On the Theoretical Studies about the Vortex Motion of Perfect Fluid I[1,2]

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Introduction and Summary

The investigation about the theory of vortex motion[3] with *vorticity continuously distributed* in the perfect fluid, was mainly carried out about 90 years ago, by Clebsch[1)] and Helmholtz[2)] etc., and soon after Kelvin[3)] established his famous circulation theorem. So it is considered that in these periods of 19th century the theoretical study of the vortex motion reached the vertex of its studies. In the following periods it was even said that there were few contributions about theoretical studies of vortex motion except Bjerknes' circulation theorem[4)] of baroclinic fluid.

Besides the only useful method about analytical expression of vortex motion in incompressible fluid, is Stokes' stream function in the cases of plane motion and axial symmetric one. Even in these cases soluble ones are constrained to linear or some special types of partial differential equations. Many people have considered that the theory of vortex motion in perfect fluid reaches the limit of its analytical method.

Causes of the above-mentioned incorrect considerations may consist in the following circumstances.

(i) Vector potential method[4] introduced by Helmholtz has been prevailed in the theoretical view on account of his fame. But the application of this method to vortex motion is incorrect as shown below.

(ii) In the treatment about the vortex motion of perfect fluid by the method of his expression, Clebsch commited a gross mistake, as pointed out in **5** of this paper. This fact prevented the theoretical development of vortex motion by Clebsch's expression. It is one of the objects of this paper to rectify the above mistake.

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3 The property of vortex motion to be discussed in this paper is limited to the one with vorticity spread over a finite region. The fluid to be treated is perfect fluid, but may be either compressible or incompressible one.

4 Vector potential with scalar potential display its full feature in electrodynamics, but its rôles in hydrodynamics are few as discussed in this paper.

(iii) To solve generally the differential equations for vortex motion is very difficult, because they have non-linear form.

(iv) The interest of hydrodynamical researches has been directed recently to the turbulent problems on hydraurics and aerodynamics and the high speed problems on aerodynamics.

However in the domain of meteorology the effects of rotation of the earth to motion of the atmosphere are essential, by which effects its motion may be transformed necessarily into vortex motion. So to solve the fundamental problem of meteorological dynamics, in the first place, we must treat the problem how to solve the vortex motion of perfect fluid from the pure theoretical point of view, especially the problem of solution of non-linear type from the analytical point of view (cf. **7**). While the author consulted the classical literature about vortex motion such as Helmholtz and Clebsch, he has found the hope to solve and develop these problems by amending their errors in the use of Clebsch's expression and applying a method of contact transformation.

In this paper, differing from the usual treatments of vortex motion, by employing exclusively the method of Clebsch's expression for velocity we shall develop the theory of fluid motion with spread vorticity. In **Part I** its general theory is developed. In **1–3** a preliminary note for further discussions is delivered. In **4–5** the extension of Kelvin's circulation theorem and its relation with Bernoulli's equation are also discussed. By using Clebsch's in **6–7** the partial differential equation for stream function is transformed from the second order to the first order, but the general solutions of the latter will be discussed in the next paper. Considering our following discussions this Clebsch's expression seems to be the most powerful method to accomplish the theoretical study of vortex motion either in perfect fluid or in viscous fluid.[5]

PART I

General Theory of Vortex Motion in Perfect Fluid by Use of Clebsch's Expression

1 Equations of motion and continuity in perfect fluid

As a preliminary note to treat the vortex motion, we shall sum up the fundamental equations of perfect fluid here. In this report forces to affect the fluid motion are assumed simply to be forces with a potential $\Omega\ (=\Omega(x, y, z))$. Let us put on $\mathbf{q}\ (u, v, w)$ the velocity fields, p the pressure, and ρ the density, then equations of motion are as follows, in either Lagrangian form or Eulerian form,

[5] Discussions of vortex motion in viscous fluid and baroclinic fluid will be treated in later papers.

$$\frac{d\mathbf{q}}{dt} = -\frac{1}{\rho}\nabla p - \nabla\Omega\,. \tag{1.1}$$

Especially in Eulerian form the acceleration is written as

$$\frac{d\mathbf{q}}{dt} = \frac{\partial\mathbf{q}}{\partial t} + u\frac{\partial\mathbf{q}}{\partial x} + v\frac{\partial\mathbf{q}}{\partial y} + w\frac{\partial\mathbf{q}}{\partial z}\,.$$

Equation of continuity in Eulerian form is,

$$\frac{d\rho}{dt} + \rho\nabla\mathbf{q} = 0\,, \tag{1.2}$$

and in the special case of incompressible fluid is

$$\nabla\mathbf{q} = 0\,, \tag{1.3}$$

Equation of continuity in Lagrangian form is

$$\frac{\rho}{\rho_0}\frac{\partial(x,y,z)}{\partial(a,b,c)} = 1\,. \tag{1.4}$$

and in the special case of incompressible fluid is

$$\frac{\partial(x,y,z)}{\partial(a,b,c)} = \text{const}\,. \tag{1.5}$$

2 Clebsch's expression about velocity fields

(I) Comparison of different methods for the expression of velocity

To express the velocity of fluid motion in the form convenient to the theoretical discussion, many methods such as those of stream function and vector potential etc. are used as the expressions by the potential function type. With respect to our special adoptation of Clebsch's expression[6] of velocity among these methods we shall make clear its reason in this section by comparing some important properties of three representative methods classified as below with each other,

a) Method of Stokes' stream function,
b) Method by Clebsch's expression,
c) Method of Helmholtz's vector potential.

