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# The Lattice Approach to Five Dimensional Gauge Theories

Ochanomizu University
Advanced Sciences
Graduate School of Humanities and Sciences

YONEYAMA Kyoko

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Abstract

The aim of the particle physics is to reveal fundamental particles and their interactions. The Standard Model (SM) of particle physics explains the interactions between fundamental particles well and is consistent with experimental results so far. However, more fundamental theory is considered to exist because SM still has some problems. A variety of theories such as String theory, Super Symmetric theory, Extra-dimensional theory so on are studied as beyond the SM. In this thesis, I explain a study of 5-dimensional theory which is one of the Extra-dimensional theories. The goal of this study is to find out whether there is Spontaneous Symmetry Breaking (SSB) and dimensional reduction in non-perturbative region of 5-dimensional pure $SU(2)$ lattice gauge theory for orbifold. This study has done by Mean-Field expansion and Monte Calro simulation.

5-dimensional gauge theories are being studied well as extensions of SM. 5-dimensional theories here mean the theory of one time dimension and four spatial dimensions. We can only perceive one time dimension and three spatial dimensions and still we can consider one extra dimension existing in a way we cannot recognize. The motivations of considering 5-dimensional theory are that the quadratic divergence of Higgs mass which is one of the problem of SM can be avoided and that the origin of Higgs field is explained by identifying Higgs field with some of the 5th components of gauge field. This identification is called Gauge-Higgs Unification (GHU). Higgs field can cause SSB and particles obtain masses. Many perturbative studies of GHU model have been done. However the perturbative study can deal with only weak coupling region. Therefore, I have done the non-perturbative study by using lattice gauge theory in the case that the 5th dimension has orbifold boundary conditions. Mean-Field study indicates that SSB occurs with orbifold but not with torus boundary conditions. The

parameters of the model are the size of 5th-dimension $N$, the lattice coupling $\beta$ and anisotropy parameter $\gamma$. The parameter $\gamma$ shows the difference of the scale size (lattice spacing) between 5th dimension and other dimensions. When $\gamma > 1$, the scale along 5th dimension is larger than other dimensions.

The lattice gauge theory is the gauge theory defined on discretized space-time. The physical observables are obtained by taking continuum limit if it exist. Otherwise an effective theory for finite lattice spacing might exist. The advantage of the lattice gauge theory is that it can study large parameter region and can introduce gauge invariant cut-off.

From the Mean-Field study, I will show that the static potential along 4-dimensional hyperplane on the orbifold boundary is 4-dimensional Yukawa potential and gauge boson mass can be extracted from the potential. This means there is SSB and the result is different from the one of perturbative study in which fermions are needed for SSB. I also found that there is dimensional reduction to 4-dimensional gauge-scalar theory near the phase transition. Higgs boson mass which is consistent with the experimental result is easily obtained. This is also the difference with perturbative study where Higgs boson mass tends to be too small. Moreover, there is 2nd order phase transition lines for $\gamma < 0.6$ and one can take a continuum limit which does not depend on ultraviolet cut-off in this region. I show that taking the continuum limit around $\gamma = 0.5$. I can get the 1st excited Z boson mass around 1 TeV. Although the convergence of Mean-Field expansion has to be verified, the Monte Calro study also shows that there is SSB and confirms Mean-Field study.

The advantage of this model is that it has only three parameters and at least in the Mean-Field has the parameter region in which renormalizable continuum limit exists and one can have a physical Higgs boson mass. Also because the 1st excited Z boson mass is around 1 TeV, it is possible to be verified by experiments.

概要

素粒子物理学は物質の最も基本的な構成要素である素粒子が従う物理法則の探求を目的としている．現在までに提唱されている素粒子標準模型は粒子の相互作用を良く説明し，素粒子実験との矛盾もない．しかしながら，この素粒子標準模型はいくつかの問題を含んでおり，より根本的な素粒子理論が存在すると考えられている．より根本的な素粒子理論として，弦理論，超対称性理論，余剰次元理論など様々な理論が研究されているが，本研究では余剰次元理論である 5 次元ゲージ理論を扱った．本研究の目的はオービフォールド境界条件をもつ 5 次元純粋 $SU(2)$ 格子ゲージ理論の非摂動領域における自発的対称性の破れと次元低減の有無を平均場展開とモンテカルロシミュレーションを用いて調べることである．

5 次元ゲージ理論は素粒子標準模型の拡張として広く研究されている．ここでの 5 次元理論とは時間 1 次元，空間 4 次元からなる 5 次元理論である．我々は通常時間 1 次元，空間 3 次元を認識するが，もう一つの空間次元が通常認識できない形で存在していると考えることができる．5 次元理論を研究する動機としては主に，1) 標準模型にはヒッグスポテンシャルが 2 次発散してしまう問題があるが，5 次元理論ではこの 2 次発散を回避できることと，2) 標準模型ではヒッグスの起源についての説明がないが，5 次元理論ではゲージ場の第 5 次元成分をヒッグス場と見なすことができる (ゲージ・ヒッグス統一) ことが挙げられる．このヒッグス場によって自発的対称性の破れ (SSB) が起こるとゲージ場やフェルミオン (物質を構成する場) が質量を持つ．ゲージ・ヒッグス統一模型の摂動論的研究は数多く行われているが，摂動論的研究ではゲージ結合定数が非常に小さい場合，つまり相互作用が非常に小さい場合しか扱うことができない．そこで本研究では第 5 次元がオービフォールド境界条件をもつ場合について格子ゲージ理論を用いた非摂動論的研究を行った．平均場を用いた研究によって SSB がトーラス境界条件下では起こらず，オービフォールド条件下では起こりうることが示唆されている．このモデルのパラメータは第 5 次元の大きさ $N$，格子結合定数 $\beta$，非等方パラメータ $\gamma$ の 3 つである．$\gamma$ は第 5 次元とその他の縮尺の違いを表し，$\gamma > 1$ では第 5 次元がその他の次元より大きい場合を表す．

格子ゲージ理論とは格子状に離散化した時空で定義される理論であり，連続極限が存在する場合には，連続極限をとることで実際の連続空間における物理量などを求めることができる．また，連続極限が存在しない場合には有限格子間隔を持つ有効理論となることが期待される．格子ゲージ理論を用いることの利点としては，広いパラメータ領域を検証することが可能であることに加え，紫外カットオフをゲージ対称性を保った形で導入できることが挙げられる．

本研究では平均場を用いた研究により，オービフォールド境界上の 4 次元超平面に沿う静電ポテンシャルが 4 次元超平面上の静電ポテンシャルが 4 次元湯川型ポテンシャルであり，この静電ポテンシャルからゲージボソンの質量が導けることを示した．このことは SSB の存在を意味し，この結果はフェルミオンの存在無しに SSB が起こるという点で摂動論的研究結果と異なる．さらに，相転移付近でモデルが 4 次元ゲージスカラー理論に帰着する傾向があること，つまり 4 次元理論への次元低減を確認した．また，このモデルでは $\gamma < 1$ のパラメータ領域で実験結果に合うヒッグス質量を得ることができた．さらに，$\gamma < 0.6$ で二次相転移線の存在を確認し，$\gamma = 0.5$ 付近で連続極限をとると $Z$ ボソンの一次励起状態が約 1 TeV となることを示した．平均場の収束性については保証されていないため，モンテカルロシミュレーションを用いた計算で SSB の存在を確認することにより平均場近似による結果の妥当性を確かめた．

このモデルの優れた点としては，パラメータが 3 つと少ないこと，少なくとも平均場ではくり込み可能な連続極限が存在し，実験に合うようなヒッグス質量が得られることが挙げられる．また，一次励起状態の $Z$ ボソン質量が約 1TeV であることから実験による検証も期待できる．

# Contents

Chapter 1 Introduction 1

1.1 Standard Model . . . . . . . . . . . . . . . . . . . . . . . . . . 1

1.1.1 $SU(2) \times U(1)$ gauge symmetry . . . . . . . . . . . . . . . . 2

1.1.2 Higgs mechanism . . . . . . . . . . . . . . . . . . . . . . . 3

1.2 Hierarchy problem . . . . . . . . . . . . . . . . . . . . . . . . 5

1.3 Beyond the Standard Model . . . . . . . . . . . . . . . . . . . . 7

Chapter 2 Gauge-Higgs Unification model (Continuum) 9

2.1 Higgs field as extra dimensional gauge field . . . . . . . . . . . . 9

2.2 Orbifold Projection . . . . . . . . . . . . . . . . . . . . . . . . 9

2.3 Hosotani Mechanism . . . . . . . . . . . . . . . . . . . . . . . 11

2.4 Non-perturbative Gauge-Higgs Unification . . . . . . . . . . . . 13

Chapter 3 Lattice formulation of pure gauge theory 15

3.1 Continuum gauge theory . . . . . . . . . . . . . . . . . . . . . . 15

3.2 Lattice gauge theory . . . . . . . . . . . . . . . . . . . . . . . 16

3.3 Continuum limit . . . . . . . . . . . . . . . . . . . . . . . . . . 17

3.4 Lagrangian for orbifold . . . . . . . . . . . . . . . . . . . . . . 18

3.5 Observables for pure SU(2) lattice gauge theory on the orbifold 19

3.5.1 Higgs boson Operators . . . . . . . . . . . . . . . . . . . . . 19

3.5.2 Z boson Operators . . . . . . . . . . . . . . . . . . . . . . . 21

3.5.3 Static potential . . . . . . . . . . . . . . . . . . . . . . . . 23
3.6 Determination of energies . . . . . . . . . . . . . . . . . . . . . . . 23
3.6.1 Correlation function . . . . . . . . . . . . . . . . . . . . . . . 23
3.6.2 Generalized eigenvalue problem . . . . . . . . . . . . . . . . 23

Chapter 4 Mean-Field formulation 25
4.1 Mean-Field expansion in 1st order . . . . . . . . . . . . . . . . . . 28
4.2 Mean-Field expansion in 2nd order . . . . . . . . . . . . . . . . . 31
4.3 Observables . . . . . . . . . . . . . . . . . . . . . . . . . . . . 33
4.3.1 Higgs and Z boson mass . . . . . . . . . . . . . . . . . . . . 33
4.3.2 The static potential . . . . . . . . . . . . . . . . . . . . . . . 34

Chapter 5 Results from Mean-Field calculation 37
5.1 The phase diagram and phase transition . . . . . . . . . . . . . 37
5.2 The masses . . . . . . . . . . . . . . . . . . . . . . . . . . . . 38
5.2.1 Higgs boson mass . . . . . . . . . . . . . . . . . . . . . . . . 38
5.2.2 Direct Z boson mass . . . . . . . . . . . . . . . . . . . . . . 39
5.3 Spontaneous Symmetry Breaking . . . . . . . . . . . . . . . . . 39
5.3.1 Isotropic lattice . . . . . . . . . . . . . . . . . . . . . . . . . 40
5.3.2 Anisotropic lattice ($\gamma = 0.55$) . . . . . . . . . . . . . . . . . 42
5.4 Dimensional reduction . . . . . . . . . . . . . . . . . . . . . . . 44
5.5 Lines of Constant Physics and the $Z'$ . . . . . . . . . . . . . . . 45

Chapter 6 Results from Monte Carlo simulation 51
6.1 Hypercubic(HYP) smearing on the orbifold . . . . . . . . . . . . 51
6.2 Spectrum . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 53

Chapter 7 Conclusion 55

Bibliography 57

# Chapter 1

# Introduction

The aim of the particle physics is to reveal the fundamental particle of the matter and describe their interactions. Currently, the fundamental particles which construct matter are considered as quarks and leptons. It is known that there are four kind of forces between the particles which are gravity, weak force, strong force, and electro magnetic force. The Standard Model is the theory which describes three kinds of interactions without gravity by using gauge symmetry assuming the quarks and lepton as point particles. Most of the experimental results have been explained by the Standard Model. In this chapter, I explain the Standard Model shortly [1, 2, 3] and discuss some problems of the model.

## 1.1 Standard Model

6 quarks and 6 leptons are discovered up to now. Table 1.1 is the summary of the particles. The quarks and leptons have three generations and the indices $i$ represent the generations. The Higgs field is a scalar field and it gives masses to quarks and leptons. The standard model explains the interactions between these particles by gauge theory.

| | 1st generation | 2nd generation | 3rd generation |
|---|---|---|---|
| $u_i$ | up quark $u$ | charm quark $c$ | top quark $t$ |
| $d_i$ | down quark $d$ | strange quark $s$ | botom quark $b$ |
| $\nu_i$ | electron neutrino $\nu_e$ | muon neutrino $\nu_\mu$ | tauon neutrino $\nu_\tau$ |
| $e_i$ | electron $e$ | muon $\mu$ | tauon $\tau$ |

Table1.1 Generations of quarks and leptons

| | field | $SU(3)_c, SU(2)_L, U(1)_Y$ |
|---|---|---|
| quark | $Q_i = (u_{Li}, d_{Li})$ | $(3, 2, \frac{1}{6})$ |
| | $u_{Ri}$ | $(3, 1, \frac{2}{3})$ |
| | $d_{Ri}$ | $(3, 1, -\frac{1}{3})$ |
| lepton | $L_i = (\nu_{Li}, e_{Li})$ | $(1, 2, -\frac{1}{2})$ |
| | $e_{Ri}$ | $( 1, 1, -1)$ |
| Higgs | $H = (H^+, H^0)$ | $(1, 2, \frac{1}{2})$ |

Table1.2 Matter fields and Higgs. The electric charge is $Q_{\text{el}} = L^3 + Y$

### 1.1.1 $SU(2) \times U(1)$ gauge symmetry

Table 1.2 shows that only left handed quarks and leptons are $SU(2)$ doublet and right handed quarks and leptons do not have $SU(2)$ charge. The SU(2) charge is called isospin charge. The upper component of the fundamental representation has isospin 1/2 and the lower component has isospin $-1/2$. The U(1) charge is called hyper charge.

The gauge transformations are

$$L_i(x) \to L'_i(x) = \exp(-iL^a\theta^a - iY\theta)L_i(x) \tag{1.1}$$

$$e_{Ri}(x) \to e'_{Ri}(x) = \exp(-iY\theta)e_{Ri}(x), \tag{1.2}$$

where

$$L^a = \frac{1}{2}\sigma^a \qquad (a = 1, 2, 3), \tag{1.3}$$

$\sigma$ is Pauli matrix, $Y$ is hyper charge and $\theta^a (a = 1, 2, 3)$ and $\theta$ are the functions of $x$. $Q_i$ transform same as $L_i$, and $u_{Ri}$ and $d_{Ri}$ transform same as $e_{Ri}$. When we write gauge fields of $SU(2)$ and $U(1)$ as $W^a_\mu$ and $B_\mu$ respectively, the Lagrangian having $SU(2) \times U(1)$ symmetry is written as

$$\begin{aligned}
\mathcal{L} = &-\frac{1}{4}W^a_{\mu\nu}W^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \\
&+ \bar{L}_i i\gamma_\mu(\partial^\mu - igL^a W^a_\mu - ig'YB_\mu)L_i + \bar{Q}_i i\gamma_\mu(\partial^\mu - igL^a W_\mu - ig'YB_\mu)Q_i \\
&+ \bar{e}_{Ri} i\gamma_\mu(\partial^\mu - ig'YB^\mu)e_{Ri} + \bar{u}_{Ri} i\gamma_\mu(\partial^\mu - ig'YB^\mu)u_{Ri} \\
&+ \bar{d}_{Ri} i\gamma_\mu(\partial^\mu - ig'YB^\mu)d_{Ri}
\end{aligned} \tag{1.4}$$

where $W_{\mu\nu}$ and $B_{\mu\nu}$ are field strength which are written as

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon^{bca}W^b_\mu W^c_\nu \tag{1.5}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{1.6}$$

where $\epsilon^{abc}$ is the completely antisymmetric tensor.

