Note on the Arc Sine Law in the Theory of Probability[1]

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The asymptotic distribution of the number of positive sums of independent random variables has been first established by P. Lévy in the binomial case of the component random variables. Erdös and Kac have generalized Lévy's result in a series of recent papers, introducing a unified method for proving limit theorems related to sums of independent random variables. In this note we shall be concerned with the same problem for unequal components and, for simplicity, we shall deal with rather special random variables which depend on symmetric laws and obey a central limit theorem with special norming constants. In the case of equal components, Kunisawa has proved the arc sine law in the most general case by showing that the "invariance principle" of Erdös and Kac holds for any sums of random variables obeying the central limit theorem. The proof of our theorem also goes along the same line.

Let $X_1, X_2, \cdots$ be independent random variables which depend on symmetric distribution laws $F_\nu(x)$, $\nu=1, 2, \cdots$ and obey the central limit theorem:

$$P\left\{\frac{S_n}{\sqrt{\log n}} \leqq x\right\} \to (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-u^2/2}\,du\,, \qquad n \to \infty\,,$$

where $S_n=\sum_{\nu=1}^{n} X_\nu$. If we denote by N_n the number of positive partial sums S_ν, $\nu=1, 2, \cdots, n$, we have the

Theorem. *If we let* $n \to \infty$

$$P\left\{\frac{1}{\log n}\sum_{\nu=1}^{n}\left(\frac{N_\nu}{\nu}\right)\frac{1}{\nu} \leqq x\right\} \to \frac{2}{\pi}\text{ arc sin }\sqrt{x}\,, \quad 0 \leqq x \leqq 1\,,$$

and

$$P\left\{\frac{N_n}{n} \leqq x\right\} \to \sigma(x)\,,$$

where $\sigma(x)$ *is a distribution function satisfying* $\sigma(+0)-\sigma(-0)=\sigma(1+0)-\sigma(1-0)=1/2$.

Proof. Let us put

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$$\psi(x)=\begin{cases}1 & \text{if } x>0,\\ 0 & \text{if } x\leqq 0,\end{cases}$$

$$n_i=[n^{i/k}], \qquad i=1,2,\cdots,k,$$

then we can write $N_n=\sum_{\nu=1}^{n}\psi(S_\nu)$ and, in our case, the invariance principle consists in proving that

$$(1) \qquad D_n=\frac{1}{\log n}\sum_{i=1}^{k}\sum_{r=n_{i-1}+1}^{n_i}\big(\psi(S_{n_i})-\psi(S_r)\big)\frac{1}{r}$$

is small in probability, whenever k is sufficiently large and $n\to\infty$. Indeed, if we put

$$y_n(t)=\frac{S[n^t]}{\sqrt{\log n}}, \qquad S_n=\sum_{\nu=1}^{n}X_\nu,$$

$y_n(t)$, $0\leqq t\leqq 1$, converges, in probability law, to the Wiener process. Now

$$(2) \qquad E(|D_n|)\leqq\frac{1}{\log n}\sum_{i=1}^{k}\sum_{r=n_{i-1}+1}^{n_i}E\big(|\psi(S_{n_i})-(S_r)|\big)\frac{1}{r},$$

and remembering that the distribution of X_ν is symmetric

$$(3) \quad E\big(|\psi(S_{n_i})-\psi(S_r)|\big)\leqq P\{|S_{n_i}-S_r|\geqq\varepsilon\sqrt{\log n_i}\}$$
$$+P\{|S_{n_i}|<\varepsilon\sqrt{\log n_i}\}.$$