Usually the case a) is considered to relate to the case c). Though it is right for plane motion, according to our opinion expressed in the following discussions (cf. (ii), b)) it is also in close connection with the case b). So among relations of these three methods we shall discuss fully relations between the case b) and c).

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6 Descriptions of Clebsch's expression are found in the textbooks on hydrodynamics such as Basset,[5)] Lamb,[6)] Appell[7)] and Batemann,[8)] but in the most textbooks written in 20th century there are no descriptions.

(i) Conciseness for expressing the quantity related to the vortex

For this point Clebsch's method is superior.[7]

(ii) Applicability for obtaining special solutions

a) Many two dimensional solutions for vortex motion known till now, are derived by the method of stream function. These solutions are also derived by the method of Clebsch as shown in this paper.

b) Three dimensional solutions, for which vector potential method should exhibit its feature, are scarcely obtained by its method.

c) Clebsch's method displays its features for three dimensional solutions by the help of contact transformation. For two dimensional solutions there are wider domains than the method of stream function. So its applicability is far large.

(iii) General theory such as combined generalizations of Bernoulli's equation with Kelvin's theorem is developed easily in the Clebsch's case, but in vain in vector potential method. (cf. **4** and **5**)

(iv) Symmetry properties for the space rotation[8]

Clebsch's expression has some properties resembled spinor symmetry, which conserves property of vorticity for the inversion of space. Contrary to this fact vector potential method does not exhibit true symmetry for the inversion of space.

(II) Definition of Clebsch's expression

Properties of Clebsch's expression are not known for many hydrodynamists, so we shall now describe this expression following the old Basset's[5)] book, as its description is lacking in usual textbooks.

Velocity fields $\mathbf{q}\,(u, v, w)$ in the general motion of fluid are divided into two parts, one is well treated potential motion with scalar velocity potential $\varphi\,(x,y,z)$, and the other the remaining parts u', v', w', that is,

$$u=-\frac{\partial\varphi}{\partial x}+u',\quad v=-\frac{\partial\varphi}{\partial y}+v',\quad w=-\frac{\partial\varphi}{\partial z}+w'. \tag{2.1}$$

As this division of the velocity fields into two parts is not performed uniquely, we can impose a relation upon u', v', w', for instance,

$$u'\left(\frac{\partial w'}{\partial y}-\frac{\partial v'}{\partial z}\right)+v'\left(\frac{\partial u'}{\partial z}-\frac{\partial w'}{\partial x}\right)+w'\left(\frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y}\right)=0\,, \tag{2.2}$$

which is nothing but the condition that a total differential form $u'dx+v'dy+w'dz$ has a integrating factor. Then changing this total differential form into a form $\alpha d\beta$, where α and β are functions of x, y, z and t, i.e., $\alpha=\alpha\,(x,y,z,t)$, $\beta=\beta\,(x,y,z,t)$, we have,

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7 Compare the description of 3 with that of vector potential method in the usual textbooks as Lamb[6)] etc.

8 Its detail is discussed in Appendix.

$$\mathbf{q}\cdot d\boldsymbol{r}=udx+vdy+wdz=-d\varphi+\alpha d\beta\,, \tag{2.3}$$

which is the form discussed by Phaff. Then velocity fields are expressed as

$$\mathbf{q}=-\nabla\varphi+\alpha\nabla\beta\,,\qquad \left(\nabla=\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\,.$$

$$\text{or}\begin{cases} u=-\dfrac{\partial\varphi}{\partial x}+\alpha\dfrac{\partial\beta}{\partial x}\,,\\ v=-\dfrac{\partial\varphi}{\partial y}+\alpha\dfrac{\partial\beta}{\partial y}\,,\\ w=-\dfrac{\partial\varphi}{\partial z}+\alpha\dfrac{\partial\beta}{\partial z}\,,\end{cases} \tag{2.4}$$

Further, calculating the procession in a infinitesimal distance $d\boldsymbol{r}$, by using (2.4) we have again

$$\mathbf{q}\cdot d\boldsymbol{r}=-d\varphi+\alpha d\beta. \tag{2.5}$$

3 Further properties of Clebsch's expression

(I) Rôles in vortex motion

As there are simple and close relations between the Clebsch's expression α,β and physical quantities appearing in the vortex motion, we shall describe them in detail following Appell's book[7] for the further discussion of our theory in Eulerian case.