### 1.1.2 Higgs mechanism

Here I explain Higgs in the Standard Model. The Lagrangian of $SU(2) \times U(1)$ symmetry with Higgs field is

$$\mathcal{L}_{higgs} = (D_\mu H)^\dagger (D^\mu H) - V(H^\dagger H) + \mathcal{L}_{yukawa} \tag{1.7}$$

$$D_\mu = \partial_\mu - igL^a W^a_\mu - ig'YB_\mu \tag{1.8}$$

$$\mathcal{L}_{yukawa} = -G_{ei}\bar{L}_i H e_{Ri} - G_{di}\bar{Q}_{Li} H d_{Ri} - G_{ui}\bar{Q}_{Li} H^\dagger u_{Ri} + h.c \tag{1.9}$$

where $G_e, G_d, G_u$ are free paremeters. $V(H^\dagger H)$ is the potential of Higgs scalar field, and we assume it as

$$V = \mu^2 H^\dagger H + \lambda(H^\dagger H)^2. \tag{1.10}$$

with $\mu^2 < 0,\ \lambda > 0$. Then the potential has minimum when

$$\sqrt{H^\dagger H} = \sqrt{\frac{-\mu^2}{\lambda}} \equiv \frac{v}{\sqrt{2}}. \tag{1.11}$$

This is the true vacuum. Now the Higgs field can be expanded around $v$ as,

$$H = \frac{1}{\sqrt{2}} \left( \begin{array}{c} (\xi_2 + i\xi_1)/2 \\ v + h - i\xi_3/2 \end{array} \right), \tag{1.12}$$

where $\xi_1, \xi_2, \xi_3, h$ are real fields. When we assume $\xi_1, \xi_2, \xi_3, h << v$ it is written as

$$H = \Big(1 + i\frac{\xi^k \tau^k}{2v}\Big) \left( \begin{array}{c} 0 \\ \frac{v+h}{\sqrt{2}} \end{array} \right) \simeq \exp\Big(i\frac{\xi^k \tau^k}{2v}\Big) \left( \begin{array}{c} 0 \\ \frac{v+h}{\sqrt{2}} \end{array} \right). \tag{1.13}$$

This is an $SU(2)$ gauge transformation. Therefore, we can write Higgs field as

$$H = \left( \begin{array}{c} 0 \\ \frac{v+h}{\sqrt{2}} \end{array} \right) \tag{1.14}$$

Now we replace $W^a_\mu, B_\mu$ with $W^+_\mu, W^-_\mu, Z_\mu, A_\mu$ as follows,

$$W^+_\mu = \frac{1}{\sqrt{2}}(W^1_\mu - iW^2_\mu) \tag{1.15}$$

$$W^-_\mu = \frac{1}{\sqrt{2}}(W^1_\mu + iW^2_\mu) \tag{1.16}$$

$$Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(-gW^3_\mu + g'B_\mu) \tag{1.17}$$

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(gW^3\mu + g'B_\mu) \tag{1.18}$$

Because $W^3_\mu$ and $B_\mu$ have same quantum numbers they can be mixed. The mixing is done so that $A_\mu$ represents the photon field. Inserting (1.14) - (1.18), to (1.7) we get

$$\begin{aligned} \mathcal{L}_{higgs} = & \frac{1}{2}\partial_\mu h \partial^\mu h - \mu^2 h^2 + \frac{v^2}{8}(g^2 + g'^2) Z_\mu Z^\mu + \frac{v^2 g^2}{4} W^+_\mu W^{-\mu} \\ & + (\text{higher order terms}) + \mathcal{L}_{yukawa}. \end{aligned} \tag{1.19}$$

We can see that $h, W^{\pm}, Z$ have masses.

$$m_h = \sqrt{-2\mu^2}, \quad m_{W^\pm} = \frac{1}{2}vg, \quad m_Z = \frac{1}{2}v\sqrt{g^2 + g'^2} \tag{1.20}$$

Next, let us see Yukawa term which is the interaction term between leptons, quarks and Higgs boson. Inserting (1.7) to the Lagrangian is

$$\mathcal{L}_{yukawa} = -\frac{G_e v}{\sqrt{2}}(\bar{e}_L e_R + \bar{e}_R e_L) - \frac{G_e v}{\sqrt{2}}(\bar{u}_L u_R + \bar{u}_R u_L) - \frac{G_e v}{\sqrt{2}}(\bar{d}_L d_R + \bar{d}_R d_L) + (\text{higher order terms}) \tag{1.21}$$

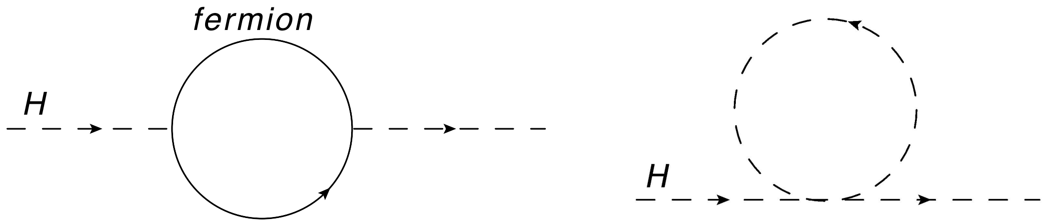
This is same for 2nd and 3rd generations. Therefore, the masses of electrons and quarks are

$$m_e = \frac{G_e v}{\sqrt{2}}, \quad m_u = \frac{G_u v}{\sqrt{2}}, \quad m_d = \frac{G_d v}{\sqrt{2}}. \tag{1.22}$$

In this way, leptons, quarks and gauge bosons $W^{\pm}_{\mu}$ and $Z_{\mu}$ obtain masses because Higgs field has vacuum expectation value. This is the mechanism of Spontaneous Symmetry Breaking (SSB).

## 1.2 Hierarchy problem

The Standard Model seems explaining the behavior of the particles well, however it contains some problems. In this section I explain the Hierarchy problem. The typical energy scale of the Standard Model is about 100GeV. The model explains the phenomena of the fundamental particles very well around this scale. The Standard Model has a limit for energy scale, and we need another theory for higher energy scale. One of the candidate of the higher energy theory is Grand Unification Theory (GUT). However, there is a problem when we assume the Standard Model is applicable up to the GUT scale ($10^{16}$GeV). This is the fine-tuning problem originating from the correction to Higgs boson mass term. The 1 loop correction to Higgs mass term contains the fermion loops and the scalar loop.




Considering the contribution of fermion and scalar loops, the correction to the mass is

$$\Delta m_h^2 = \frac{\mid \lambda_f \mid^2}{16\pi}\left[-2\Lambda^2 + 6m_f^2 \log\frac{\Lambda}{m_f}\right] + \frac{\lambda_h}{16\pi^2}\left[\Lambda^2 - 6m_h^2 \log\frac{\Lambda}{m_h}\right] + ..., \quad (1.23)$$

where $m_f$ is fermion masses and $\Lambda$ is cut-off scale.

Higgs mass is $m_h = 126.5\text{GeV}$ according to the results from experiments [4]. The correction becomes very big if we assume the cut-off scale is around $10^{16}$GeV which is the GUT scale. In order to have $m_h^2 \sim (100\text{GeV})^2$ we should obtain $(100\text{GeV})^2$ from sums and subtraction of the $(10^{16}\text{GeV})^2$ order terms. That is we need 28 order fine-tuning and it seems unnatural. It means that there are phenomena which can not be explained by the Standard Model in that scale.

## 1.3 Beyond the Standard Model

There are many attempts to solve the hierarchy problem introducing new theories such as extra-dimensional theories and supersymmetric theories. In this study I worked on Gauge-Higgs Unification model which is one of the extra dimensional theories. I explain Gauge Higgs Unification model in chapter 2. The Gauge-Higgs Unification model has been studied in perturbative region very well. In this time I focused on non-perturbative region applying Lattice gauge theory (chapter 3). First, I used mean-field expansion to calculate physical quantities (chapter 4 and 5) and also applied the Monte Carlo simulation (chapter 6).

# Chapter 2

# Gauge-Higgs Unification model (Continuum)

## 2.1 Higgs field as extra dimensional gauge field

In this section I explain the Gauge-Higgs Unification model which identifies the Higgs field as the extra dimensional Gauge field. In this case the Higgs boson mass is protected by 5-dimensional gauge symmetry. Thus it can be a solution of hierarchy problem explaining the origin of the Higgs field. When the extra dimension has torus boundary conditions ($S_1$) Higgs field is adjoint representation. To get fundamental Higgs field one can consider orbifold boundary conditions ($S_1/Z_2$).

## 2.2 Orbifold Projection

In this section I explain "orbifold projection" along 5th dimension [5, 6]. First we start with the torus boundary condition. The $SU(N)$ gauge field on torus requires two open charts. And different $SU(N)$ gauge fields ($A_M^{(-)}$ and $A_M^{(+)}$) are defined on each of these charts . And also a transition function $\mathcal{G} \in SU(N)$ is required on the overlaps of these charts.

$$A_M^{(-)} = \mathcal{G} A_M^{(+)} \mathcal{G}^{-1} + \mathcal{G} \partial_M \mathcal{G}^{-1} \tag{2.1}$$

Then we impose the orbifold projection

$$\mathcal{R}A_M^{(+)} = A_M^{(-)}. \tag{2.2}$$

Here reflection $\mathcal{R}$ is

$$\begin{aligned} z = (x_\mu, x_5) &\rightarrow \bar{z} = (x_\mu, -x_5) \\ A_M(z) &\rightarrow \alpha_M A_M(\bar{z}), \quad \alpha_\mu = 1, \ \alpha_5 = -1. \end{aligned} \tag{2.3}$$

On the overlaps of these charts, the orbifold projection is written as

$$\mathcal{R}A_M^{(+)} = \mathcal{G}A_M^{(+)}\mathcal{G}^{-1} + \mathcal{G}\partial_M\mathcal{G}^{-1} \tag{2.4}$$

because of the relation between $A_M^{(+)}$ and $A_M^{(-)}$ in the regions Eq. (2.1). I write $A_M^{(+)}$ as $A_M$ from now on. Gauge-covariance under gauge transformation $\Omega$ requires

$$\mathcal{G} \ \rightarrow \ (\mathcal{R}\Omega)\mathcal{G}\Omega^{-1}. \tag{2.5}$$

For $\epsilon \rightarrow 0$ at the boundary, we impose

$$\mathcal{G}|_{x_5=0,\pi R} = g \tag{2.6}$$

where $g$ is a constant. $A_M$ has Dirichlet boundary condition $\alpha_M A_M = gA_M g^{-1}$ and $\partial_5 A_M$ has Neumann boundary conditions $-\alpha_M\partial_5 A_M = g\partial_5 A_M g^{-1}$. $\mathcal{G} = g$ constant implies $[g, \Omega] = 0$ on the boundary for gauge transformations $\Omega$. $g$ should be inner automorphism which assigns parities to group generator $T^a$ which transform as

$$\begin{aligned} gT^a g^{-1} &= T^a \\ gT^{\hat{a}} g^{-1} &= -T^{\hat{a}}, \end{aligned}$$

where $T^a$ are unbroken generators and $T^{\hat{a}}$ are broken generators [7]. Then the gauge symmetry $G = SU(N)$ is broken on the boundary to its subgroup depending on $g$.

$$G = SU(p+q) \quad \rightarrow \quad H = SU(p) \times SU(q) \times U(1)$$

The boundary Higgs boson mass term is

$$m_H^2 \mathrm{tr}\Big\{[A_5, g][A_5, g]\Big\}\Bigg|_{x_5=0,\pi R} \equiv 0. \tag{2.7}$$

This Higgs boson mass term is zero because

$$(D_5\mathcal{G})(D_5\mathcal{G}) \equiv 0 \tag{2.8}$$

from (2.4) [5].

For the $SU(2)$ case the gauge symmetry can be broken to $U(1)$ on the boundary. If we choose $g = \mathrm{diag}(-i, i)$, the unbroken fields on the boundaries are $A_\mu^3$, $A_5^1$ and $A_5^2$. We can assume this $A_\mu^3$ as $U(1)$ vector boson and $A_5^{1,2}$ as complex Higgs.

## 2.3 Hosotani Mechanism

The Gauge-Higgs Unification model has been studied perturbatively [8]. The simplest case is 5-dimensional $SU(2)$ gauge theory with orbifolded extra dimension $S_1/Z_2$ [9].

5th dimension is small enough to be dimensional reduction and the cut-off of this theory is $1/R$, where $R$ is the radius of 5th dimension. The fields are expanded with Fourier expansion along 5th dimension because of $S_1$.

$$\phi(x_M) = \frac{1}{\sqrt{2\pi R}} \sum_n^\infty \phi^{(n)}(x_\mu) e^{i\frac{n}{R}x_5} \tag{2.9}$$

Then, with orbifold projection $\mathcal{R} : \phi(x_\mu, -x_5) = \mathcal{R}\phi(x_\mu, x_5)$, even and odd fields are written as

$$\begin{aligned}
&\mathcal{R} = +1 : \\
&\quad \phi_+(x_M) = \frac{1}{\sqrt{2\pi R}}\phi^{(0)}(x_\mu) + \frac{1}{\sqrt{\pi R}} \sum_{n=1}^\infty \phi^{(n)}(x_\mu)\cos(nx_5/R)\ , && (2.10) \\
&\mathcal{R} = -1 : \\
&\quad \phi_-(x_M) = \frac{1}{\sqrt{\pi R}} \sum_{n=1}^\infty \phi^{(n)}(x_\mu)\cos(nx_5/R). && (2.11)
\end{aligned}$$

This expansion is called Kaluza-Klein (KK) expansion. The 4 dimensional KK masses $m_n$ are

$$(m_n R)^2 = n^2. \tag{2.12}$$

Now we consider vacuum expectation value of Higgs field $< H >=< A_5 >$. If $\alpha$ is defined as

$$\alpha = g_5 < A_5^1 > R, \tag{2.13}$$

KK mass is shifted as

$$(m_n R)^2 = n^2,\ (n \pm \alpha)^2 \quad \text{for } n \neq 0. \tag{2.14}$$

The effective potential is written as [9, 10]

$$V(\alpha) = -\frac{3 \cdot 2 \cdot P}{64\pi^6 R^4} \sum_{m=1}^{\infty} \frac{\cos(2\pi m \alpha)}{m^5} \tag{2.15}$$

where $P = 3 - 4N_f$ and $N_f$ is the number of adjoint fermions. Higgs boson mass from the potential is

$$(m_H R)^2 = R g_4^2 \left. \frac{\mathrm{d}^2 V}{\mathrm{d}\alpha^2} \right|_{\alpha=\alpha_{\min}}, \quad g_4^2 = \frac{g_5^2}{2\pi R} \tag{2.16}$$

where $\alpha_{\min}$ is the $\alpha$ value which minimizes the effective potential. The dynamical gauge boson mass is

$$m_Z = \frac{\alpha_{\min}}{R} \tag{2.17}$$

When $N_f < 3/4$, there is no SSB ($\alpha_{\min} = 0$) and $m_f = m_Z = 0$. On the other hand, when $N_f > 3/4$, there is SSB. There is no SSB for pure gauge field and more than one fermion is needed for SSB.

In this perturbative study of GHU, the experimental value of $\rho = m_H/m_Z = 1.38$ is hard to get [11]. And because it is the 5-dimensional theory, it is non-renormalizable. Thus the theory is low energy effective theory.