Since the first term on the last expression does not exceed

$$2\sum_{\nu=r}^{n_i}\int_{-\infty}^{\infty}\frac{x^2}{\varepsilon^2\log n_i+x^2}\,dF_\nu(x),$$

we obtain

$$\frac{1}{\log n}\sum_{r=n_{i-1}+1}^{n_i}P\{|S_{n_i}-S_r|\geqq\varepsilon\sqrt{\log n_i}\}\frac{1}{r}$$
$$\leqq 2\frac{1}{\log n}\sum_{\nu=n_{i-1}+1}^{n_i}\log\frac{\nu}{n_{i-1}}\int_{-\infty}^{\infty}\frac{x^2}{\varepsilon^2\log n_i+x^2}\,dF_\nu(x)$$
$$\leqq\frac{2}{k}\sum_{\nu=n_{i-1}+1}^{n_i}\int_{-\infty}^{\infty}\frac{x^2}{\varepsilon^2\log n_i+x^2}\,dF_\nu(x)\to 2\frac{1}{k}\sum_{i=1}^{k}\frac{1}{\varepsilon^2 i}$$
$$\leqq 2\frac{1+\log k}{k\varepsilon^2}, \qquad \text{as } n\to\infty.$$

Hence, on applying the central limit theorem on the second term of (3), we get

$$\varlimsup_{n\to\infty}E(|D_n|)\leqq 2\frac{1+\log k}{k\varepsilon^2}+(2\pi)^{-1/2}\int_{-\varepsilon}^{\varepsilon}e^{-u^2/2}\,du.$$

Then by the reasoning due to Erdös and Kac and partial summation, we get finally

$$P\left\{\frac{1}{\log n}\sum_{r=1}^{n}\psi(S_r)\frac{1}{r}\leqq x\right\}$$

$$=P\left\{\frac{1}{\log n}\sum_{\nu=1}^{n}\left(\frac{N_\nu}{\nu}\right)\frac{1}{\nu}\leqq x\right\}\to\frac{2}{\pi}\arcsin\sqrt{x},\quad 0\leqq x\leqq 1.$$

To prove the second half of the theorem let us write

$$\frac{N}{n}-\psi(S_n)=\frac{1}{n}\left[\sum_{\nu=1}^{[n^{1-\eta}]}+\sum_{\nu=[n^{1-\eta}]+1}^{n}\right]\left(\psi(S_\nu)-\psi(S_n)\right),$$

where $\eta>0$ is a sufficiently small positive number. Then

$$\left|\frac{1}{n}\sum_{\nu=1}^{[n^{1-\eta}]}\left(\psi(S_\nu)-\psi(S_n)\right)\right|\leqq n^{-\eta},$$

and

$$\frac{1}{n}\sum_{[n^{1-\eta}]+1}^{n}E\,|\,\psi(S_\nu)-\psi(S_n)\,|$$

$$\leqq 2\sum_{[n^{1-\eta}]}^{n}\int_{-\infty}^{\infty}\frac{x^2}{\varepsilon^2\log n+x^2}\,dF_\nu(x)+P\{\,|\,S_n\,|<\varepsilon\sqrt{\log n}\,\}$$

$$\to\frac{\eta}{\varepsilon^2}+(2\pi)^{-1/2}\int_{-\varepsilon}^{\varepsilon}e^{-u^2/2}\,du,\quad \text{as } n\to\infty.$$

Since the last expression can be made as small as we please by a suitable choice of ε and η, $E\left(\left|\frac{N_n}{n}-\psi(S_n)\right|\right)\to 0$, as $n\to\infty$. In other words

$$\lim_{n\to\infty}P\left\{\frac{N_n}{n}\leqq x\right\}=\lim_{n\to\infty}P\left\{\psi\left(\frac{S_n}{\sqrt{\log n}}\right)\leqq x\right\}=\sigma(x),$$

where $\sigma(x)$ is the distribution function stated in the theorem.

Literature

1) Erdös, P. and M. Kac. On the number of positive sums of independent random variables. Bull. Amer. Math. Soc. 53 (1947)

2) Kunisawa, K. On an analytical method in the theory of independent random variables. Annals of the Institute of Stat. Math. (Tokyo) 1 (1949)