1) Vorticity $\mathfrak{w}\,(\xi,\eta,\zeta)$

From (2.4) we have as vorticity $\mathfrak{w}$,

$$\mathfrak{w}=\frac{1}{2}\nabla\times\mathbf{q}=\frac{1}{2}\nabla\alpha\times\nabla\beta\,,$$

$$\text{or}\begin{cases}\xi=\dfrac{1}{2}(\alpha_y\beta_z-\alpha_z\beta_y)\,,\\ \eta=\dfrac{1}{2}(\alpha_z\beta_x-\alpha_x\beta_z)\,,\\ \zeta=\dfrac{1}{2}(\alpha_x\beta_y-\alpha_y\beta_x)\,.\end{cases} \tag{3.1}$$

2) Direction of vorticity

Multiplying (3.1) into $\nabla\alpha$ or $\nabla\beta$

$$\mathfrak{w}\cdot\nabla\alpha=0 \qquad\text{and}\qquad \mathfrak{w}\cdot\nabla\beta=0\,,$$

$$\left(\xi\frac{\partial\alpha}{\partial x}+\eta\frac{\partial\alpha}{\partial y}+\zeta\frac{\partial\alpha}{\partial z}=0\right)\ \left(\xi\frac{\partial\beta}{\partial x}+\eta\frac{\partial\beta}{\partial y}+\zeta\frac{\partial\beta}{\partial z}=0\right). \tag{3.2}$$

So the vorticity is directed along the intersection of the two surfaces α=const. and β=const., whose surface normals are expressed respectively by the vectors $\nabla\alpha$ and $\nabla\beta$.

3) Properties of vortex surface

Any surface above mentioned is called vortex surface. Its meaning is as follows.

Taking a differential procession $d\Gamma$ as (2.5) from velocity fields $\boldsymbol{v}$, we have,

$$d\Gamma=\mathbf{q}\cdot d\boldsymbol{r}=-d\varphi+\alpha d\beta\,.$$

Along the surface α=const.,

$$d\Gamma=d(-\varphi+\alpha\beta)\,. \tag{3.3}$$

Then for the closed circuit on the surface α=const. total processsions are,

$$\oint d\Gamma=\oint d(-\varphi+\alpha\beta)=0\,. \tag{3.4}$$

So the surface α=const. is a vortex surface.

Next on the surface β=const. we have,

$$d\Gamma=-d\varphi\,.$$

For the closed circuit on the surface β=const.,

$$\oint d\Gamma=\oint d(-\varphi)=0\,. \tag{3.5}$$

So the surface β=const. is also a vortex surface.

(II) Geometrical consideration of Clebsch's expression and generalization of stream function by author's view

In order to understand correctly the geometrical meaning of Clebsch's expression and to make clear its relation with stream function, we shall examine the geometrical behaviour of Clebsch's expression α, β in the 4-dimensional space (x, y, z, t).

The trajectory of fluid particles is expressed as a intersection of the two surfaces,

$$\psi_1(x, y, z, t)=\text{const.} \quad \text{and} \quad \psi_2(x, y, z, t)=\text{const.}, \tag{3.6}$$

where t is considered to be a sort of parameter. These surfaces ψ_1 and ψ_2 are considered to be those of moving parameter groups in the case of the variation of constants, also. But these surfaces for the trajectory are not determined uniquely. From the next consideration these quantities ψ_1 and ψ_2 may be taken to be generalizations of the stream function.[9] From the above definition the spatial parts of direction cosine of the intersection are proportional to the velocities $\mathbf{q}\,(u, v, w)$ of fluid particles,

$$\mathbf{q}\,(u, v, w) = \text{const.} \times (\nabla\psi_1 \times \nabla\psi_2)\,. \tag{3.7}$$

Taking another parameter surface orthogonal to this intersection as

$$\beta\,(x, y, z, t) = \text{const.}, \tag{3.8}$$

the velocities $\mathbf{q}$ are proportional to $\nabla\beta$. Putting this constant of proportion as α, we have

$$\mathbf{q} = \alpha\nabla\beta = \nabla\psi_1 \times \nabla\psi_2\,, \tag{3.9}$$

where $\alpha\,(x, y, z, t) = \text{const.}$ is also a sort of parameter surface. If we take another surface $\gamma\,(x, y, z, t) = \text{const.}$, parameters ψ_1 and ψ_2 will be expressed as functions of α, β and γ as

$$\psi_1 = \psi_1\,(\alpha, \beta, \gamma)\,, \qquad \psi_2 = \psi_2\,(\alpha, \beta, \gamma)\,. \tag{3.10}$$

As the two surfaces (3.6) contain the trajectory commonly, differentiating them with t along this trajectory, we have,

$$\begin{aligned} \frac{d\psi_1}{dt} &= 0 = \frac{\partial\psi_1}{\partial\alpha}\frac{d\alpha}{dt} + \frac{\partial\psi_1}{\partial\beta}\frac{d\beta}{dt} + \frac{\partial\psi_1}{\partial\gamma}\frac{d\gamma}{dt}\,, \\ \frac{d\psi_2}{dt} &= 0 = \frac{\partial\psi_2}{\partial\alpha}\frac{d\alpha}{dt} + \frac{\partial\psi_2}{\partial\beta}\frac{d\beta}{dt} + \frac{\partial\psi_2}{\partial\gamma}\frac{d\gamma}{dt}\,. \end{aligned} \tag{3.11}$$

Taking a special type without losing generality, $\psi_1 = \gamma$, $\psi_2 = \alpha$, we have,

$$\begin{aligned} &\frac{\partial\psi_1}{\partial\alpha} = \frac{\partial\psi_1}{\partial\beta} = 0\,, \qquad \frac{\partial\psi_2}{\partial\beta} = \frac{\partial\psi_2}{\partial\gamma} = 0\,, \\ \text{then} \quad &\frac{d\alpha}{dt} = 0\,, \quad \frac{d\gamma}{dt} = 0\,, \quad \text{but} \quad \frac{d\beta}{dt} \neq 0\,. \end{aligned} \tag{3.12}$$

Then this form is fit for the expression of barotropic flow.