## 2.4 Non-perturbative Gauge-Higgs Unification

In previous section we saw GHU theory in perturbative study. What happens for the GHU theory in non-perturbative region? We study GHU non-perturbatively by Lattice gauge theory. Lattice gauge theory is the calculation method of gauge theory by discretizing the space time on a Euclidean lattice. The advantage of using the lattice theory is that it is possible to introduce UV cut-off in gauge invariant form as well as it is possible to study non-perturbative region. We also apply Mean-Field expansion. Mean-Field expansion is expected to work well for higher dimension although it does not work well for 4-dimensions.

We study the structure of phase diagram and whether there can be SSB for pure gauge theory. We also study whether there is dimensional reduction or not and, if it is, what is the way of dimensional reduction. Is it compactification like perturbative region or localization? (cf. [12, 13, 14, 15])

# Chapter 3

# Lattice formulation of pure gauge theory

## 3.1 Continuum gauge theory

Lagrangian for continuum pure $SU(N)$ gauge theory is written as

$$L = \frac{1}{2g^2}\mathrm{tr}(F_{MN}F_{MN}) \tag{3.1}$$

where $F_{MN}$ is strength of the gauge fields $A_M = iA_M^a T^a \in su(N)$.

$$F_{MN} = \partial_M A_N - \partial_N A_M + [A_M,\ A_N] \tag{3.2}$$

$su(N)$ is Lie algebra of the group $SU(N)$ and $A_M^\dagger = -A_M$, $\mathrm{tr}(A_M) = 0$. Eq. (3.1) is invariant under the following gauge transformations

$$A'_N = \Omega(x)\partial_M\Omega(x)^\dagger + \Omega(x)A_N\Omega(x)^\dagger. \tag{3.3}$$

Where $\Omega(x) \in SU(N)$ is local gauge transformation. Under the gauge transformation the field strength $F_{MN}$ transforms covariantly as

$$F_{MN} = \Omega(x)F_{MN}\Omega(x)^\dagger \tag{3.4}$$

Then it is obvious that the Lagrangian Eq. (3.1) is invariant under the gauge transformation. [16, 17]

## 3.2 Lattice gauge theory

Now we consider the definition of the gauge field on lattice. Gauge field is defined on the links of the lattice because the gauge field is vector field. However, we cannot define $A_M$ directly on the links because $A_M$ is not covariant under the gauge transformation. So we consider covariant variable $U(x, M)$ which is defined as

$$U(x,M) \equiv \mathcal{P}\mathrm{exp}\{\int_0^1 ds A_M(x + a\hat{M}\cdot s)\} \tag{3.5}$$

$$= \mathcal{P}\mathrm{exp}\{\int_x^{x+a\hat{M}} dx A_M(x)\}. \tag{3.6}$$

Where $M$ is the direction of the gauge field, $n$ is the position of the lattice points and $U(x, M) \in SU(N)$. The variables for opposite direction is defined as $U(x + a\hat{M}, -M) \equiv U(x, M)^\dagger$. The link gauge variables transform as follows.

$$U(x,M)' \to \Omega(x)U(x,M)\Omega(x+aM)^\dagger$$

and the product of the link line transforms as

$$\begin{aligned} U_{\mathrm{line}} &= U(x,M_1)U(x+a\hat{M}_1,M_2)U(x+a\hat{M}_1+a\hat{M}_2,M_3)\cdots U(x_n,M_n) \\ &\to \Omega(x)U_{\mathrm{line}}\Omega(x_n+a\hat{M}_n)^\dagger. \end{aligned}$$

Then closed line transforms as

$$\begin{aligned} U_{\mathrm{loop}} &= U(x,M_1)U(x+a\hat{M}_1,M_2)U(x+a\hat{M}_1+a\hat{M}_2,M_3)\cdots U(x-a\hat{M}_n,M_n) \\ &\to \Omega(x)U_{\mathrm{loop}}\Omega(x)^\dagger. \end{aligned}$$

It means that tr $\{U_{\mathrm{loop}}\}$ is gauge invariant. The smallest closed loop is called plaquette. A plaquette is a product of four links and it is written as follows.

$$U_{M,N}(x) = U(x,M)U(x+a\hat{M},N)U^\dagger(x+a\hat{N},M)U^\dagger(x,N) \tag{3.7}$$

Using this plaquette, the Wilson plaquette action [16, 17] is defined as

$$S_W[U] = \frac{\beta}{2N}\sum_p \mathrm{Re}\ \mathrm{tr}\{1 - U(p)\}. \tag{3.8}$$

Where $\beta$ is a lattice coupling and $\sum_p$ means sum over all plaquettes:

$$\sum_p = \sum_x \sum_{M\neq N} \tag{3.9}$$

## 3.3 Continuum limit

Here I show that the Wilson plaquette action corresponds to continuum action for $a \to 0$.

$$\begin{aligned}
U_{MN}(x) = \mathcal{P}\exp\{&\int_x^{x+a\hat{M}} dx' A_M(x')\} \cdot \mathcal{P}\exp\{\int_{x+a\hat{M}}^{x+a\hat{M}+a\hat{N}} dx' A_N(x')\} \\
&\cdot\mathcal{P}\exp\{-\int_{x+a\hat{N}}^{x+a\hat{N}+a\hat{M}} dx' A_M(x')\} \cdot \mathcal{P}\exp\{-\int_x^{x+a\hat{N}} dx' A_N(x')\} \\
&= \exp\{a^2\big((\partial_M A_N(x) - \partial_N A_M(x)) + [A_M(x), A_N(x)] + a^3 X_3 + a^4 X_4 + \mathcal{O}(a^5)\big) \\
&= 1 + a^2 F_{MN} + a^3 X_3 + a^4 X_4 + a^4 F_{MN}^2 + \mathcal{O}(a^5).
\end{aligned} \tag{3.10}$$

where $X_3$ and $X_4$ are $a^3$ and $a_4$ term. Because $\mathrm{tr}\{T^a\} = 0$, $\mathrm{tr}\{F_{MN}\} = \mathrm{tr}\{X_3\} = \mathrm{tr}\{X_4\} = 0$. Then Wilson plaquette action is

$$\begin{aligned}
S_W[U] &= \frac{\beta}{2N}\sum_p \mathrm{Re}\ \mathrm{tr}\{1 - U(p)\} \\
&= \frac{\beta}{4N}\sum_{x,M\neq N} \mathrm{tr}\{1 - \frac{1}{2}(U_{MN}(x) + U_{MN}^{\dagger}(x))\} \\
&= \frac{\beta}{4N}\sum_{x,M\neq N} \mathrm{tr}\{a^4 g^2 F_{MN}(x)^2 + \mathcal{O}(a^5)\}.
\end{aligned} \tag{3.11}$$

It follows

$$\lim_{a\to 0} S_W[U] = \lim_{a\to 0}\frac{\beta g^2}{4N} a^4 \sum_x \sum_{M\neq N} \mathrm{tr}\{F_{MN}(x)^2\}. \tag{3.12}$$

On the other hand the continuum action is

$$\lim_{a\to 0} S_{YM}[U] = \frac{1}{2}\int dx^4 \sum_x \sum_{M\neq N} \mathrm{tr}\{F_{MN}(x)^2\}. \tag{3.13}$$

Thus, Wilson plaquette action is consistent with continuum action when $\beta = \frac{2N}{g^2}$. [16, 17]

## 3.4 Lagrangian for orbifold

Now we consider anisotropic 5-dimensional pure $SU(2)$ gauge theory where 5th dimension is orbifolded. The Wilson plaquette is

$$\begin{aligned} S_{\mathrm{W}} = &-\frac{\beta_4}{2}\sum_{n_M}\sum_{n_5=1}^{N_5-1}\left[\sum_{M<N}\mathrm{Re\ tr}\ U_{p\notin\mathrm{bound}}(n;M,N)\right] \\ &-\frac{\beta_5}{2}\sum_{n_M}\sum_{n_5=0}^{N_5-1}\left[\sum_{M}\mathrm{Re\ tr}\ U_{p\notin\mathrm{bound}}(n;M,5)\right] \\ &-\frac{\beta_4}{4}\sum_{n_M}\left[\sum_{M<N}\sum_{n_5=0,N_5}\mathrm{Re\ tr}\ U_{p\in\mathrm{bound}}(n;M,N)\right]. \end{aligned} \tag{3.14}$$

The lattice coupling is defined as

$$\beta_4 = \frac{2Na_5}{g_5^2}, \qquad \beta_5 = \frac{2Na_4^2}{g_5^2 a_5}\,. \tag{3.15}$$

In this study we parameterized the anisotropic lattice by $\beta$ and $\gamma$ where $\beta_4 = \beta/\gamma$ and $\beta_5 = \beta\gamma$. Then $\gamma = a_4/a_5$ at classical limit. The gauge transformation of the bulk links is

$$U(n,M) \longrightarrow \Omega^{(SU(2))}(n)U(n,M)\Omega^{(SU(2))\dagger}(n+\hat{M}), \tag{3.16}$$

the gauge transformation of the links on the boundaries is

$$U(n,M) \longrightarrow \Omega^{(U(1))}(n)U(n,M)\Omega^{(U(1))\dagger}(n+\hat{M}) \tag{3.17}$$

and the gauge transformation of the links which one end is in the bulk and the other touch the boundary is

$$U(n,M) \longrightarrow \Omega^{(U(1))}(n)U(n,M)\Omega^{(SU(2))\dagger}(n+\hat{M})\,. \tag{3.18}$$

In this set up, general links satisfy following orbifold projection condition

$$\Gamma\, U(n,M) = U(n,M), \qquad \Gamma = \mathcal{T}_g\mathcal{R} \tag{3.19}$$

where the reflection property about the origin of the fifth dimension is

$$\begin{aligned} \mathcal{R}\, U(n,M) &= U(\overline{n},M) \\ \mathcal{R}\, U(n,5) &= U^{\dagger}(\overline{n}-\hat{5},5) \end{aligned} \tag{3.20}$$

with

$$n = (n_M, n_5), \qquad \overline{n} = (n_M, -n_5). \tag{3.21}$$

The transformation under the group conjugation is

$$\mathcal{T}_g U(n,M) = g\, U(n,M)\, g^{-1} \tag{3.22}$$

where $g = -i\sigma^3$.

## 3.5 Observables for pure SU(2) lattice gauge theory on the orbifold

### 3.5.1 Higgs boson Operators

Polyakov loop along 5th dimension can be Higgs boson operator. In order to construct Higgs boson operator for orbifold, I start from the Polyakov loop for torus $P(n_\mu)^{(\mathrm{torus})}$which is parametrized by the coordinates $n_5 = 0, 1, \cdots, 2N_5-1$.

$$P(n_\mu)^{(\mathrm{torus})} = U((n_\mu,0),5)U((n_\mu,1),5)\cdots U((n_\mu,2N_5-1),5) \tag{3.23}$$

Then the Polyakov loop for orbifold $P(n_\mu)$ is obtained by applying orbifold projection

$$\text{Orbifold projection } : \; U(n,M) = \Gamma U(n,M), \tag{3.24}$$

where $\Gamma = \mathcal{R}\mathcal{T}_g$. The link $U((n_\mu, n_5), 5)$ transforms under $\Gamma$ as

$$\Gamma U((n_\mu, n_5), 5) = gU^\dagger((n_\mu, -n_5-1), 5)g^{-1} = gU^\dagger((n_\mu, 2N_5 - n_5 - 1), 5)g^{-1}.$$

Then, the Polyakov loop for torus is

$$\begin{aligned} P(n_\mu) &= U((n_\mu,0),5)U((n_\mu,1),5)\cdots U((n_\mu,N_5-1),5) \\ &\quad \cdot U((n_\mu,N_5),5)U((n_\mu,N_5+1),5)\cdots U((n_\mu,N_5+1),5) \\ &= U((n_\mu,0),5)U((n_\mu,1),5)\cdots U((n_\mu,N_5-1),5) \\ &\quad \cdot \Gamma U((n_\mu,N_5),5)\Gamma U((n_\mu,N_5+1),5)\cdots \Gamma U((n_\mu,N_5+1),5) \\ &= U((n_\mu,0),5)U((n_\mu,1),5)\cdots U((n_\mu,N_5-1),5) \\ &\quad \cdot gU((n_\mu,N_5-1),5)U((n_\mu,N_5-2),5)\cdots U((n_\mu,0),5)g^\dagger \\ &= l(n_\mu)gl^\dagger(n_\mu)g^\dagger, \end{aligned} \tag{3.25}$$

where $l(n_\mu)$ is the line $l(n_\mu) = U((n_\mu,0),5)U((n_\mu,1),5)\cdots U((n_\mu,N_5-1),5)$. Then we obtain two Higgs boson operators with the Polyakov loop

$$\mathcal{O}_H^1(t) = \frac{1}{L^3}\sum_{n_k} \mathrm{tr}(P(t,n_k)) \tag{3.26}$$

and

$$\mathcal{O}_H^2(t) = \frac{1}{L^3}\sum_{n_k} \mathrm{tr}(\Phi(n_\mu)\Phi^\dagger(n_\mu)), \tag{3.27}$$

where $\Phi(n_\mu) = \frac{1}{4N_5}[P(n_\mu) - P^\dagger(n_\mu), g]$. If we choose gauge transformation as

$$\begin{aligned} \Omega(n_\mu,0) &= V \\ \Omega(n_\mu,1) &= U((n_\mu,0),5) \\ \Omega(n_\mu,2) &= V^\dagger U((n_\mu,0),5)U((n_\mu,1),5) \\ \Omega(n_\mu,3) &= (V^\dagger)^2 U((n_\mu,0),5)U((n_\mu,1),5)U((n_\mu,2),5) \\ &\;\;. \\ &\;\;. \\ &\;\;. \\ \Omega(n_\mu,n_5) &= (V^\dagger)^{n_5-1}U((n_\mu,0),5)U((n_\mu,1),5)\cdots U((n_\mu,n_5-1),5) \end{aligned} \tag{3.28}$$

where $V = e^{aA_5^{\text{lat}}}$, the gauge links along 5th dimension transform as

$$
\begin{aligned}
U((n_\mu, 0), 5) &\rightarrow \Omega(n_\mu, 0)U((n_\mu, 0), 5)\Omega(n_\mu, 1)^\dagger = V \\
U((n_\mu, 1), 5) &\rightarrow \Omega(n_\mu, 1)U((n_\mu, 1), 5)\Omega(n_\mu, 2)^\dagger = V \\
&\cdot \\
&\cdot \\
&\cdot \\
U((n_\mu, 2N_5 - 1), 5) &\rightarrow \Omega(n_\mu, 2N_5 - 1)U((n_\mu, 2N_5 - 1), 5)\Omega(n_\mu, 0)^\dagger = V.
\end{aligned}
$$

Thus, Polyakov loop $P(n_\mu)$ can be written as $P(n_\mu) = V^{2N_5}$. Then we see that

$$
\begin{aligned}
\Phi(n_\mu) &= \frac{1}{4N_5}[P(n_\mu) - P^\dagger(n_\mu), g] \\
&= a[A_5^{\text{lat}}, g] + \mathcal{O}(a^3).
\end{aligned} \tag{3.29}
$$

$\Phi(n_\mu)$ has components only for broken generators $\sigma^1$ and $\sigma^2$. Because of the orbifold projection, gauge components of $A_5$ which commute with $g$ vanish. Here, $\mathcal{O}_H^1(t)$ and $\mathcal{O}_H^1(t)$ have spin $J = 0$, 3-dimensional parity $P = 0$ and charge conjugation $C = 1$. [18, 19]

### 3.5.2 Z boson Operators

First we consider 4-dimensional $SU(2)$ Higgs Model. We write the complex SU(2) Higgs doublet as

$$
\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}
$$

Then the gauge invariant gauge boson operators can be written as [20]

$$
W_k^B = -i \ \mathrm{tr}\{\sigma^B \varphi^\dagger(x + a\hat{k})U(x, \hat{k})\varphi(x)\},
$$

where

$$
\tilde{\Phi} = i\sigma_3\Phi, \qquad \varphi = \begin{pmatrix} \tilde{\Phi} & \Phi \end{pmatrix} = \begin{pmatrix} \tilde{\phi}_1 & \phi_1 \\ \tilde{\phi}_2 & \phi_2 \end{pmatrix} = \text{constant} \cdot SU(2) \text{ matrix},
$$

$k = 1, 2, 3$ is Lorentz index and $B = 1, 2, 3$ is adjoint gauge index. Under the isospin transformation $\Lambda \in SU(2)$ (global transformation) $\varphi$ and $U$ transform as follows:

$$\begin{aligned} \varphi &\to \Lambda\varphi\Lambda^{-1} \\ U &\to \Lambda U \Lambda^{-1}. \end{aligned}$$

The action is invariant under the transformation but $W_k^B$ transform as isospin triplet(adjoint representation of $SU(2)$). On the other hand, their gauge transformations $\Omega$ (local transformation) are as follows.

$$\begin{aligned} \varphi(x) &\to \Omega(x)\varphi(x) \\ U(x) &\to \Omega(x + a\hat{\mu})U(x, \mu)\Omega(x) \end{aligned}$$

Then, the action and $W_k^B$ are both invariant under the gauge transformation.

