of measurable functions on S, we shall make use of, for the sake of brevity, the properties of measurable functions.

Since our integral is identical with the ordinary one for simple functions, we have

$$\int_S f(x)\ dm(x) = \sum_{j=1}^{n} c_j\, m(X_j)$$

where $S = X_1 + \cdots + X_n$, $f(x) \equiv c_j$ for $x \in X_j$ $(j=1,\cdots,n)$.

By Egoroff's Theorem or others, we can show that if $\{f_n\}$ is a sequence of uniformly bounded measurable functions and satisfies $\lim\limits_{n\to\infty} f_n(x) = f(x)$, then we have

$$\lim \int_S f_n(x)\ dm(x) = \int_S f(x)\ dm(x). \tag{5}$$

Moreover, if $f(x) \geqq 0$ for $x \in S$ and $\int_S f(x)\ dm(x) = 0$ then $f(x) = 0$ almost everywhere, namely

$$f \geqq 0, \quad \int_S f = 0 \Rightarrow m(\{x \mid f(x) \neq 0\}) = 0. \tag{6}$$

By the above preparation, we can prove the following

Theorem 3 *If the function F of Theorem 1 is uniquely determined by f, then f is measurable.*

Proof. Let $\varDelta$ be a partition of S where $S = X_1 + \cdots + X_n$. From the assumption, we have

$$\inf_{\varDelta} \sum \sup X_j^f\, m(X_j) = \sup_{\varDelta} \sum \inf X_j^f\, m(X_j)$$

which we shall denote by s.

For each positive integer n, there exist partitions of S such that

$$(7)\ \varDelta_n:\quad S = X_1^{(n)} + \cdots + X_{k_n}^{(n)}, \qquad s - \frac{1}{n} < \sum m_j^{(n)}\, m(X_j^{(n)}),$$

$$(8)\ \varDelta_n':\quad S = Y_1^{(n)} + \cdots + Y_{k_n'}^{(n)}, \qquad s + \frac{1}{n} > \sum M_j^{(n)}\, m(Y_j^{(n)}),$$

where

$$(9)\qquad m_j^{(n)} = \inf (X_j^{(n)})^f, \qquad M_j^{(n)} = \sup (Y_j^{(n)})^f.$$

We compose these partitions by putting $Z_{ij}^{(n)} = X_i^{(n)} \frown Y_j^{(n)}$ and find

$$\sum m_j^{(n)}\, m(X_j^{(n)}) \leqq \sum_{i,j} m_{ij}^{(n)}\, m(Z_{ij}^{(n)}),$$

$$\sum M_j^{(n)}\, m(Y_j^{(n)}) \geqq \sum_{i,j} M_{ij}^{(n)}\, m(Z_{ij}^{(n)}),$$

where $M_{ij}^{(n)}$ and $m_{ij}^{(n)}$ are defined similarly after (9).

Thus, we see that as the partitions of (7) and (8), we may choose the same one, so that $\varDelta_n'$ will be replaced by $\varDelta_n$. Hence

$$(10)\qquad s - \frac{1}{n} < \sum m_j^{(n)}\, m(X_j^{(n)}) \leqq \sum M_j^{(n)}\, m(X_j^{(n)}) < s + \frac{1}{n}.$$

Similarly, we may suppose that in the partitions $\varDelta_1, \varDelta_2, \cdots$, each preceding one is the refinement of the previous one.

Defining simple functions $\bar{f}_n(x)$ and $\underline{f}_n(x)$ by

$$\bar{f}_n(x) \equiv M_j^{(n)} \quad \text{and} \quad \underline{f}_n(x) \equiv m_j^{(n)} \quad \text{for} \quad x \in X_j^{(n)} \quad (j = 1, \cdots, k),$$

we observe

$$\bar{f}_1(x) \geqq \bar{f}_2(x) \geqq \cdots, \qquad \underline{f}_1(x) \leqq \underline{f}_2(x) \leqq \cdots.$$

Writing

$$\lim_{n\to\infty} \bar{f}_n(x) = \bar{f}(x) \quad \text{and} \quad \lim_{n\to\infty} \underline{f}_n(x) = \underline{f}(x),$$

we have, by

$$M_j^{(n)} \geqq f(x) \geqq m_j^{(n)} \quad \text{for} \quad x \in X_j^{(n)},$$

$$(11)\qquad \bar{f}(x) \geqq f(x) \geqq \underline{f}(x).$$

In view of (10), we find

$$\lim \int_S \bar{f}_n(x)\, dm(x) = \lim \int_S \underline{f}_n(x)\, dm(x) = s$$

from which also

$$\lim \int_S (\bar{f}_n - \underline{f}_n)\ dm(x) = 0\,.$$

But, as $\bar{f}_n - \underline{f}_n \to \bar{f} - \underline{f}$, we must have, by the uniform boundedness of $\{\bar{f}_n - \underline{f}_n\}$ and (7)

$$\int_S (\bar{f}(x) - \underline{f}(x))\ dm(x) = 0\,,$$

whence, in view of (11) and (6), we have at last

$$\bar{f}(x) = \underline{f}(x)$$

almost everywhere on S, which completes the proof by (11).

Though we have discussed so far under the assumption that f be bounded and the total measure of S be finite, these restrictions may, of course, be removed in a well known way, so that we shall not go further.

As our references, one or two of the following books may be sufficient:

S. Saks: Theory of the Integrals (1937)
H. Halmos: Measure Theory (1950)
J. von Neumann: Functional Operators Vol 1. (1950)