It is easily shown that the velocity expression of the form (3.9) satisfies the equation of continuity. For

$$\begin{aligned} \nabla\mathbf{q} &= \nabla\,(\nabla\psi_1 \times \nabla\psi_2) \\ &= \nabla\psi_2 \cdot (\nabla \times \nabla\psi_1) - \nabla\psi_1 \cdot (\nabla \times \nabla\psi_2) \\ &= 0\,. \qquad (\text{as } \nabla \times \nabla\psi_1 = 0 \text{ and } \nabla \times \nabla\psi_2 = 0) \end{aligned} \tag{3.13}$$

Special case I Plane motion

[9] This expression was also considered by Liabouchinsky,[11] but he did not develop it in the direction of this paper.

Taking this plane as xy-plane and putting $\psi_1=z$, $\psi_2=\psi(x,y)$ in (3.9), we have,

$$\begin{cases} u=\alpha \dfrac{\partial\beta}{\partial x} = -\dfrac{\partial\psi}{\partial y}, \\ v=\alpha \dfrac{\partial\beta}{\partial y} = \dfrac{\partial\psi}{\partial x}. \end{cases} \tag{3.14}$$

This is well known Stokes' expression of velocity by a stream function ψ. If we could choose arbitrarily $\psi(x,y)$ and β orthogonal to ψ, the velocities by the above formula (3.14) satisfy the equation of continuity.

Special case II Motion on the surface for rotation.

Taking x, ρ and φ as the co-ordinates of the surface of rotation, where are $\rho^2=y^2+z^2$ and $\varphi=\tan^{-1}\dfrac{y}{z}$ and putting $\psi_1=\varphi$, $\psi_2=\psi(x,\rho)$, we have for the velocity components u, v along x and ρ,

$$\begin{cases} u=\dfrac{\alpha}{\rho}\dfrac{\partial\beta}{\partial x} = -\dfrac{1}{\rho}\dfrac{\partial\psi}{\partial \rho}, \\ v=\dfrac{\alpha}{\rho}\dfrac{\partial\beta}{\partial y} = \dfrac{1}{\rho}\dfrac{\partial\psi}{\partial x}. \end{cases} \tag{3.15}$$

This case is also nothing but the one treated by Stokes' stream function.

4 Energy relations with pressure (Modification of Kelvin's circulation theorem)

Assuming Clebsch's expression for velocity fields $\mathbf{q}$, we have as an infinitesimal procession in a direction $\delta \boldsymbol{r}$ as (2.5),

$$\mathbf{q}\cdot\delta\boldsymbol{r}=-\delta\varphi+\alpha\delta\beta. \tag{4.1}$$

Differentiating it by t along the path refering to fluid particles we have

$$\frac{d\mathbf{q}}{dt}\,\delta\boldsymbol{r}+\mathbf{q}\delta\left(\frac{d\boldsymbol{r}}{dt}\right)=-\delta\left(\frac{d\varphi}{dt}\right)+\frac{d\alpha}{dt}\,\delta\beta+\alpha\delta\left(\frac{d\beta}{dt}\right). \tag{4.2}$$

On the other hand from equations of motion (1.1) for barotropic motion under the action of external force with a potential $\Omega(x,y,z)$, the acceleration $\dfrac{d\mathbf{q}}{dt}$ is replaced by

$$\frac{d\mathbf{q}}{dt}=-\nabla\Omega-\frac{1}{\rho}\nabla p. \tag{4.3}$$

Further using next relations, where $V=|\mathbf{q}|$,

$$\begin{aligned} &\mathbf{q}\delta\left(\frac{d\boldsymbol{r}}{dt}\right)=\mathbf{q}\cdot\delta\mathbf{q}=\delta\left(\frac{V^2}{2}\right), \\ &\alpha\delta\left(\frac{d\beta}{dt}\right)=\delta\left(\alpha\,\frac{d\beta}{dt}\right)-\frac{d\beta}{dt}\,\delta\alpha, \end{aligned} \tag{4.4}$$

and introducing (4.3) and (4.4) into (4.2), we have,

$$\delta\left(\Omega-\frac{V^2}{2}-\frac{d\varphi}{dt}+\alpha\frac{d\beta}{dt}\right)+\frac{1}{\rho}\delta p=\frac{d\beta}{dt}\delta\alpha-\frac{d\alpha}{dt}\delta\beta\,. \quad (4.5)$$

Specially when the fluid is barotropic $\rho=f(p)$, we have,

$$\delta\left(\int\frac{dp}{\rho}+\Omega-\frac{V^2}{2}-\frac{d\varphi}{dt}+\alpha\frac{d\beta}{dt}\right)=\frac{d\beta}{dt}\delta\alpha-\frac{d\alpha}{dt}\delta\beta\,. \quad (4.6)$$

These relations (4.5) and (4.6) correspond to the differential form of Bernoulli's equation as shown in the next special cases.