We can make following replacement for 5-dimensions.

| 4-dimensions | | 5-dimensions for orbifold on the boundary |
|---|---|---|
| $\varphi$ | $\to$ | $\frac{[P(x)-P^\dagger(x),g]}{\sqrt{\det([P(x)-P^\dagger(x),g])}} \equiv \alpha(x)$ |
| $\Omega(x)\varphi(x)$ | $\to$ | $\Omega(x)\alpha(x)\Omega^{-1}(x)$ |
| $-i\ \mathrm{tr}\{\sigma^B\varphi^\dagger(x + a\hat{k})U(x, k)\varphi(x)\}$ | $\to$ | $\mathrm{tr}\{gU(x, k)\alpha(x + a\hat{k})U^\dagger(x, k)\alpha(x)\}$ |

We have $Z$ operators for 5-dimensional $SU(2)$ orbifold.

$$\mathcal{O}_Z^1(t) = \frac{1}{L^3}\sum_{x_1,x_2,x_3}\Big\{\ \mathrm{tr}\{gU(x, k)\alpha(x + a\hat{k})U^\dagger(x, k)\alpha(x)\} - \mathrm{tr}\{k \to -k\}\Big\}/2 \tag{3.30}$$

$$\begin{aligned} \mathcal{O}_Z^2(t) &= \frac{1}{L^3}\sum_{x_1,x_2,x_3}\Big\{\ \mathrm{tr}\{U(x, k)|_{n_5=0}l(x + a\hat{k})U^\dagger(x, k)|_{n_5=N_5}gl^\dagger(x)\} - \mathrm{tr}\{k \to -k\}\Big\}/2 \\ &= \frac{1}{L^3}\sum_{x_1,x_2,x_3}\Big\{\ \mathrm{tr}\{gl^\dagger(x)U(x, k)|_{n_5=0}l(x + a\hat{k})U^\dagger(x, k)|_{n_5=N_5}\} - \mathrm{tr}\{k \to -k\}\Big\}/2 \end{aligned} \tag{3.31}$$

$\mathcal{O}^1_Z(t)$ and $\mathcal{O}^2_Z(t)$ are defined at $n_5 = 0$ and are gauge invariant because $[\Omega, g] = 0$ on the boundaries. $\mathcal{O}^{(1,2)}_Z(t)$ have $J = 1$, $P = -1$ and $C = -1$.[18, 19]

### 3.5.3 Static potential

Static potential is the energy of a pair of infinitely heavy quark and anti quark. It is extracted from Wilson loop

$$< W(r,t) >= \sum_{n=1}^{\infty} d_n \mathrm{e}^{-V_n(r)\cdot t} \tag{3.32}$$

where $V_1(r)$ is the ground state static potential and $V_n(r),\ n > 1$ are its excitations.

## 3.6 Determination of energies

### 3.6.1 Correlation function

We denote operators projected to $\overrightarrow{p} = 0$ by $\mathcal{O}(t)$. Then the connected time correlation function is written as

$$\begin{aligned} C(t) &= < \mathcal{O}(t)\mathcal{O}(0)^* > - < \mathcal{O}(t) >< \mathcal{O}(0)^* > \\ &= \sum_{n=1}^{\infty} c_n e^{-E_n\cdot t}, \end{aligned} \tag{3.33}$$

where $E_1, E_2, \cdots$ are energies of states created by the operator $\mathcal{O}$. Since $\overrightarrow{p} = 0$, energies are the masses $(E_1 = m_1, E_2 = m_2, \cdots)$.

### 3.6.2 Generalized eigenvalue problem

We construct basis of the operator. We can use more than one operators to calculate the masses. For example, we have Higgs boson operators $\mathcal{O}_1 = \mathrm{tr}(P)$ and $\mathcal{O}_2 = \mathrm{tr}(\Phi\Phi^\dagger)$ and we can have more operators by using fat links, See section 6.1. We require that these operators $\mathcal{O}_i,\ i = 1, 2, \cdots N$ have the same quantum numbers (Parity(P), Charge(C), Spin(J)). Then the matrix correlation function is constructed with these operators as

$$\begin{aligned} C_{ij}(t) &= < \mathcal{O}_i(t)\mathcal{O}_j(0)^* > - < \mathcal{O}_i(t) >< \mathcal{O}_j(0)^* > \\ &= \sum_{n=1}^{\infty} c_n^{(i,j)} e^{-E_n \cdot t} \end{aligned} \tag{3.34}$$

For a given time $t$, $C_{ij}(t)$ is a $N \times N$ matrix. The generalized eigenvalue problem is defined as

$$C(t)v = \lambda C(t_0)v. \tag{3.35}$$

$\lambda_n(t, t_0),\ n = 1, 2, \cdots, N$ are the generalized eigenvalues which are the eigenvalues of $C(t_0)^{1/2}C(t)C(t_0)^{1/2}$. They are related to the energies $E_n$ by [21]

$$\lambda_n(t, t_0) = e^{-E_n(t-t_0)}(1 + \text{corrections}) \tag{3.36}$$

Then, the effective masses $E_n^{\text{eff}}$ are

$$\begin{aligned} aE_n^{\text{eff}}(t, t_0) &= -\ln \frac{\lambda_n(t + a, t_0)}{\lambda_n(t, t_0)} \\ &\sim -\ln \frac{e^{-E_n(t+a-t_0)}}{e^{-E_n(t-t_0)}} \\ &\xrightarrow[t\ \text{large}]{} -\ln e^{-E_n \cdot a} = E_n \cdot a \end{aligned} \tag{3.37}$$

with correction $\sim e^{-\Delta \cdot t}$ where $\Delta = \min_{n \neq m} |E_n - E_m|$. When $2t_0 > t$ the correction is $\sim e^{-(E_{N+1}-E_n)\cdot t}$. [22]

# Chapter 4

# Mean-Field formulation

The partition function of the gauge theory on lattice is

$$Z = \int \mathrm{D}U \mathrm{e}^{-S_W[U]}\,, \tag{4.1}$$

where $S_W[U]$ is Wilson plaquette action.

Using the Fourier representation of delta function

$$\delta(f(x)) = \int_{-i\infty}^{i\infty} \frac{\mathrm{d}\alpha(x)}{2\pi i} \mathrm{e}^{-\alpha(x)f(x)}, \tag{4.2}$$

link variables $U$ are replaced by complex matrices $V$ and Lagrange multiplier $H$.

$$\begin{aligned} Z &= \int \mathrm{D}U \int \mathrm{D}V\ \delta(V-U)\mathrm{e}^{-S_W[U]} \\ &= \int \mathrm{D}U \int \mathrm{D}V \int \mathrm{D}H\ \mathrm{e}^{(1/2)\mathrm{Re\,tr}\{H(U-V)\}}\mathrm{e}^{-S_W[V]}. \end{aligned} \tag{4.3}$$

After integration of original links $U$, the partition function [23] is written as

$$Z = \int \mathrm{D}V \int \mathrm{D}H\, \mathrm{e}^{-S_{\mathrm{eff}}[V,H]}\,, \quad S_{\mathrm{eff}}[V,H] = S_W[V] + u(H) + (1/N)\mathrm{Re\,tr}\{HV\}\,, \tag{4.4}$$

where

$$\mathrm{e}^{-u(H)} \equiv \int \mathrm{D}U \mathrm{e}^{(1/2)\mathrm{Re\,tr}\{UH\}}. \tag{4.5}$$

Then, the action Eq. (3.11) is written as

$$\begin{aligned}
S_{\text{eff}} = &-\frac{\beta_4}{2}\sum_{n_\mu}\sum_{n_5=1}^{N_5-1}\left[\sum_{\mu<\nu}\text{Re tr } V_{p\notin\text{bound}}(n;\mu,\nu)\right] \\
&-\frac{\beta_5}{2}\sum_{n_\mu}\sum_{n_5=0}^{N_5-1}\left[\sum_{\mu}\text{Re tr } V_{p\notin\text{bound}}(n;\mu,5)\right] \\
&-\frac{\beta_4}{4}\sum_{n_\mu}\left[\sum_{\mu<\nu}\sum_{n_5=0,N_5}\text{Re tr } V_{p\in\text{bound}}(n;\mu,\nu)\right] \\
&+\sum_{n_\mu}\sum_{n_5=1}^{N_5-1}\sum_{\mu}\left[u_2(\rho(n,\mu)) + \sum_\alpha h_\alpha(n,\mu)v_\alpha(n,\mu)\right] \\
&+\sum_{n_\mu}\sum_{n_5=0}^{N_5-1}\left[u_2(\rho(n,5)) + \sum_\alpha h_\alpha(n,5)v_\alpha(n,5)\right] \\
&+\sum_{n_\mu}\sum_{\mu}\sum_{n_5=0,N_5}\left[u_1(\rho(n,mu)) + \sum_\alpha h_\alpha(n,\mu)v_\alpha(n,\mu)\right]\,. \qquad (4.6)
\end{aligned}$$

In this study the fluctuating fields in the bulk are parametrized as

$$\begin{aligned}
V(n,M) &= v_0(n,M) + i\sum_{A=1}^{3} v_A(n,M)\sigma^A\,, \\
H(n,M) &= h_0(n,M) - i\sum_{A=1}^{3} h_A(n,M)\sigma^A\,, \qquad (4.7)
\end{aligned}$$

and on the boundaries are parametrized as

$$\begin{aligned}
V(n,M) &= v_0(n,M) + iv_3(n,M)\sigma^3\,, \\
H(n,M) &= h_0(n,M) - ih_3(n,M)\sigma^3\,, \qquad (4.8)
\end{aligned}$$

where $\sigma^A$ is the Pauli matrices and $v_{0,A} \in \mathbb{C}$. The effective mean-field actions

$u_1$ and $u_2$ are defined as

$$\mathrm{e}^{-u_2(H)} = \int_{SU(2)} \mathrm{D}U \ \mathrm{e}^{\frac{1}{2}\mathrm{Re}\{\mathrm{tr}(UH)\}} = \frac{2}{\rho} I_1(\rho), \quad \rho = \sqrt{\sum_\mu (\mathrm{Re}\ h_\mu)^2}, \tag{4.9}$$

$$\mathrm{e}^{-u_1(H)} = \int_{U(1)\subset SU(2)} \mathrm{D}U \ \mathrm{e}^{\frac{1}{2}\mathrm{Re}\{\mathrm{tr}(UH)\}} = I_0(\rho), \quad \rho = \sqrt{(\mathrm{Re}\ h_0)^2 + (\mathrm{Re}\ h_3)^2}. \tag{4.10}$$

The mean-field is the field which makes the effective action minimal. We can choose the mean-field proportional to the identity. Considering translational invariance in direction $\mu = 0, 1, 2, 3$, we parameterize the mean-field as follows: for $n_5 = 0, 1, \ldots, N_5$ (4-dimensional links)

$$\overline{H}(n,\mu) = \overline{h}_0(n_5)\mathbf{1}\,, \qquad \overline{V}(n,\mu) = \overline{v}_0(n_5)\mathbf{1}\,, \qquad \forall n_\mu\,, \mu\,, \tag{4.11}$$

for $n_5 = 0, 1, \ldots, N_5 - 1$ (5th dimensional links)

$$\overline{H}(n,5) = \overline{h}_0(n_5 + 1/2)\mathbf{1}\,, \qquad \overline{V}(n,5) = \overline{v}_0(n_5 + 1/2)\mathbf{1}\,, \qquad \forall n_\mu\,. \tag{4.12}$$

Mean-field background can be obtained by taking derivatives of Eq. (4.6) with respect to $V$ and $H$ and requiring them to vanish

$$\left.\frac{\partial S_{eff}}{\partial H}\right|_{\overline{H},\overline{V}} = 0\,, \qquad \left.\frac{\partial S_{eff}}{\partial V}\right|_{\overline{H},\overline{V}} = 0\,. \tag{4.13}$$

$$\rightarrow \overline{V} = -\left.\frac{\partial u}{\partial H}\right|_{\overline{H}}, \qquad \overline{H} = -\left.\frac{\partial S_W[V]}{\partial V}\right|_{\overline{V}}. \tag{4.14}$$

From these minimization equations lead to the following relations [6]: for $n_5 = 0$

$$\overline{v}_0(0) = -u_1'(\overline{h}_0(0)) = \frac{I_1(\overline{h}_0(0))}{I_0(\overline{h}_0(0))}\,, \tag{4.15}$$

$$\overline{h}_0(0) = \beta_4 \left[(d-2)(\overline{v}_0(0))^3 + \gamma^2(\overline{v}_0(1/2))^2 \overline{v}_0(1)\right]\,. \tag{4.16}$$

A prime on $u_1$ or $u_2$ denotes differentiation with respect to its argument. Similarly, for $n_5 = N_5$ we have

$$\overline{v}_0(N_5) = -u_1'(\overline{h}_0(N_5)) = \frac{I_1(\overline{h}_0(N_5))}{I_0(\overline{h}_0(N_5))}\,, \tag{4.17}$$

$$\overline{h}_0(N_5) = \beta_4\left[(d-2)(\overline{v}_0(N_5))^3 + \gamma^2\overline{v}_0(N_5-1)(\overline{v}_0(N_5-1/2))^2\right]\,. \tag{4.18}$$

For $n_5 = 1, \ldots, N_5 - 1$ (four-dimensional links)

$$\overline{v}_0(n_5) = -u_2'(\overline{h}_0(n_5)) = \frac{I_2(\overline{h}_0(n_5))}{I_1(\overline{h}_0(n_5))}\,, \tag{4.19}$$

$$\begin{aligned}\overline{h}_0(n_5) = \beta_4\Big[&2(d-2)(\overline{v}_0(n_5))^3 + \gamma^2\Big((\overline{v}_0(n_5+1/2))^2\overline{v}_0(n_5+1)\\ &+\overline{v}_0(n_5-1)(\overline{v}_0(n_5-1/2))^2\Big)\Big]\,.\end{aligned} \tag{4.20}$$

For $n_5 = 0, \ldots, N_5 - 1$ (extra-dimensional links)

$$\overline{v}_0(n_5+1/2) = -u_2'(\overline{h}_0(n_5+1/2)) = \frac{I_2(\overline{h}_0(n_5+1/2))}{I_1(\overline{h}_0(n_5+1/2))}\,, \tag{4.21}$$

$$\overline{h}_0(n_5+1/2) = 2\beta_5(d-1)\overline{v}_0(n_5)\overline{v}_0(n_5+1/2)\overline{v}_0(n_5+1)\,. \tag{4.22}$$

The mean-field is obtained by solving these equations iteratively.

