On a Theorem of the Phragmén-Lindelöf Type[1]

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The following theorem is one of the Phragmén-Lindelöf type, which may find some future applications, perhaps in the analytic number-theory. The proof is modelled on that for Satz 405 of Landau's Vorleusungen über Zahlentheorie (Leipzig, 1927), vol. 2.

Theorem. *Let s be a complex variable with $\sigma = \Re s$ and $t = \Im s$; a, b, c, A and B real constants; $\alpha < \beta$, $t_0 > 3$. Let $f(s)$ be a function regular in the half-strip E defined by $\alpha \leqq \sigma \leqq \beta$ and $t \geqq t_0$, such that*

$$|f(\alpha+ti)| < Kt^a (\log t)^A \qquad (t \geqq t_0),$$
$$|f(\beta+ti)| < Kt^b (\log t)^B \qquad (t \geqq t_0),$$
$$|f(s)| < Kt^c \qquad (s \in E).$$

Then we have, for $s \in E$,

$$|f(s)| < Lt^{a(\beta-\sigma)/(\beta-\alpha)+b(\sigma-\alpha)/(\beta-\alpha)} (\log t)^{A(\beta-\sigma)/(\beta-\alpha)+B(\sigma-\alpha)/(\beta-\alpha)},$$

where L is a suitable positive number independent of s.

Proof. The function $\log s$ is regular in the s-plane cut along the non-positive axis ($\sigma \leqq 0$, $t = 0$), if we take the branch which is real for positive s. Since the unique solution of the equation

$$\log s - \frac{\pi}{2} i = 0$$

is $s = i$, the function $\log(\log s - \frac{\pi}{2}i)$ is regular in the half-plane $t > 2$, if we take the branch which is real for $s = it$, $t > 2$.

This being so, let us put

$$\Phi(s) = a\frac{\beta-s}{\beta-\alpha} + b\frac{s-\alpha}{\beta-\alpha}, \qquad \Psi(s) = A\frac{\beta-s}{\beta-\alpha} + B\frac{s-\alpha}{\beta-\alpha}.$$

Then the function

$$g(s) = \exp\{\Phi(s)(\log s - \frac{\pi}{2}i) + \Psi(s)\log(\log s - \frac{\pi}{2}i)\}$$

is regular and vanishes nowhere, in the half-plane $t > 2$, by what has been said above.

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Now, we have in the half-strip $|\sigma| \leqq |\alpha| + |\beta|$, $t > 2$,

$$|\log s - \log t - \frac{\pi}{2}i| = |\log s - \log(ti)|$$
$$= \left| \int_{ti}^{\sigma+ti} \frac{du}{u} \right| \leqq \frac{|\sigma|}{t} \leqq \frac{|\alpha| + |\beta|}{t},$$

the integral being taken along a horizontal path. Hence, in the same half-strip,

$$\log s - \frac{\pi}{2}i = \log t + \frac{\theta}{t} \qquad (|\theta| \leqq |\alpha| + |\beta|),$$
$$\log(\log s - \frac{\pi}{2}i) = \log\log t + \log\left(1 + \frac{\theta}{t\log t}\right)$$
$$= \log\log t + \frac{\xi}{t\log t} \qquad (|\xi| \leqq C_1),$$

where $\log\log t$ is real and C_1 denotes a positive number independent of s (similarly for C_2, C_3, ... in the sequel).

Hence, in the given half-strip E, we find

$$\Phi(s)(\log s - \frac{\pi}{2}i) + \Psi(s)\log(\log s - \frac{\pi}{2}i)$$
$$= \left(\Phi(\sigma) + \frac{b-a}{\beta-\alpha}ti\right)\left(\log t + \frac{\theta}{t}\right) + \left(\Psi(\sigma) + \frac{B-A}{\beta-\alpha}ti\right)\left(\log\log t + \frac{\xi}{t\log t}\right)$$
$$= \Phi(\sigma)\log t + \Psi(\sigma)\log\log t + i\frac{b-a}{\beta-\alpha}t\log t$$
$$+ i\frac{B-A}{\beta-\alpha}t\log\log t + \eta \quad (|\eta| \leqq C_2),$$

$$|g(s)| = |e^\eta| \exp\{\Phi(\sigma)\log t + \Psi(\sigma)\log\log t\}$$
$$\leqq C_3 t^{\Phi(\sigma)}(\log t)^{\Psi(\sigma)} \qquad (C_3 = e^{C_2}),$$
$$\left|\frac{1}{g(s)}\right| \leqq C_3 t^{-\Phi(\sigma)}(\log t)^{-\Psi(\sigma)}.$$

We now put $F(s) = \dfrac{f(s)}{g(s)}$. Then $F(s)$ is regular in E, and we have for $t \geqq t_0$, since $\Phi(\alpha) = a$ and $\Psi(\alpha) = A$,

$$|F(\alpha + ti)| \leqq Kt^a(\log t)^A C_3 t^{-\Phi(\alpha)}(\log t)^{-\Psi(\alpha)} = C_3K,$$

and similarly

$$|F(\beta + ti)| \leqq C_3K.$$

Also, in the set E,

$$|F(s)| \leqq Kt^c \cdot C_3 t^{-\Phi(\sigma)}(\log t)^{-\Psi(\sigma)} < C_4 t^{C_5}.$$

Hence we obtain for $s \in E$, by the Phragmén-Lindelöf theorem (see Satz 404 of Landau's book cited above),

$$|F(s)| < C_6,$$

$$|f(s)| = |F(s)| \cdot |g(s)| < C_6 C_3 t^{\Phi(\sigma)} (\log t)^{\Psi(\sigma)}$$
$$= L t^{a(\beta-\sigma)/(\beta-\alpha) + b(\sigma-\alpha)/(\beta-\alpha)} (\log t)^{A(\beta-\sigma)/(\beta-\alpha) + B(\sigma-\alpha)/(\beta-\alpha)} \quad (L = C_3 C_6);$$

which proves our theorem.