The derivation of this formula (4.5) or (4.6) is compared with that of Kelvin's circulation theorem. In his case the right-hand side and the last two terms of left-hand side of (4.6) are summed up as time rate of circulation. Then considering the above derivation, we know there is a close connection[10] between Bernoulli's formula and Kelvin's circulation theorem. This relation has been made clear for the first time by the above description and it resembles to the relation for Lagrangian and Hamiltonian functions of Dynamics (Bernoulli's to H, Kelvin's to L).

5 Special classification of flow pattern (Bernoulli's formula and its generalization)

With respect to the result above obtained (4.6), we shall examine some special cases of the flow pattern.

(I) Potential flow

In this case it is $\dfrac{d\alpha}{dt}=0$, $\dfrac{d\beta}{dt}=0$, and only φ is not zero, then (4.5) is

$$\delta\left(\Omega-\frac{V^2}{2}-\frac{d\varphi}{dt}\right)+\frac{\delta p}{\rho}=0\,. \quad (5.1)$$

As the first term in the left-hand side of (5.1) is an exact differential, the second term mu t be an exact differential. Then ρ must be a function of the pressure p, i. e., this case is barotropic one. Integrating (5.1), we obtain the following formula,

$$\int\frac{dp}{\rho}+\Omega-\frac{V^2}{2}-\frac{d\varphi}{dt}=\text{const}\,, \quad (5.2)$$

where the constant of right-hand side is independent of x, y, z and t. Considering the relation

10 Compare this relation with the note in the book of Prantl-Tietjens, Hydro- und Aeromechanik, Erster Bd. S. 178. Though they remarked the distinction between these expressions, they did not explain the above relation.

$$\frac{d\varphi}{dt}=\frac{\partial\varphi}{\partial t}+u\frac{\partial\varphi}{\partial x}+v\frac{\partial\varphi}{\partial y}+w\frac{\partial\varphi}{z\partial}=\frac{\partial\varphi}{\partial t}-V^2, \tag{5.2'}$$

we obtain

$$\int\frac{dp}{\rho}+\Omega+\frac{V^2}{2}-\frac{\partial\varphi}{\partial t}=\text{const}. \tag{5.3}$$

This formula is nothing but the Bernoulli's formula for barotropic fluid motion with velocity potential φ. It should be noticed that in (5.2) the sign of $V^2/2$ is negative, but in (5.3) this sign is positive. Then (5.3) is equivalent to the usual interpretation as the sum of kinetic energy and potential energy Ω etc.

(II) Barotropic vortex motion

As a second special case we shall consider the vortex motion under the assumption that the right-hand side of (4.5) is the exact differential form of a function $f(\alpha,\beta)$ of α and β. The right-hand side of (4.5) is expressed as

$$\frac{d\beta}{dt}\delta\alpha-\frac{d\alpha}{dt}\delta\beta=-\delta f(\alpha,\beta).^{11} \tag{5.4}$$

Then in the formula (4.5) the term $\delta p/\rho$, which is apparently not an exact differential, must be an exact differential, then $\rho=\rho(p)$. So this type of fluid flow must be also caused by barotropic motion. Integrating (4.5) by using (5.4) along fluid path, we have,

$$\int\frac{dp}{\rho(p)}+\Omega-\frac{V^2}{2}-\frac{d\varphi}{dt}+\alpha\frac{d\beta}{dt}+f(\alpha,\beta)=\text{const}. \tag{5.5}$$

The constant of the right-hand side is also independent of x,y,z and t. Considering the relation (5.2') we have finally the following result,

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11 (i) In his book Basset[5)] proved the results $\frac{d\alpha}{dt}=0$ and $\frac{d\beta}{dt}=0$ in the general barotropic flow following Clebsch's paper, but these results are considered to be wrong by comparing them with our descriptions in this section.

(ii) In his book Lamb[6)] proved the results $\frac{D\lambda}{Dt}=0$ and $\frac{D\mu}{Dt}=0$, (in his book $\alpha=\lambda$, $\beta=\mu$) in the general barotropic flow, and these results are considered to contradict the assumption of his next page, $\frac{D\lambda}{Dt}\neq 0$ etc.

(iii) It is conceived that the wrong results of Lamb and Basset would be the cause to interfere the study of the vortex motion in barotropic fluids.

(iv) Batemann[8)] discussed the case of vortex motion of inviscid fluid, by taking a Lagrangian function with a form similar to the above equation (5. 6). In his description in p. 164 his adopting case is the special one of our results by taking $f(\alpha,\beta)=0$, consequently $\frac{d\alpha}{dt}=0$ and $\frac{d\beta}{dt}=0$. This is the case Clebsch's proof was incorrect, so the very limited vortex motion is permitted. In p. 166 the special case of our result is treated by taking $f(\alpha,\beta)=s^2$ etc.

$$\int \frac{dp}{\rho(p)} + \Omega + \frac{V^2}{2} - \frac{\partial \varphi}{\partial t} + \alpha \frac{\partial \beta}{\partial t} + f(\alpha, \beta) = \text{const}. \tag{5.6}$$

This formula is regarded as a generalization of Bernoulli's formula in the case of barotropic vortex motion.