## 4.1 Mean-Field expansion in 1st order

Here, we introduce Gauss fluctuation around the mean-field

$$H = \bar{H} + h \quad\text{and}\quad V = \bar{V} + v\,. \tag{4.23}$$

Gauge fixing is necessary for computing fluctuations. It has been discussed in [24, 25, 26]. We write the second derivative of the effective action as follows.

$$\left.\frac{\delta^2 S_{\rm eff}}{\delta H^2}\right|_{\overline{V},\overline{H}} h^2 = h_i K^{(hh)}_{ij} h_j = h^T K^{(hh)} h$$
$$\left.\frac{\delta^2 S_{\rm eff}}{\delta V \delta H}\right|_{\overline{V},\overline{H}} vh = v_i K^{(vh)}_{ij} h_j = v^T K^{(vh)} h$$
$$\left.\frac{\delta^2 S_{\rm eff}}{\delta V^2}\right|_{\overline{V},\overline{H}} v^2 = v_i K^{(vv)}_{ij} v_j = v^T K^{(vv)} v \tag{4.24}$$

Then, mean-field expansion up to second derivative is

$$\begin{aligned} S_{eff} &= S_{eff}[\overline{V},\overline{H}] + \frac{1}{2}\Big( \left.\frac{\delta^2 S_{\rm eff}}{\delta H^2}\right|_{\overline{V},\overline{H}} h^2 + \left.\frac{\delta^2 S_{\rm eff}}{\delta V \delta H}\right|_{\overline{V},\overline{H}} vh + \left.\frac{\delta^2 S_{\rm eff}}{\delta V^2}\right|_{\overline{V},\overline{H}} v^2 \Big) \\ &= S_{eff}[\overline{V},\overline{H}] + \frac{1}{2}(h^T K^{(hh)} h + 2v^T K^{(vh)} h + v^T K^{(vv)} v) \\ &= S_{eff}[\overline{V},\overline{H}] + S^{(2)}[v,h], \end{aligned} \tag{4.25}$$

where $S^{(2)}[v,h] \equiv \frac{1}{2}(h^T K^{(hh)} h + 2v^T K^{(vh)} h + v^T K^{(vv)} v)$. The partition function is also expanded as

$$\begin{aligned} Z &= \int \mathrm{D}v \int \mathrm{D}h\, \mathrm{e}^{-(S_{eff}[\overline{V},\overline{H}] + S^{(2)}[v,h])} \\ &= Z[\overline{V},\overline{H}] \cdot z, \end{aligned} \tag{4.26}$$

where

$$z = \int \mathrm{D}v \int \mathrm{D}h\, \mathrm{e}^{-S^{(2)}[v,h]} \tag{4.27}$$
$$= \int \mathrm{D}v \int \mathrm{D}h\, \mathrm{e}^{-\frac{1}{2}h^T K^{(hh)} h - v^T K^{(vh)} h - \frac{1}{2} v^T K^{(vv)} v} \tag{4.28}$$
$$= \frac{(2\pi)^{|h|/2}}{\sqrt{\det[K^{(hh)}]}} \int \mathrm{D}v\, \mathrm{e}^{-\frac{1}{2}v^T(-K^{(vh)} K^{(hh)^{-1}} K^{(vh)} + K^{(vv)})v} \tag{4.29}$$
$$= \frac{(2\pi)^{|h|/2}(2\pi)^{|v|/2}}{\sqrt{\det[(-\mathbf{1} + K^{(hh)}(-K^{(vh)} K^{(hh)^{-1}} K^{(vh)} + K^{(vv)})]}}. \tag{4.30}$$

Using Eq. (4.25) and Eq. (4.26), the expectation value of an observable

$$\langle \mathcal{O} \rangle = \frac{1}{Z}\int \mathrm{D}U\, \mathcal{O}[U] \mathrm{e}^{-S_W[U]} \tag{4.31}$$
$$= \frac{1}{Z}\int \mathrm{D}V \mathrm{D}H\, \mathcal{O}[V] \mathrm{e}^{-S_{eff}[V,H]}. \tag{4.32}$$

is expanded as

$$\begin{aligned}
\langle\mathcal{O}\rangle &= \frac{1}{Z[\overline{V},\overline{H}]\cdot z}\int \mathrm{D}v\int \mathrm{D}h\Big(\mathcal{O}[\overline{V}]+\frac{\delta\mathcal{O}}{\delta V}\Big|_{\overline{V}}v+\frac{\delta^2\mathcal{O}}{\delta V^2}\Big|_{\overline{V}}v^2\Big)\mathrm{e}^{-(S_{eff}[\overline{V},\overline{H}]+S^{(2)}[v,h])}\\
&= \mathcal{O}[\overline{V}]+\frac{1}{2}\frac{\delta^2\mathcal{O}}{\delta V^2}\Big|_{\overline{V}}\frac{1}{z}\int \mathrm{D}v\int \mathrm{D}h\ v^2\ \mathrm{e}^{-S^{(2)}[v,h]}.
\end{aligned}$$

The link 2-point function can be integrated to

$$\begin{aligned}
\langle v_i v_j\rangle &= \frac{1}{z}\int \mathrm{D}v\int \mathrm{D}h\ v^2\ \mathrm{e}^{-S^{(2)}[v,h]}\\
&= \frac{1}{z}\frac{(2\pi)^{|h|/2}}{\sqrt{\det[K^{(hh)}]}}\int \mathrm{D}v\ \ v_i v_j \mathrm{e}^{-\frac{1}{2}v^T(-K^{(vh)}K^{(hh)^{-1}}K^{(vh)}+K^{(vv)})v}\\
&= (K)^{-1}_{ij},
\end{aligned} \tag{4.33}$$

where $K$ is the propagator $K=-K^{(vh)}{K^{(hh)}}^{-1}K^{(vh)}+K^{(vv)}$. Then $\langle\mathcal{O}\rangle$ is expanded as

$$\langle\mathcal{O}\rangle = \mathcal{O}[\overline{V}]+\frac{1}{2}\mathrm{tr}\left\{\frac{\delta^2\mathcal{O}}{\delta V^2}\Big|_{\overline{V}}(K)^{-1}\right\}. \tag{4.34}$$

In order to extract the mass associated with an operator $\mathcal{O}(t)$, we need mean-field expansion of the connected correlator

$$\begin{aligned}
C(t) &= <\mathcal{O}(t_0+t)\mathcal{O}(t_0)> - <\mathcal{O}(t_0+t)><\mathcal{O}(t_0)>\\
&= C^{(0)}(t)+C^{(1)}(t)+\cdots.
\end{aligned} \tag{4.35}$$

The mean-field expansion of each part of $C(t)$ are

$$\begin{aligned}
<\mathcal{O}(t_0+t)\mathcal{O}(t_0)> &= \mathcal{O}^{(0)}(t_0+t)\mathcal{O}^{(0)}(t_0)+\frac{1}{2}\mathrm{tr}\left\{\frac{\delta^2(\mathcal{O}(t_0+t)\mathcal{O}(t_0))}{\delta^2 v}K^{-1}\right\}+\cdots\\
<\mathcal{O}(t_0+t)> &= \mathcal{O}^{(0)}(t_0+t)+\frac{1}{2}\mathrm{tr}\left\{\frac{\delta^2\mathcal{O}(t_0+t)}{\delta^2 v}K^{-1}\right\}+\cdots\\
<\mathcal{O}(t_0)> &= \mathcal{O}^{(0)}(t_0)+\frac{1}{2}\mathrm{tr}\left\{\frac{\delta^2\mathcal{O}(t_0)}{\delta^2 v}K^{-1}\right\}+\cdots.
\end{aligned}$$

Then 0th order and 1st order correction of the mean-field of $C(t)$ are the following.

$$
\begin{aligned}
C^{(0)}(t) &= 0 \\
C^{(1)}(t) &= \frac{1}{2}\mathrm{tr}\left\{\frac{\delta^2(\mathcal{O}(t_0+t)\mathcal{O}(t_0))}{\delta^2 v}K^{-1}\right\} - \frac{1}{2}\mathcal{O}^{(0)}(t_0+t)\mathrm{tr}\left\{\frac{\delta^2\mathcal{O}(t_0)}{\delta^2 v}K^{-1}\right\} \\
&\quad -\frac{1}{2}\mathcal{O}^{(0)}(t_0)\mathrm{tr}\left\{\frac{\delta^2\mathcal{O}(t_0+t)}{\delta^2 v}K^{-1}\right\} \\
&= \mathrm{tr}\left\{\frac{\delta\mathcal{O}(t_0+t)}{\delta v}\frac{\delta\mathcal{O}(t_0)}{\delta v}K^{-1}\right\}
\end{aligned} \tag{4.36}
$$

A gauge invariant correlator can be expanded in terms of the energy eigenvalues of the states as

$$
C(t) = \sum_{\lambda} e^{-E_\lambda t} \tag{4.37}
$$

where $E_0 = m,\ E_1 = m^*,\ \cdots$. Then, the mass is obtained for $t \to \infty$ as,

$$
m \simeq \lim_{t\to\infty} \ln \frac{C^{(1)}(t)}{C^{(1)}(t-1)}. \tag{4.38}
$$

## 4.2 Mean-Field expansion in 2nd order

In order to extract gauge boson masses, we need 2nd order mean-field expansion. The effective action is expanded as

$$
\begin{aligned}
S_{eff} = S_{eff}[\overline{V},\overline{H}] &+ \frac{1}{2}\Big(\left.\frac{\delta^2 S_{\mathrm{eff}}}{\delta H^2}\right|_{\overline{V},\overline{H}} h^2 + \left.\frac{\delta^2 S_{\mathrm{eff}}}{\delta V\delta H}\right|_{\overline{V},\overline{H}} vh + \left.\frac{\delta^2 S_{\mathrm{eff}}}{\delta V^2}\right|_{\overline{V},\overline{H}} v^2\Big) \\
&+\frac{1}{6}\Big(\left.\frac{\delta^3 S_{\mathrm{eff}}}{\delta H^3}\right|_{\overline{V},\overline{H}} h^3 + \left.\frac{\delta^3 S_{\mathrm{eff}}}{\delta V^3}\right|_{\overline{V},\overline{H}} v^3\Big) \\
&+\frac{1}{24}\Big(\left.\frac{\delta^4 S_{\mathrm{eff}}}{\delta H^4}\right|_{\overline{V},\overline{H}} h^4 + \left.\frac{\delta^4 S_{\mathrm{eff}}}{\delta V^4}\right|_{\overline{V},\overline{H}} v^4\Big) + \cdots
\end{aligned}
$$

The cross terms in the cubic and quartic terms vanish because of the special form of $S_{eff}$. The observables are also expanded as

$$
\mathcal{O}[V] = \mathcal{O}[\overline{V}] + \left.\frac{\delta\mathcal{O}}{\delta V}\right|_{\overline{V}} v + \frac{1}{2}\left.\frac{\delta^2\mathcal{O}}{\delta V^2}\right|_{\overline{V}} v^2 + \frac{1}{6}\left.\frac{\delta^3\mathcal{O}}{\delta V^3}\right|_{\overline{V}} v^3 + \frac{1}{24}\left.\frac{\delta^4\mathcal{O}}{\delta V^4}\right|_{\overline{V}} v^4 + \cdots. \tag{4.39}
$$

Then, tadpole-free contributions to the expectation values of an observable are

$$\begin{aligned}\langle\mathcal{O}\rangle &= \frac{1}{Z[\overline{V},\overline{H}]\cdot z}\int \mathrm{D}v\int \mathrm{D}h\Big(\mathcal{O}[\overline{V}] + \frac{1}{2}\left.\frac{\delta^2\mathcal{O}}{\delta V^2}\right|_{\overline{V}} v^2 \\ &\qquad\qquad + \frac{1}{24}\left.\frac{\delta^4\mathcal{O}}{\delta V^4}\right|_{\overline{V}} v^4\Big)\cdot \mathrm{e}^{-(S_{eff}[\overline{V},\overline{H}]+S^{(2)}[v,h])} \\ &= \mathcal{O}[\overline{V}] + \frac{1}{2}\left.\frac{\delta^2\mathcal{O}}{\delta V^2}\right|_{\overline{V}}\frac{1}{z}\int \mathrm{D}v\int \mathrm{D}h\ v^2\ \mathrm{e}^{-S^{(2)}[v,h]} \\ &\quad + \frac{1}{24}\left.\frac{\delta^4\mathcal{O}}{\delta V^4}\right|_{\overline{V}}\frac{1}{z}\int \mathrm{D}v\int \mathrm{D}h\ v^4\ \mathrm{e}^{-S^{(2)}[v,h]}.\end{aligned}$$

The link 4-point function can be integrated as

$$\begin{aligned}\langle v_i v_j v_l v_m\rangle &= \frac{1}{z}\int \mathrm{D}v\int \mathrm{D}h\ v_i v_j v_l v_m\ \mathrm{e}^{-S^{(2)}[v,h]} \\ &= (K)^{-1}_{ij}(K)^{-1}_{lm} + (K)^{-1}_{il}(K)^{-1}_{jm} + (K)^{-1}_{im}(K)^{-1}_{lj}.\end{aligned} \tag{4.40}$$

Finally we obtain 2nd order correction

$$\begin{aligned}\langle\mathcal{O}\rangle &= \mathcal{O}[\overline{V}] + \frac{1}{2}\left(\left.\frac{\delta^2\mathcal{O}}{\delta V^2}\right|_{\overline{V}}\right)_{ij}(K^{-1})_{ij} \\ &\quad + \frac{1}{24}\sum_{i,j,l,m}\left(\left.\frac{\delta^4\mathcal{O}}{\delta V^4}\right|_{\overline{V}}\right)_{ijlm} \\ &\quad \cdot\Big((K^{-1})_{ij}(K^{-1})_{lm} + (K^{-1})_{il}(K^{-1})_{jm} + (K^{-1})_{im}(K^{-1})_{jl}\Big).\end{aligned} \tag{4.41}$$

The 2nd order correction of the connected correlation function is

$$\begin{aligned}C^{(2)}(t) &= \frac{1}{24}\sum_{i,j,l,m}\left(\frac{\delta^2\mathcal{O}(t_0+t)}{\delta v^2}\right)_{ij}\left(\frac{\delta^2\mathcal{O}(t_0)}{\delta v^2}\right)_{lm} \\ &\quad \cdot\Big((K^{-1})_{ij}(K^{-1})_{lm} + (K^{-1})_{il}(K^{-1})_{jm} + (K^{-1})_{im}(K^{-1})_{jl}\Big).\end{aligned} \tag{4.42}$$

The extracted mass is

$$m \simeq \lim_{t\to\infty}\ln\frac{C^{(1)}(t)+C^{(2)}(t)}{C^{(1)}(t-1)+C^{(2)}(t-1)}. \tag{4.43}$$

