On a Relation between two Non-Euclidean Connexions of Einstein Spaces[1]

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In this note we shall show that there exists a close relation between the generalized Poincaré's and Klein's representations of Einstein spaces 1), 2)[2]. As these representations may be regarded to be non-Euclidean connexions of Einstein spaces which are special cases of those of arbitrary Riemann spaces 3), the same result holds good also for the latter.

1. Let E_n be an Einstein space with a positive definite metric tensor g_{ij} $(i, j, k, \cdots = 1, 2, \cdots, n)$ and R be its scalar curvature. We define a constant c by $c = -R/2n(n-1)$ as in previous paper 3) and assume that $c > 0$, hence $R < 0$. We construct the space C_n with normal conformal connexion from the given E_n 5), then the equations which define the connexion are given by

$$(1)\qquad \begin{cases} dA_0 = dx^i A_i , \\ dA_j = cg_{jk}\, dx^k A_0 + \{{}^{\,i}_{jk}\} dx^k A_i + g_{jk}\, dx^k A_\infty , \\ dA_\infty = c dx^i A_i , \end{cases}$$

with respect to the Veblen's repères corresponding to the E_n. The group of holonomy of C_n fixes a hypersphere A: $A_\infty - cA_0$, 1), 4). On the other hand we construct the space P_n with normal projective connexion from the E_n, then the equations which define the connexion are given by

$$(2)\qquad dR_0 = dx^i R_i , \qquad dR_j = 2cg_{jk}\, dx^k R_0 + \{{}^{\,i}_{jk}\} dx^k R_i$$

with respect to semi-natural repères corresponding to the E_n. We can easily see that the group of holonomy of P_n fixes a hyperquadric B defined by: $2cg_{ij} X^i X^j - (X^0)^2 = 0$, where $X^\lambda (\lambda = 0, 1, \cdots, n)$ are current coordinates of tangent spaces of P_n, 2).

2. We shall define a hypersphere $\bar{A}$ by $2c\bar{A} = A_\infty + cA_0$, then we get from $(1)_1$, $(1)_3$

$$(3)_1 \qquad d\bar{A} = dx^i A_i .$$

From $(1)_2$ it follows that

$$(3)_2 \qquad d\bar{A} = 2cg_{jk}\, dx^k \bar{A} + \{{}^{\,i}_{jk}\} dx^k A_i .$$

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2 The number in bracket denotes the paper in literature.

Our object is to give the reason why (3) is the same form as (2).

3. The n-dimensional tangent Möbius' space M_n which is attached to each point of C_n may be considered to be in $(n+1)$-dimensional projective space P^{n+1} in which the $(n+2)$-simplex of coordinates is $[A_0, A_i, A_\infty]$ which may be also regarded as repères of $(n+2)$-hyperspherical coordinates of M_n. In the following we shall denote A's by A^*'s when they are regarded as ones of P^{n+1}. Möbius' transformations in M_n is projective ones in P^{n+1} which fix a hyperquadric

$$Q_n: \ g_{ij} X^i X^j - 2X^0 X^\infty = 0,$$

where (X^0, X^i, X^∞) are current coordinates with respect to $[A_0^*, A_i^*, A_\infty^*]$. The coordinates of A_∞^* are $(0, 0, \cdots, 0, 1)$. By the stereographic projection from A_∞^* and the polarity with respect to Q_n, there arises a $(1-1)$-correspondence between hyperspheres in M_n and points in P^{n+1}. We shall denote this correspondence by $*$. The operation $*$ is linear and commutative with the differential operation d. The hypersphere A corresponds to a point A^* whose coordinates are $(-c, 0, 1)$.

The polar plane $\overline{P}^n$ of A^* with respect to Q_n is then given by $X^0 = cX^\infty$. Hence points A_i^* corresponding to hyperspheres A_i lie in $\overline{P}^n$. As the point $\overline{A}^*$ corresponding to $\overline{A}$ has coordinates $(c, 0, 1)$, it lies also in $\overline{P}^n$. The intersection of $\overline{P}^n$ and Q_n is expressible by

$$\overline{B}: \ g_{ij} X^i X^j - 2c(X^\infty)^2 = 0.$$

Let $\Phi = \phi^\alpha A_\alpha \,(\alpha = 0, 1, \cdots, n, \infty)$ be a hypersphere in M_n. If the point Φ^* is on $\overline{P}^n$, then

$$\Phi^* = (\phi^\alpha A_\alpha)^* = \phi^\alpha A_\alpha^* = 2c\phi^\infty \overline{A}^* + \phi^i A_i^*,$$

for $\phi^0 = c\phi^\infty$. Therefore, under the $*$ operation hyperspheres, whose components are $(\phi^0, \phi^i, \phi^\infty)$ with respect to $[A_0, A_i, A_\infty]$, correspond to points whose coordinates are $(2c\phi^\infty, \phi^i)$ with respect to $(\overline{A}^*, A_i^*)$ on $\overline{P}^n$. Now we denote coordinates of points on $\overline{P}^n$ with respect to $(\overline{A}^*, A_i^*)$ by (X^*, X^{*i}). Then from the equation of $\overline{B}$ as $\overline{B}$ is on $\overline{P}^n$, we get

$$2cg_{ij} X^{*i} X^{*j} - (X^*)^2 = 0,$$

which is the same form as the equation of B. $\overline{P}^n$ is a projective space, and Möbius' transformations which fix A induce projective ones in $\overline{P}^n$ which fix $\overline{B}$. Hence if we regard $\overline{P}^n$ as the tangent n-dimentional projective space and regard $\overline{A}^*$ as the point of contact of underlying space, then the connexion of C_n induces a projective connexion which is the same form as (2).

Literature

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