(III) General flow

In the general case that the right-hand side of (4.5) is not an exact differential, the left-hand side of (4.5) is not also an exact differential. Then the density is not a unique function of pressure, which is the case of baroclinic fluid. These flows are realized by taking some boundary barriers prohibiting to flow cyclically. The pressure of some cyclic flow at the starting point and the end point with the same space point is not equal. This general case of baroclinic flow will be discussed in later papers.

6 Soluble cases of incompressible flow

We shall examine the cases, where vortex motions of incompressible flow are soluble analytically and compare their results with those in classical literature.

Conditions to be prescribed for the flow pattern of incompressible fluid are two. One is the continuity condition, which is expressed as follows by using Clebsch's expression

$$\nabla \mathbf{q} = \nabla (\alpha \nabla \beta) = 0, \tag{6.1}$$

which is satisfied automatically in the case that (3.9) holds.

Otherwise (6.1) may be satisfied by the choice of two functions $h_1(x, y, z, t)$ and $h_2(x, y, z, t)$, if

$$\beta = \tan^{-1} \frac{h_1}{h_2} \quad \text{and} \quad \alpha = h_1^2 + h_2^2, \tag{6.2}$$

where is

$$\nabla^2 h_1 = \lambda h_1 \quad \text{and} \quad \nabla^2 h_2 = \lambda h_2. \tag{6.3}$$

Then

$$\nabla (\alpha \nabla \beta) = \nabla (h_2 \nabla h_1 - h_1 \nabla h_2) = 0.$$

Next the other condition that the density ρ is a unique function of pressure, is by using the relation (5.4), as follows,

$$\frac{d\beta}{dt} \delta\alpha - \frac{d\alpha}{dt} \delta\beta = -\delta f(\alpha, \beta) \left(= -\frac{\partial f}{\partial \alpha} \delta\alpha - \frac{\partial f}{\partial \beta} \delta\beta \right). \tag{6.4}$$

To simplify the affairs without losing generality, we can assume that the surface $\alpha = \text{const.}$ contains the paths of fluid particles. Differentiating with t along these paths, we have,

$$\frac{d\alpha}{dt}=0, \qquad \therefore \frac{\partial f}{\partial \beta}=0, \tag{6.5}$$

then $f(\alpha, \beta)$ is only a function of α and

$$\frac{d\beta}{dt}=\frac{df(\alpha)}{d\alpha}. \tag{6.6}$$

Or analytically the condition (6.4) is expressed as

$$\begin{aligned} \frac{d\alpha}{dt} &= \frac{\partial \alpha}{\partial t}+\mathbf{q}\cdot\nabla\alpha=0, \\ \frac{d\beta}{dt} &= \frac{\partial \beta}{\partial t}+\mathbf{q}\cdot\nabla\beta=f'(\alpha). \end{aligned} \tag{6.7}$$

In the stationary case the relation $\frac{\partial \alpha}{\partial t}=0$ is further assumed, for as the flow pattern is not changed, the surface $\alpha=$const. is fixed in the space. But $\frac{d\beta}{dt}$ is not zero generally as the path of fluid particles go through the surface $\beta=$const., though $\frac{\partial \beta}{\partial t}$ can be zero. Then the conditions (6.7) are replaced by

$$\begin{aligned} &\mathbf{q}\cdot\nabla a=-\nabla\varphi\cdot\nabla\alpha+\alpha\nabla\beta\cdot\nabla\alpha=0, \\ &\frac{\partial \beta}{\partial t}-\nabla\varphi\cdot\nabla\beta+\alpha(\nabla\beta)^2=f'(\alpha). \end{aligned} \tag{6.8}$$

To satisfy the former condition of (6.8), it is sufficient that the surfaces $\beta=$const. and $\varphi=$const. are orthogonal to the flow surface $\alpha=$const.,

$$\nabla\beta\cdot\nabla\alpha=0 \quad \text{and} \quad \nabla\varphi\cdot\nabla\alpha=0.$$

Then the latter of (6.8) becomes

$$\frac{\partial \beta}{\partial t}+\alpha(\nabla\beta)^2=f'(\alpha), \tag{6.9}$$

if $\frac{\partial \beta}{\partial t}=$const., then

$$\alpha(\nabla\beta)^2=f'(\alpha)-c. \tag{6.10}$$

But it is not necessary to assume that the surfaces $\varphi=$const. and $\beta=$const. are orthogonal each other.

On the other hand from equation of continuity (6.1)

$$\nabla(\alpha\nabla\beta)=\nabla\alpha\cdot\nabla\beta+\alpha\nabla^2\beta=0,$$

then

$$\nabla^2\beta=0.$$

To obtain the analytical solutions of (6.10), it is necessary to assume further a relation between two vectors $\nabla\alpha$ and $\nabla\beta$. This process may be performed by adding the assumption to specify one of stream functions ψ_1 and ψ_2, some of whose examples are already shown in **3** (II). This assumption corresponds to specify the flow pattern.

In these cases the equation (6.10) is transformed into the form

$$F\left(\frac{\partial\alpha}{\partial x},\ \frac{\partial\alpha}{\partial y},\ \frac{\partial\alpha}{\partial z},\ \alpha\right)=0, \tag{6.11}$$

which is the partial differential equation of the first order, whose examples are shown in the next section by discussing the relation between ψ and α, β with stream function.