## 4.3 Observables

### 4.3.1 Higgs and Z boson mass

In order to construct observables, we define the line

$$l^{(n_5)}(t_0, \vec{m}) = \prod_{m_5=0}^{n_5-1} V((t_0, \vec{m}, m_5); 5) \tag{4.44}$$

and introduce the matrices

$$\sigma^\alpha = \{\mathbf{1},\ i\sigma^A\}, \qquad \overline{\sigma}^\alpha = \{\mathbf{1},\ -i\sigma^A\}, \qquad A = 1, 2, 3\,. \tag{4.45}$$

The orbifold projected Polyakov loop is writen as

$$P^{(0)}(t, \vec{m}) = l^{(N_5)}(t, \vec{m})\, g\, l^{(N_5)\dagger}(t, \vec{m})\, g^\dagger\,. \tag{4.46}$$

We define the displaced Polyakov loop

$$Z_k^{(0),A}(t, \vec{m}) = \overline{\sigma}^A\, V((t, \vec{m}, 0); k)\, \Phi^{(0)\dagger}(t, \vec{m} + \hat{k})\, V((t, \vec{m}, 0); k)^\dagger\, \Phi^{(0)}(t, \vec{m})\,, \tag{4.47}$$

which assigns a vector and a gauge index to the observable appropriate to a gauge boson where $\Phi^{(0)}(t, \vec{m}) = P^{(0)}(t, \vec{m}) - P^{(0)\dagger}(t, \vec{m})$. The Higgs boson observable is derived from the averaged over space and time location connected correlator

$$\mathcal{O}_H(t_0 + t)\mathcal{O}_H(t_0) = \frac{1}{L^6 T} \sum_{t_0} \sum_{\vec{m}', \vec{m}''} \mathrm{tr}\{P^{(0)}(t_0, \vec{m}')\}\mathrm{tr}\{P^{(0)}(t_0 + t, \vec{m}'')\} \tag{4.48}$$

and the Z-boson from the correlator

$$\mathcal{O}_Z(t_0 + t)\mathcal{O}_Z(t_0) = \frac{1}{L^6 T} \sum_{t_0} \sum_{\vec{m}', \vec{m}''} \sum_{k} \mathrm{tr}\{Z_k^{(0),3}(t_0, \vec{m}')\}\, \mathrm{tr}\{Z_k^{(0),3}(t_0 + t, \vec{m}'')\}\,. \tag{4.49}$$

From Eq. (4.36), 1sr order correlation function of Higgs boson mass is

$$C_H^{(1)}(t) = \frac{8}{\mathcal{N}^{(4)}} (P_0^{(0)})^2 \Pi^{(1)}_{\langle 1,1\rangle}(0, 0)\,, \tag{4.50}$$

where

$$\Pi^{(1)}_{\langle 1,1\rangle}(\alpha,\beta)$$
$$= 2\sum_{p'_0}\cos p'_0 t \sum_{n'_5,n''_5} \Delta_1^{(N_5)}(n'_5) K^{(-1)}(p'_0,\vec{0},n'_5,5,\alpha;p'_0,\vec{0},n''_5,5,\beta)\Delta_1^{(N_5)}(n''_5) \tag{4.51}$$

and

$$\Delta_1^{(N_5)}(n_5) = \sum_{r=0}^{N_5-1}\frac{\delta_{n_5,r}}{\overline{v}_0(r+1/2)} = (1-\delta_{n_5,N_5})\frac{1}{\overline{v}_0(n_5+\frac{1}{2})}. \tag{4.52}$$

This correlation function does not contain torons. Because the 1st order mean-field expansion of the correlation function is zero, we need 2nd order expansion for $Z$ mass. From Eq. (4.42),

$$C_Z^{(2)}(t) = \frac{4096}{(\mathcal{N}^{(4)})^2}(P_0^{(0)})^4(v_0(0))^4\sum_{\vec{p}'}\sum_k \sin^2 p'_k \Pi^{(2)}_{\langle 1,1\rangle}(1,1)^2\,, \tag{4.53}$$

where

$$\Pi^{(2)}_{\langle 1,1\rangle}(\alpha,\beta)$$
$$= \sum_{p'_0} e^{ip'_0 t}\sum_{n'_5,n''_5}\Delta_1^{(N_5)}(n'_5)K^{(-1)}(p'_0,\vec{p}',n'_5,5,\alpha;p'_0,\vec{p}',n''_5,5,\beta)\Delta_1^{(N_5)}(n''_5)\,. \tag{4.54}$$

This correlation function contains regularizable torons, whose contribution vanishs in the infinite lattice volume limit.

### 4.3.2 The static potential

There are three types of potential for orbifold boundary conditions. Here we are interested in the potential along 4 dimensional hyper plane on the boundary. We consider the Wilson loops with size $r$ in one direction and take average over all directions. The exchange contribution ($\delta^2\mathcal{O}^c/\delta V^2$) is

$$\mathcal{O}_{\rm ex} \equiv \frac{t^2}{L^3T} 2(\overline{v}_0(0))^{2(t+n_3)-2}\delta_{M'0}\delta_{M''0}$$
$$\delta_{n_5 0}(\delta_{\alpha'0}\delta_{\alpha''0} + \delta_{\alpha'3}\delta_{\alpha''3})\delta_{p'_0 0}\delta_{p''_0 0}\left(\prod_{M=1,2,3}\delta_{p'_M p''_M}\right)\frac{1}{3}\sum_{k=1}^{3}2\cos\left(p_k r\right)\,\delta_{n'_5 0}\delta_{n''_5 0} \tag{4.55}$$

The Self energy contribution is

$$\mathcal{O}_{\rm se} \equiv \frac{t^2}{L^3T} 2(\overline{v}_0(0))^{2(t+n_3)-2}\delta_{M'0}\delta_{M''0}$$
$$\delta_{n_5 0}(\delta_{\alpha'0}\delta_{\alpha''0} - \delta_{\alpha'3}\delta_{\alpha''3})\delta_{p'_0 0}\delta_{p''_0 0}\left(\prod_{M=1,2,3}\delta_{p'_M p''_M}\right) 2\,\delta_{n'_5 0}\delta_{n''_5 0}\,. \tag{4.56}$$

The first order correlation function is written as

$$C_W^{(1)} = \frac{1}{2}\sum_{\alpha',\alpha''}\sum_{p'_k}\sum_{n'_5,n''_5}$$
$$\mathcal{O}\Big(0,p'_k,n'_5,0,\alpha';0,p'_k,n''_5,0,\alpha''\Big)K^{-1}\Big(0,p'_k,n'_5,0,\alpha';0,p'_k,n''_5,0,\alpha''\Big)\,, \tag{4.57}$$

where $\mathcal{O} = \mathcal{O}_{\rm ex} + \mathcal{O}_{\rm se}$. Then, the potential is writen with the correlation function as

$$V = \text{const.} - \lim_{t\to\infty}\frac{1}{t}\frac{C_W^{(1)}}{\mathcal{O}[\overline{V}]}\,. \tag{4.58}$$

Therefore, the potential along 4-dimensional hyper plane on boundary is

$$V_4(0) = -\log(\overline{v}_0(0)^2) - \frac{1}{2}\frac{1}{L^3T}\frac{1}{(\overline{v}_0(0))^2}\sum_{p'_k}$$
$$\left\{\frac{1}{3}\sum_k\Big[2\cos\left(p'_k r\right)+2\Big]K^{-1}\left(0,p'_k,0,0,0;0,p'_k,0,0,0\right)\right.$$
$$\left.+\frac{1}{3}\sum_k\Big[2\cos\left(p'_k r\right)-2\Big]K^{-1}\left(0,p'_k,0,0,3;0,p'_k,0,0,3\right)\right\}. \tag{4.59}$$

# Chapter 5

# Results from Mean-Field calculation

## 5.1 The phase diagram and phase transition

Fig. 5.1 is the phase diagram which is based on the value of the mean-field. We can see there are three phases. The red region is the confined phase where $\overline{v}_0(n_5) = \overline{v}_0(n_5 + 1/2) = 0$ for all $n_5$, the blue region is the layered phase where $\overline{v}_0(n_5) \neq 0$ and $\overline{v}_0(n_5 + 1/2) = 0$ for all $n_5$ and the white region is Coulomb phase (or deconfined phase) where $\overline{v}_0(n_5) \neq 0$ and $\overline{v}_0(n_5 + 1/2) \neq 0$ for all $n_5$. The green region is a kind of cross over phase. We can analyze only for the Coulomb phase by mean-field expansion, because when the background is zero, we can not obtain any information. Now, we are interested in the order of the phase transition between Coulomb phase and the other two phases. We can find out the order of the phase transition by the critical exponent $\nu$ which can be obtained as follows

$$a_4 m_H = A\Big(\frac{\beta - \beta_c}{\beta}\Big)^{\nu}, \tag{5.1}$$

where $m_H$ is the Higgs boson mass obtained from $C_H^{(1)}$. We see that the critical exponent $\nu \simeq 1/2$ for $\gamma \leq 0.6$ and $\nu \simeq 1/4$ for $\gamma \geq 0.65$. It means that phase transition for $\gamma \geq 0.65$ is 1st order and for $\gamma \leq 0.6$ is 2nd order. It means that the phase transition between Coulomb phase and layerd phase is 2nd order and

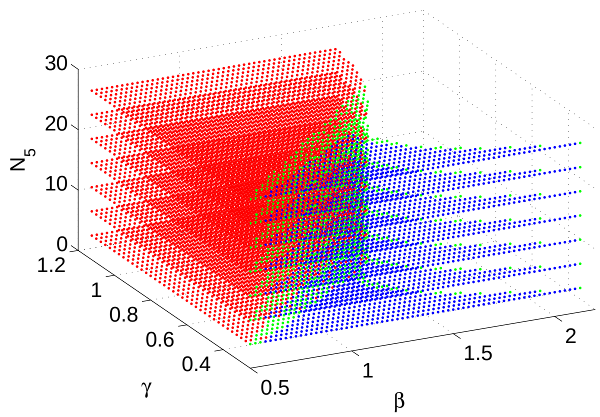



Figure5.1 The mean-field phase diagram of the $SU(2)$ orbifold theory in the $(\beta, \gamma, N_5)$ space. The color code is explained in the text.

between Coulomb phase and confined phase is 1st order. When the bulk phase transition is 1st order, the 4-dimensional lattice spacing $a_4$ does not go to zero and it is impossible to take a continuum limit. In this case the theory could be a low energy effective theory that must be defined with a finite cut-off in the effective action. When the phase transition is 2nd order, one expects that the lattice spacing goes to zero at the phase boundary. In this case a cut-off does not need in the effective action and the theory could be non-perturbatively renormalizable.

## 5.2 The masses

### 5.2.1 Higgs boson mass

The Higgs boson mass in units of the lattice spacing $M_H = a_4 m_H$ is extracted from $C_H^{(1)}$ in Eq. (4.50). The Higgs boson mass depends on the parameters $\beta$, $\gamma$ and $N_5$. Using $M_H$, we can get the Higgs boson mass in units of the radius of the 5th dimension $F_1$.

$$F_1 = m_H \, R = M_H \, \frac{N_5}{\gamma \, \pi} \, . \tag{5.2}$$

The left plot in Fig. 5.2 is the $N_5$-dependence of $M_H$ for $\gamma = 1$(isotropic lattice) at $\beta = 1.677$. I choose $\beta = 1.677$ to be near the phase transition. The line in the left plot in Fig. 5.2 is a quadratic fit. On the other hands in perturbation theory, the Higgs boson mass from the one-loop result [27] for $SU(N)$ is expressed as

$$M_H^{\text{pert.}} = \frac{c\,\gamma\,\pi}{N_5^{3/2}\,\beta^{1/2}}, \qquad c = \frac{3}{4\pi^2}\sqrt{N\,\zeta(3)\,C_2(N)} \tag{5.3}$$

where $C_2(N) = (N^2-1)/(2N)$. This plot also shows that $M_H$ cannot be lowered to zero but approaches around 0.7. Therefore we can see that the phase transition is the 1st order.

### 5.2.2 Direct Z boson mass

The Z boson mass in units of the lattice spacing $M_Z^{\text{dir.}} = a_4 m_Z^{\text{dir.}}$ is extracted from the correlator $C_Z^{(2)}$ in Eq. (4.53). $M_Z^{\text{dir.}}$ does not depend on $\beta$, $\gamma$ or $N_5$. This means that the masses from the correlator $C_Z^{(2)}$ is always infinite $N_5$ limit value.

The dependence on $L$ is

$$M_Z^{\text{dir.}} = \frac{4\pi}{L}. \tag{5.4}$$

This expression shows that this observable describes two non interacting gauge bosons.

## 5.3 Spontaneous Symmetry Breaking

We can find out whether there is SSB by calculating the wilson loop. We expect that the boundary gauge theory can be described in four-dimensional term. So, if the boundary $U(1)$ symmetry is spontaneously broken the corresponding static potential extracted from $C_W^{(1)}$ should be fitted by a 4-dimensional Yukawa form. The 4-dimenaional Yukawa potential is

$$V_4(r) = -b\frac{e^{-m_Z r}}{r} + \text{const.}, \tag{5.5}$$

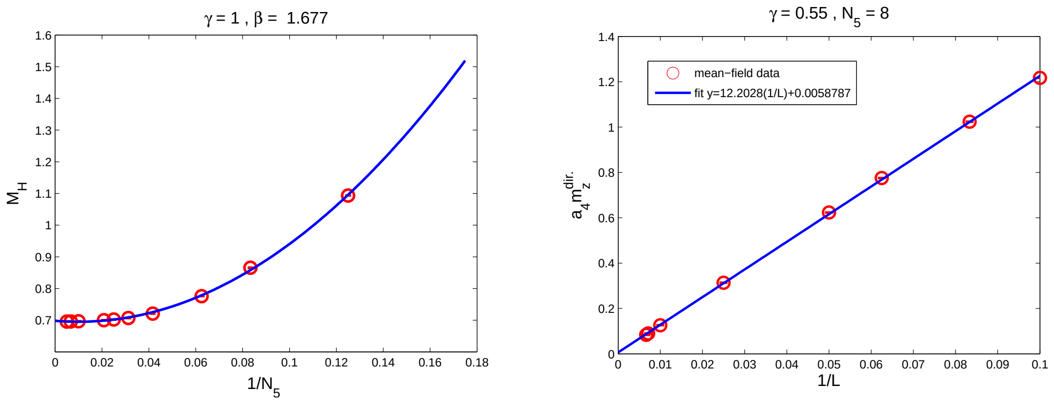



Figure5.2 The left plot is Higgs boson mass $M_H$ as a function of $1/N_5$ at $\gamma = 1$ for $\beta = 1.677$ with the line of a quadratic fit. The right plot is direct Z boson mass $M_Z^{\text{dir.}}$ as a function of $1/L$ at $\gamma = 0.55$ with the line of a linear fit.

where $b$ is a constant. The corresponding static force is

$$F_4(r) = \frac{\mathrm{d}\, V_4(r)}{\mathrm{d}\, r} = b\frac{e^{-m_Z r}}{r}(m + \frac{1}{r}). \tag{5.6}$$

To extract the Yukawa mass, we define the quantity $y(r) = \log(r^2 F_4(r))$ from which we form the combination

$$a_4 y'(r) = -M_Z + \frac{M_Z}{m_Z r + 1}\,, \tag{5.7}$$

where $M_Z$ is the $Z$ mass in lattice units defined as $M_Z = a_4 m_4$. Then we determine $M_Z$ iteratively so that the plot $-a_4 y'(r) + M_Z/(m_Z r + 1)$ has plateau at $M_Z$. The plateaus do not depend on $L$ if $L$ is large enough. So $M_Z$ depends only on $\beta$, $\gamma$ and $N_5$ for infinite $L$.