7 Relations between differential equation for the stream function and that for Clebsch's expression

In classical hydrodynamics the problem of vortex motion with the vorticity spread over finite regions has been treated generally by the method of stream function ψ. In this section as an alternative discussion of Clebsch's expression, we shall derive the differential equation for the α from the condition imposed for the pressure in a special flow pattern, utilizing the close relation between these methods of stream function and our's in **3**.

(I) Incompressible stationary flow (two dimensional case)

For this case it is shown that the partial differential equation of the second order about the stream function $\psi(x, y)$ is equal to the partial differential equation of the first order about $\alpha(x, y)$. Neglecting the potential flow the velocity fields (u, v) are expressed by both the stream function ψ and Clebsch's expression α and β as follows,

$$\begin{cases} u=-\dfrac{\partial\psi}{\partial y}=\alpha\dfrac{\partial\beta}{\partial x}, \\ v=\dfrac{\partial\psi}{\partial x}=\alpha\dfrac{\partial\beta}{\partial y}. \end{cases} \tag{7.1}$$

The condition that the pressure for the stream function of two dimensional flow is unique, i. e., the condition of integrability for the pressure, is as well known,[9] as follows

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=F(\psi). \tag{7.2}$$

By considering the discussion of **3** (II), we shall assume that the stream function is combined with the α by the relation $\psi=g(\alpha)$. Then (7.1) is

$$\begin{cases} \dfrac{\partial \psi}{\partial x} = g'(\alpha) \dfrac{\partial \alpha}{\partial x} = \alpha \dfrac{\partial \beta}{\partial y}, \\ \dfrac{\partial \psi}{\partial y} = g'(\alpha) \dfrac{\partial \alpha}{\partial y} = -\alpha \dfrac{\partial \beta}{\partial x}. \end{cases} \tag{7.3}$$

From (7.3) we obtain

$$\frac{\partial \psi}{\partial x}\frac{\partial \beta}{\partial x} + \frac{\partial \psi}{\partial y}\frac{\partial \beta}{\partial y} = 0, \qquad \left(\text{or } \frac{\partial \alpha}{\partial x}\frac{\partial \beta}{\partial x} + \frac{\partial \alpha}{\partial y}\frac{\partial \beta}{\partial y} = 0\right).$$

This relation expresses that surface ψ=const. (or α=const.) is orthogonal to the surface β=const. Replacing (7.3) into the left-hand side of (7.2),

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial \alpha}{\partial x}\frac{\partial \beta}{\partial y} - \frac{\partial \alpha}{\partial y}\frac{\partial \beta}{\partial x},$$

then using (7.3) we have,

$$= \frac{\alpha}{g'(\alpha)}\left\{\left(\frac{\partial \beta}{\partial x}\right)^2 + \left(\frac{\partial \beta}{\partial y}\right)^2\right\}. \tag{7.4}$$

As the relation

$$\left(\frac{\partial \beta}{\partial x}\right)^2 + \left(\frac{\partial \beta}{\partial y}\right)^2 = G(\alpha),$$

consists about α and β from (6.10), the left-hand side of (7.2) becomes

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{\alpha}{g'(\alpha)} G(\alpha) = F(\psi).$$

This equation is nothing but the formula (7.2). On the other hand using (7.3), the formula (7.4) becomes

$$\left(\frac{\partial \beta}{\partial x}\right)^2 + \left(\frac{\partial \beta}{\partial y}\right)^2 = \left(\frac{g'(\alpha)}{\alpha}\right)^2 \left\{\left(\frac{\partial \alpha}{\partial x}\right)^2 + \left(\frac{\partial \alpha}{\partial y}\right)^2\right\} = G(\alpha),$$

$$\therefore \left(\frac{\partial \alpha}{\partial x}\right)^2 + \left(\frac{\partial \alpha}{\partial y}\right)^2 = G_1(\alpha). \tag{7.5}$$

Thus in our case the partial differential equation of the second order (7.2) for the stream function ψ is replaced by the non-linear partial differential equation of the first order (7.5) for the Clebsch's α. Mathematically to seek the general solution of the latter is possible analytically, contrary to the former, which has solutions only for the special form of $F(\psi)$ and its general solutions are not obtained analytically.

(II) Incompressible stationary flow (3-dimensional case with axial symmetry)

In this case, including the typical Hill's spherical vortex, we shall take symmetry axis as x-axis, and let be $\rho^2=y^2+z^2$. The velocity fields (u, v) (components of velocity along x-axis and ρ-axis) are expressed as, using Stokes' stream function $\psi(x, \rho)$ and Clebsch's expression $\alpha(x, \rho)$ and $\beta(x, \rho)$

$$\begin{cases} u=-\dfrac{1}{\rho}\dfrac{\partial\psi}{\partial\rho}=\dfrac{\alpha}{\rho}\dfrac{\partial\beta}{\partial x}, \\ v=\dfrac{1}{\rho}\dfrac{\partial\psi}{\partial x}=\dfrac{\alpha}{\rho}\dfrac{\partial\beta}{\partial y}. \end{cases} \tag{7.6}$$

Then by the same reasons as the case (I), we have that the partial differential equation of second order for ψ

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}=F(\psi), \tag{7.7}$$ [10]

is equivalent to the partial differntial equation of the first order for α

$$\left(\frac{\partial\alpha}{\partial x}\right)^2+\left(\frac{\partial\alpha}{\partial\rho}\right)^2-\frac{\alpha}{\rho}\frac{\partial\alpha}{\partial\rho}=\rho^2G(\alpha) \tag{7.8}$$ [12]

where is $\psi=g(\alpha)$ and $G(\alpha)=\dfrac{\alpha}{g'}F\big(g(\alpha)\big)$.

Appendix

Symmetry property of velocity expression expressed by groups of space rotation

In this appendix we shall consider the symmetry property of the velocity expression of incompressible perfect fluid from the point of view of the theory of group for space rotation. According to the representation theory of space group, the co-ordinates x, y, z, the velocity fields u, v, w and the differential operators $\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}$ belong to ϑ_1-representation. Then the symmetry property of the physical quantities appearing in our paper is discussed as follows.

(a) Equation of continuity

The left-hand side of the equation of continuity in incompressible fluid

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{A.1}$$

belongs to the ϑ_0-representation in the three irreducible representations produced from the product of two ϑ_1-representation.

12 The solutions of (7.5) and (7.8) will be discussed fully in **Part II** following this paper.

$$\vartheta_1\times\vartheta_1=\vartheta_0+\vartheta_1^-+\vartheta_2.$$

The equation (A.1) expresses that the ϑ_0-repesentation in the product representation produced from the velocity and differential operators is zero.

(b) Irrotational motion

In the solution of irrotational motion

$$u=-\frac{\partial\varphi}{\partial x},\qquad v=-\frac{\partial\varphi}{\partial y},\qquad w=\frac{\partial\varphi}{\partial z},\tag{A.2}$$

the scalar potential φ is a ϑ_0-representation. The right-hand side of the above formula (A.2) is ϑ_1-representation and this expression coincides with the representation of velocity (A.2).

(c) Expression of velocity by vector potential

In the expression of velocity by vector potentials (F, G, H)

$$u=\frac{\partial H}{\partial y}-\frac{\partial G}{\partial z},\quad v=\frac{\partial F}{\partial z}-\frac{\partial H}{\partial x},\quad w=\frac{\partial G}{\partial x}-\frac{\partial F}{\partial y},\tag{A.3}$$

if we assume that F, G and H form a vector, i. e., ϑ_1-representation, the right-hand side of (A.3) is the ϑ_1^--representation in the product representation

$$\vartheta_1\times\vartheta_1=\vartheta_2+\vartheta_1^-+\vartheta_0.$$

But this is not equal to the velocity completely, for, as shown in the notation, the symmetry of these two expressions does not coincide for the inversion of space. For the investigation of rotational property for the vortex motion of fluid, the inversion property is important. However in the vector potential expression, in which F, G and H form a vector, the property of velocity expression by this method does not express the true symmetry property for the inversion of space. Alternatively, to make this property hold, we must take F, G and H as the ϑ_1^--representation, whose type is not used in the usual hydrodynamics. So in the form of its usual employment this potential does not fit itself to the expression of velocity in fluid motion.

(d) Clebsch's expression

In the Clebsch's expression the velocities u, v, w are expressed as

$$\begin{cases} u=\alpha\dfrac{\partial\beta}{\partial x}=\lambda\dfrac{\partial\mu}{\partial x}-\mu\dfrac{\partial\lambda}{\partial x} \\ v=\alpha\dfrac{\partial\beta}{\partial y}=\lambda\dfrac{\partial\mu}{\partial y}-\mu\dfrac{\partial\lambda}{\partial y} \\ w=\alpha\dfrac{\partial\beta}{\partial z}=\lambda\dfrac{\partial\mu}{\partial z}-\mu\dfrac{\partial\lambda}{\partial z} \end{cases}\tag{A.4}$$

where is $\alpha=\lambda^2+\mu^2$ and $\beta=\tan^{-1}\lambda/\mu$, (A.5)

and the condition of continuity is expressed as

$$\lambda \nabla^2 \mu-\mu \nabla^2 \lambda=0 . \tag{A.6}$$

If we assume that the two potentials λ and μ have the symmetry property of spinor, which is well known in quantum mechanics, that is, λ and μ have symmetry property of $\vartheta_{\frac{1}{2}}$-representation, the above equation may be considered as the ϑ_1-representation from the product $\vartheta_{\frac{1}{2}}$, $\vartheta_{\frac{1}{2}}$ and ϑ_1-representations

$$\vartheta_{\frac{1}{2}} \times \vartheta_1 \times \vartheta_{\frac{1}{2}}=\vartheta_{\frac{1}{2}} \times \vartheta_{\frac{3}{2}}+\vartheta_{\frac{1}{2}} \times \vartheta_{\bar{\frac{1}{2}}}=\vartheta_0+\vartheta_{\bar{1}}+\vartheta_1+\vartheta_2 .$$

If we assume that λ and μ compose a spinor, the above expression of velocity expresses true property for the inversion of space. As for the symmetry property of α and β we must seek it from the relation (A.5) and their symmetry property is not so simple as λ and μ.

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