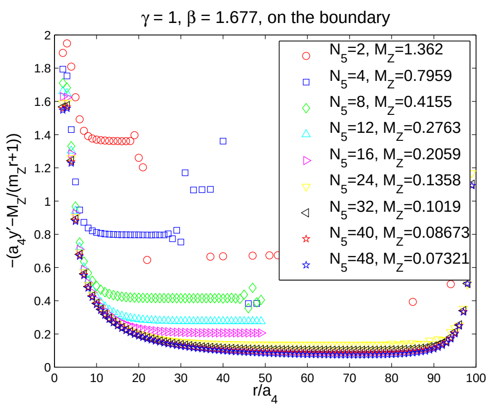
### 5.3.1 Isotropic lattice

The left plot of Fig. 5.3 is the plots of $-a_4 y'(r) + M_Z/(m_Z r + 1)$ for various $N_5$ at fixed $\beta = 1.677$ and $\gamma = 1$ near the bulk phase transition. The plateau values do not depend on $L$ for $L \geq 200$. The right plot of Fig. 5.3 is the plateau values of $M_Z$ from the right plot of Fig. 5.3 as a function of $1/N_5$. The blue line is a linear fit with slope 3.32, which is very close to $\pi$ and it describes the data very well. These plot shows that gauge boson is massive on the boundary and it means that there is the dynamical spontaneous breaking of the $U(1)$ symmetry. Note that, since $\beta$ and $\gamma$ are kept fixed and the location of the phase transition $\beta_c$ depends on $N_5$, the masses in Fig. 5.3 correspond to different lattice spacings.

In Fig. 5.4, the blue squares are plot of Higgs and Z boson mass ratio

$$\rho_{HZ} = \frac{m_H}{m_Z}\,, \tag{5.8}$$

for $N_5 = 4, 6, 8$ and $L = 200$. The ratio does not depend on $N_5$ for these parameters. We can see that the Higgs and the $Z$ boson masses are almost same so that $\rho_{HZ} \simeq 1$ for $\gamma = 1$ and $F_1$ in the range $[0.08, 0.4]$. In Fig. 5.4, the results from Monte Carlo simulations at $N_5 = 4$(diamonds) and at $N_5 = 6$(circle) for $L = 12$ and $T = 96$ are also plotted. There is good agreement between the mean-field data and the Monte Carlo data on isotropic lattice.



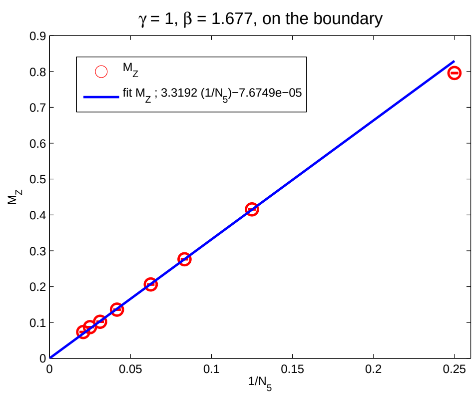




Figure5.3 The left plot is the combination $-a_4 y'(r) + M_Z/(m_Z r + 1)$, cf. Eq. (5.7) plotted for different values of $N_5$ at $\gamma = 1$ for $\beta = 1.677$. The right plot is the $Z$ boson mass $M_Z$ extracted from the left plot as a function of $1/N_5$.

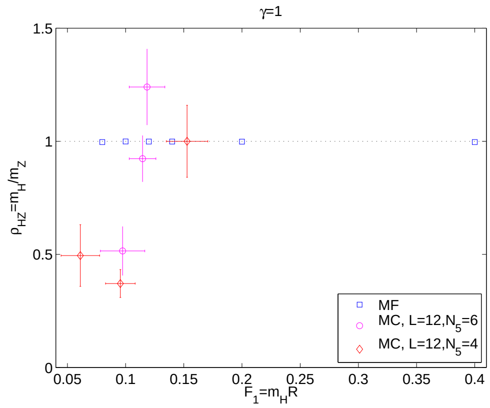



Figure5.4 The ratio of the Higgs boson mass to the $Z$ boson mass Eq. (5.8). Comparison of Monte Carlo (diamonds [18, 28] and circles [29, 30]) and mean-field data (squares) at $\gamma = 1$.

### 5.3.2 Anisotropic lattice ($\gamma = 0.55$)

We are interested in the parameter region where there is 2nd order phase transition($\gamma \leq 0.6$). So, I study static potential on the boundary and in the middle of the orbifold at $\gamma = 0.55$ to find out whether there is SSB or not. In this calculation, I choose $\beta$ so that $F_1 = 0.2$ constant, which means that $M_H \propto 1/N_5$, cf. Eq. (5.2). Fig. 5.5 is the plots of $-a_4 y'(r) + M_Z/(m_Z r + 1)$ (see Eq. (5.7)) extracted from the boundary potential (left plot) and the potential in the middle of the bulk (right plot). In the left plot, there are two plateaus for $N_5 \geq 6$. These plateaus correspond to masses $M_{Z'}(> M_Z)$ which do not depend on $N_5$ and $Z'$ are the 1st excited vector boson state. We also checked that the Yukawa masses are independent of $L$ for $L \geq 200$. It means that there is SSB and the boundary $U(1)$ gauge symmetry is broken. We checked that the Yukawa masses are independent of $L$ for $L \geq 200$. These data say that the boundary $U(1)$ gauge symmetry is broken.

The left plot of Fig. 5.6 is the plots of $\rho_{HZ}$ corresponding the plateaus in the left plot of Fig. 5.5. It shows that we get $\rho_{HZ} < 1$ for that parameter region.

In the left plot of Fig. 5.6(potential in the middle), there is only one plateau for each $N_5$. These plateau corresponds $Z$ boson mass $M_Z$. The $M_Z$ is decreasing as $N_5$ increases and does not depend on $L$ for $L \geq 200$. It means that there is SSB also in the bulk. This result is completely different from the result for the torus where there is no SSB. We also observe a difference between the Yukawa masses in the bulk as compared to those on the boundary. This situation is different from the one for the isotropic lattice, where we found the boundary and bulk Yukawa masses to be the same.

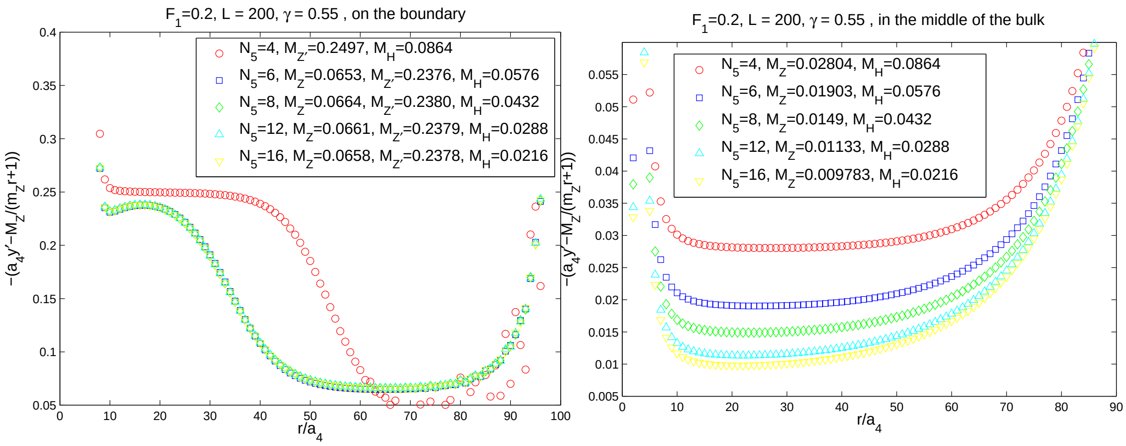



Figure5.5 The combination $-a_4 y'(r) + M_Z/(m_Z r + 1)$, cf. Eq. (5.7) is plotted for different values of $N_5$ at $\gamma = 0.55$ and $F_1 = 0.2$ (for the boundary potential at $N_5 = 4$ we use $M_{Z'}$). Boundary potential (left plot) and bulk potential (right plot).

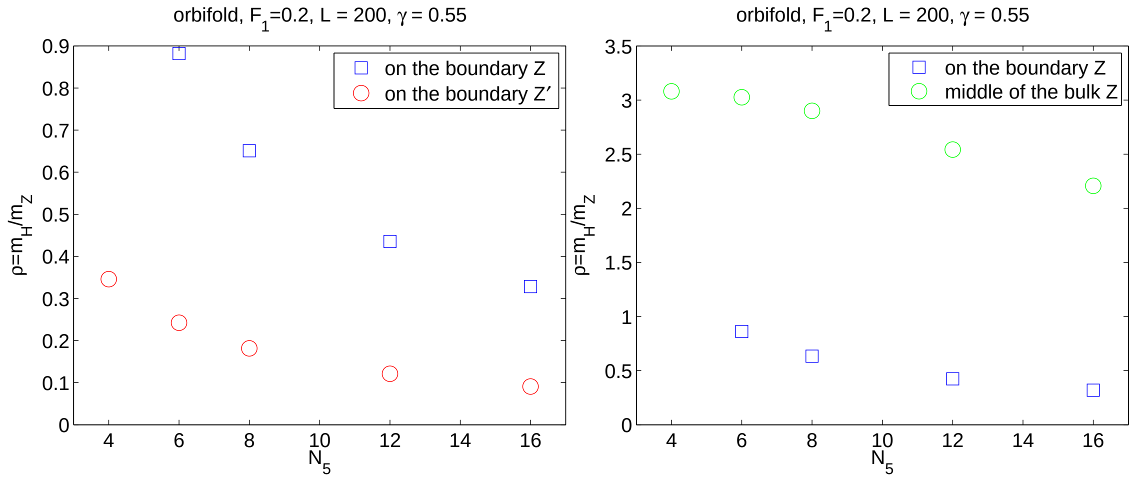



Figure5.6 The ratio of the Higgs boson mass to the $Z$ boson mass Eq. (5.8) in the mean-field extracted from the static potential. On the boundary (left plot) and in the bulk (right plot).

## 5.4 Dimensional reduction

Here I defined the ratio of the Higgs boson mass to the mass of the first excited vector boson state

$$\rho_{HZ'} = \frac{m_H}{m_{Z'}} . \tag{5.9}$$

In the previous section the static potential is fitted by 4-dimensional Yukawa potential. Such a fit makes sense if the spectrum can be interpreted as an effective four-dimensional theory. However, it is not a precise definition of the dimensional reduction. More constrained criteria for dimensional reduction are the following.

The definition of the dimensional reduction

- The static potential along 4-dimensional hyperplane can be fitted by the 4-dimensional Yukawa potential Eq. (5.5) with $m_Z \neq 0$.

  This ensures that there is SSB, signaled by the presence of the massive $U(1)$ gauge boson. Otherwise the gauge boson is massless and only a

Coulomb fit is possible.

- the quantities $M_H = a_4 m_H$ and $M_Z = a_4 m_Z$ are $< 1$.

  This ensures that the observables are not dominated by the cut-off.

- we have $m_H R < 1$ and $\rho_{HZ} > 1$.

  These two conditions ensure that the Higgs boson and the $Z$ boson masses are lighter than the Kaluza-Klein scale $1/R$ and Higgs boson is heavier than the $Z$ boson. we will target the value

  $$\rho_{HZ} = 1.38\,, \tag{5.10}$$

  which is (approximately) the currently favored value of the analogous quantity in the SM, based on recent LHC data [4].

Here I have three observables $F_1$, $\rho_{HZ}$ and $\rho_{HZ'}$ and all three observables depend on the three dimensionless parameters $\beta$, $\gamma$ and $N_5$. I consider to taking continuum limit satisfying above criteria and keeping physical value same.

The procedure is the followings. First, I fixed $F_1$ to a given value and $\rho_{HZ}$ to the value Eq. (5.10). With these two conditions the value $\rho_{HZ'}$ becomes a function of one parameter which I choose to be $N_5$. Then we obtain the value of the second excited $Z$ boson mass $m_{Z'}$ for each $N_5$. We call such a trajectory on the phase diagram a Line of Constant Physics (LCP) [31, 32]. In this calculation I inserted the SM experimental value for $m_H$ and $m_Z$. I checked that both $M_H$ and $M_Z$ decrease as approaching phase transition. So, to obtain $M_H, M_Z < 1$, we need to stay near the critical point. I check also only for small $\gamma$ regime where we can get $\rho_{HZ} > 1$. Thus I calculated LCP for small $\gamma$ near the critical point.

## 5.5 Lines of Constant Physics and the $Z'$

The first LCP I construct is one where

$$F_1 = m_H R = 0.61\,, \qquad \rho_{HZ} = 1.38 \tag{5.11}$$

| $F_1$ | $N_5$ | $\gamma^*$ | $\beta^*$ |
|---|---|---|---|
| 0.61 | 12 | 0.5460(33) | 1.343501425 |
| | 14 | 0.5320(10) | 1.34442190 |
| | 16 | 0.5228(7) | 1.34664820 |
| | 20 | 0.5028(18) | 1.35582290 |
| | 24 | 0.4844(32) | 1.36940695 |
| 0.20 | 6 | 0.5113(15) | 1.351160631 |

Table5.1 Bare parameters of the LCP defined by $\rho_{HZ} = m_H/m_Z = 1.38$ and $F_1 = m_H R = 0.61$, together with one point for a LCP with $\rho_{HZ} = 1.38$ and $F_1 = 0.20$. The lattice gauge couplings $\beta^*$ correspond to the central values $\gamma^*$ and are computed for future reference.

are kept fixed. In Fig. 5.7, I plot the corresponding points on the phase diagram, which are listed in Table 5.1.

Along this LCP, I computed $\rho_{HZ'}$ for $N_5 = 12, 14, 16, 20, 24$. In Fig. 5.7, we plot the corresponding points interpolated by a black line on the phase diagram, which are listed in Table 5.1. As I mentioned above the LCP is constructed for small $\gamma$ near the phase boundary. This region of $\gamma$, the phase transition is of second order.

For each value of $N_5$, I constructed LCP. I also computed the $Z$ and $Z'$ masses for various values of the parameter $\gamma$.

The steps of detailed calculation of the LCP are as follows. First, I chose the starting point of $N_5$. After fixed $N_5$, I determined $\beta = \beta(\gamma, N_5)$ so that $F_1 = 0.61$. The value of $L$ should be large enough to get clear plateaus so we set $L = 400$ for all $N_5$ values. Then I calculated the static potential on the boundary for several $\gamma$ values and extracted $Z$ masses and $Z'$ masses. The gauge boson masses are extracted by identifying them as Yukawa masses as in Section 5.3.

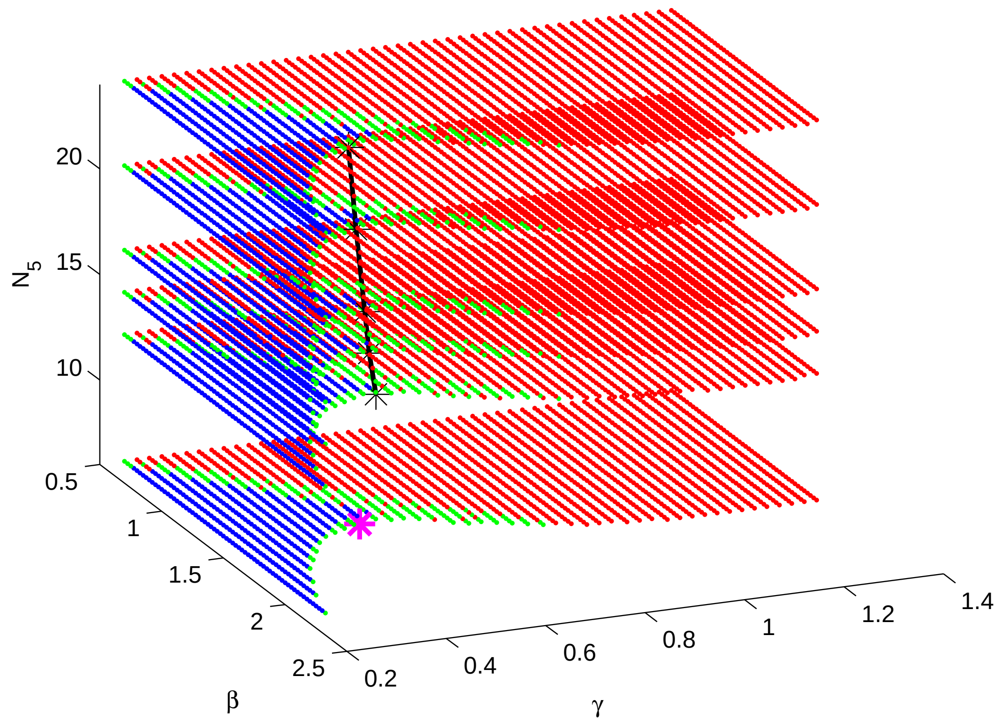



Figure5.7 LCP (black line) defined in Eq. (5.11) near the (tricritical point of the) bulk phase transition. Red: Confined phase. Blue: Layered phase. White: Deconfined phase. The magenta point (star) is on a different LCP with $F_1 = 0.2$, $\rho_{HZ} = 1.38$.

For instance, the left Fig. 5.8 is the plot of $-\left[a_4 y'(r) - M_Z/(M_Z r/a_4 + 1)\right]$ for $N_5 = 24$ and $\gamma = 0.485$. There are two plateaus. I defined $M_Z$ averaging the smaller (red points) plateau points and $M_{Z'}$ averaging the larger (blue points) ones. The ranges of $r$ values defining the plateaus are defined around the minima of the derivative of $-\left[a_4 y'(r) - M_Z/(M_Z r/a_4 + 1)\right]$. The errors of the masses are the standard deviation of the plateau points. Then I computed $\rho_{HZ}$ and $\rho_{HZ'}$ with the known values of $M_Z$ and $M_H$ for several $\gamma$ and plotted these on the right plot of Fig. 5.8 as a function of $\gamma$. The upper red circles are the values $\rho_{HZ}$ and the red line is its linear fit and the lower blue circles are the values $\rho_{HZ'}$

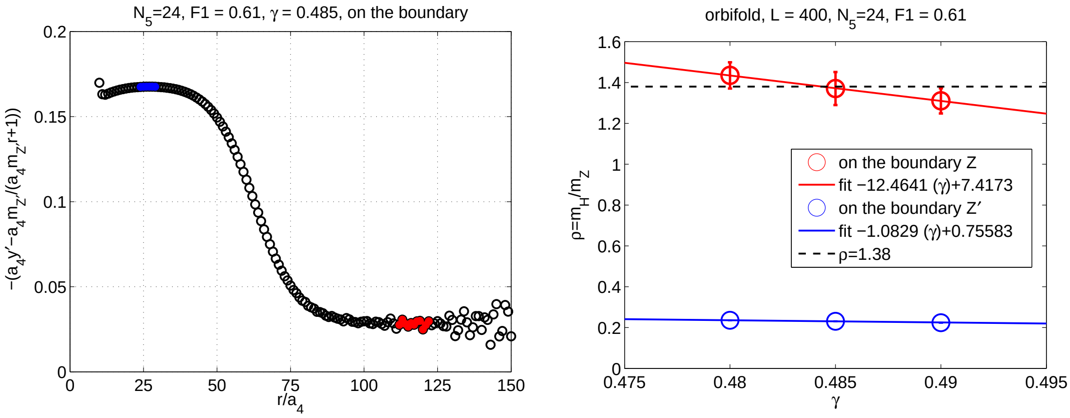



Figure5.8 Left plot: plateaus of the quantity defined in Eq. (5.7) corresponding to the $Z$ (red points) and $Z'$ (blue points) masses. Right plot: the $\rho_{HZ}$ data (upper red circles) are lineraly interpolated (red line) to the value of $\gamma$ corresponding to $\rho_{HZ} = 1.38$ (marked by the dashed horizontal line). The lower blue circles show the data for $\rho_{HZ'}$ with a linear fit (blue line).

and the blue line is its linear fit. The both data are fitted very well linearly. So, we can determine $\gamma = \gamma^*(N_5)$ such that $\rho_{HZ} = 1.38$ from the fit. In this case we get $\gamma^*(24) = 0.4844(32)$ for $N_5 = 24$. And also we get $\rho_{HZ'}$ from the linear fit (the blue line on the left Fig. 5.8) for $\gamma^*(24) = 0.4844(32)$. I have done these calculation for each $N_5$ and the summary of the LCP parameters for all $N_5$ values is given in Table 5.1.

Fig. 5.9 is the plot of $\rho_{HZ'}$ on the LCP line against $a_4 m_H$ for $N_5 = 12, 14, 16, 20, 24$. Since $F_1 = m_H R = (a_4 m_H) N_5/(\gamma^* \pi)$, $a_4 m_H$ is proportional to $\gamma^*/N_5$ on the LCP. So, $a_4 m_H$ shows the physical distance to the continuum limit. The straight line is a linear fit of the data. In principle it wound be fitted with a quadratic curve because of the Symanzik analysis of cut-off effects. The dominant contribution is expected to be from the dimension 5 boundary operator

$$\frac{\pi}{4}\left(F^1_{5\mu}F^1_{5\mu} + F^2_{5\mu}F^2_{5\mu}\right)\delta_{n_5,0} \tag{5.12}$$

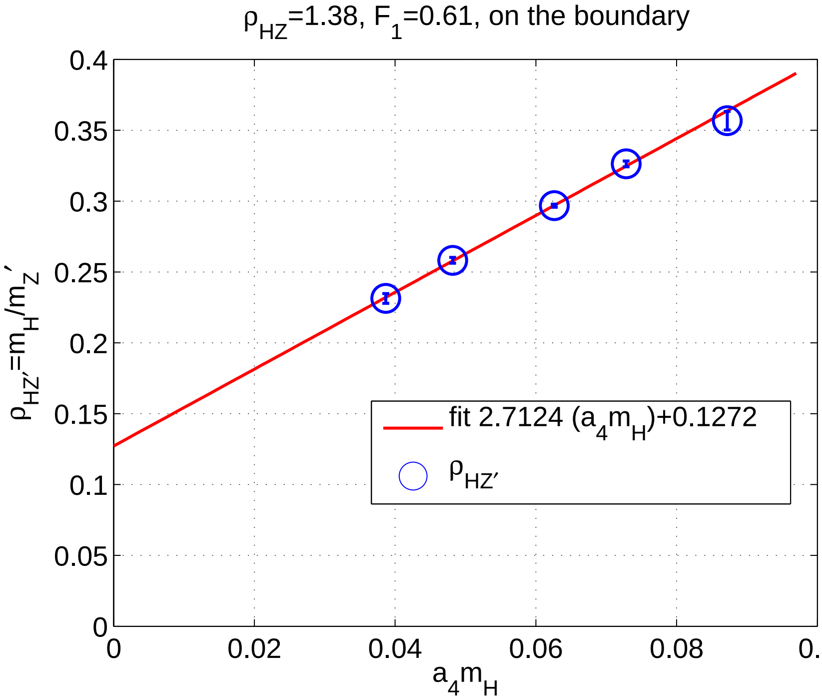



Figure5.9 Extrapolation of LCP in Eq. (5.11) to $a_4 m_H \to 0$.

multiplied by one power of the lattice spacing and from the dimension 7 bulk operator

$$\frac{1}{2g_5^2}\frac{1}{24}\sum_{M,N}\mathrm{tr}\left\{F_{MN}\left(D_M^2+D_N^2\right)F_{MN}\right\} \tag{5.13}$$

multiplied by two powers of the lattice spacing [29]. In this study, we are very close to the phase transition that is we are in a regime where the effect of the dimension 5 boundary operator dominates. Therefore the data on Fig. 5.9 is fitted lineary. By extrapolating to $a_4 m_H \to 0$ we get non-zero value of $\rho_{HZ'}$.

$$\rho_{HZ'} = 0.1272\,. \tag{5.14}$$

Inserting $m_Z = 91.19\,\mathrm{GeV}$, this implies $m_{Z'} = 989\,\mathrm{GeV}$ in the continuum limit. Here, the $\chi^2$ per degree of freedom of the fit is $0.025/3$.

# Chapter 6

# Results from Monte Carlo simulation

In this chapter I will show the result from Monte Carlo simulation. In the MC simulation, I applied Hyper cubic (HYP) smearing [33] to obtain large number operators to improve the generalized eigenvalues problem. HYP smearing is briefly explained in the next section.

## 6.1 Hypercubic(HYP) smearing on the orbifold

The fat links are constructed by adding staples around the links. We only add the staples in direction of 3 spacial dimensions and not in time and 5th dimension. The fat links along 3 spacial dimension are constructed in two steps. The fat links $V_{i,k}$ are written with decorated links $\bar{V}_{i,k;l}$ as

$$\begin{aligned} V_{i,k} = Proj_{SU(2)}[(1-\alpha_2)U_{i,k} \\ +\frac{\alpha_1}{4}\sum_{l\neq m\neq k}\{\bar{V}_{i,l;m}\bar{V}_{i+\hat{l},k;m}\bar{V}^\dagger_{i+\hat{k},l;m} + \bar{V}^\dagger_{i,l;m}\bar{V}_{i-\hat{l},k;m}\bar{V}_{i+\hat{k},l;m}\}], \end{aligned} \tag{6.1}$$

where $U_{i,k}$ is the original thin link. The decorated links $\bar{V}_{i,k:l}$ are constructed with the original thin links as

$$\bar{V}_{i,k;l} = Proj_{SU(2)}[(1-\alpha_2)U_{i,k} + \frac{\alpha_3}{2}\{U_{i,l}U_{i+\hat{l},k}U^{\dagger}_{i+\hat{k},l} + U^{\dagger}_{i,l}U_{i-\hat{l},k}U_{i+\hat{k},l}\}], \tag{6.2}$$

where $k, l, m = 1, 2, 3$. $\bar{V}_{i,k:l}$ represents the link at location $i$ in direction $k$ which with is decorated with staples in direction $l$.

The fat links along 5th dimension are constructed in three steps. The fat links $V_{i,5}$ are written with the decorated links $\tilde{V}_{i,5;k}$ and $\tilde{V}_{i,k;5}$ as

$$\begin{aligned} V_{i,5} = Proj_{SU(2)}[&(1-\alpha_1)U_{i,5} \\ &+\frac{\alpha_1}{6}\sum_k\{\tilde{V}_{i,k;5}\tilde{V}_{i+\hat{k},5;k}\tilde{V}^{\dagger}_{i+\hat{5},k;5} + \tilde{V}^{\dagger}_{i,k;5}\tilde{V}_{i-\hat{k},5;k}\tilde{V}_{i+\hat{5},k;5}\}]. \end{aligned} \tag{6.3}$$

Where $\tilde{V}_{i,5;k}$ and $\tilde{V}_{i,k;5}$ are constructed with other set of decorated links $\bar{V}_{i,M;k}$ as

$$\begin{aligned} \tilde{V}_{i,5;k} = Proj_{SU(2)}[&(1-\alpha_2)U_{i,5} \\ &+\frac{\alpha_2}{4}\sum_{l\neq m\neq k}\{\bar{V}_{i,l;m}\bar{V}_{i+\hat{l},5;m}\bar{V}^{\dagger}_{i+\hat{5},l;m} + \bar{V}^{\dagger}_{i,l;m}\bar{V}_{i-\hat{l},5;m}\bar{V}_{i+\hat{5},l;m}\}]. \end{aligned} \tag{6.4}$$

$$\begin{aligned} \tilde{V}_{i,k;5} = Proj_{SU(2)}[&(1-\alpha_2)U_{i,k} \\ &+\frac{\alpha_2}{4}\sum_{l\neq m\neq k}\{\bar{V}_{i,l;m}\bar{V}_{i+\hat{l},k;m}\bar{V}^{\dagger}_{i+\hat{k},l;m} + \bar{V}^{\dagger}_{i,l;m}\bar{V}_{i-\hat{l},k;m}\bar{V}_{i+\hat{k},l;m}\}]. \end{aligned} \tag{6.5}$$

$\tilde{V}_{i,5;k}$ is the link in direction 5 and $\tilde{V}_{i,k;5}$ is the link in direction $k$ both at location a $i$ and decorated in two spatial dimensions different from $k$. $\bar{V}_{i,M;k}$ are constructed with original thin links $U_{i,M}$ as

$$\bar{V}_{i,M;k} = Proj_{SU(2)}[(1-\alpha_3)U_{i,M} + \frac{\alpha_3}{2}\{U_{i,k}U_{i+\hat{k},M}U^{\dagger}_{i+\hat{M},k} + U^{\dagger}_{i,k}U_{i-\hat{k},M}U_{i+\hat{M},k}\}], \tag{6.6}$$

where $M = 1, 2, 3, 5$. We chose the parameters $\alpha_1 = 0.5$, $\alpha_2 = 0.4$ and $\alpha_3 = 0.2$ for $SU(2)$ orbifold.

## 6.2 Spectrum

Higgs boson masses and $Z$ boson masses are obtained by applying generalized eigenvalue problem to operators calculated in MC simulation. I use two sets of Higgs boson operators, see Eq. (3.26) and Eq. (3.27), and two sets of $Z$ boson operators, see Eq. (3.30) and Eq. (3.31) in section 3.5. The operators are calculated with certain levels of smeared fields specified later. Applying Generalized eigenvalue problem we get masses form these operators. I used two operator sets for each Higgs and $Z$ boson to improve the mass determination. I checked that the masses extracted from individual set of operators are the same as the masses from two sets of operators.

Higgs boson masses and $Z$ boson masses obtained from MC simulation are plotted on the Fig. 6.1. I have 5000 measurements and three levels of smearing between 15-45 for each operator. The blue points are the masses for $L = 32$, $N_5 = 4$, $\gamma = 1$ and $\beta = 1.66,\ 1.68,\ 1.9$ and the red points are the excited states. We cannot get excited state for $\beta = 1.9$. The green points are the masses for $L = 24$, $N_5 = 4$, $\gamma = 1$ and $\beta = 1.9$ where again we cannot get excited state. A summary of the data is in the Table 6.1.

From these data we see that the $Z$ boson has nonzero finite mass. Also comparing $L = 24$ with $L = 32$ we see that the masses do not go to zero as $L \to \infty$. This means there is SSB and supports the Mean-Field calculation. On the contrary the perturbative calculation gives zero $Z$ boson mass. For $L = 32$, $N_5 = 4$, $\gamma = 1$ and $\beta = 1.66$ the Yukawa mass extracted from the boundary static potential agrees well with Z boson mass in Table 6.1 [34].

In the right plot of Fig. 6.1, the magenta dashed line is the Higgs boson mass from perturbative formula Eq. (5.3). We see the non-perturbative Higgs boson masses are bigger than perturbative one and they seem to approach to the perturbative value as $\beta \to \infty$.

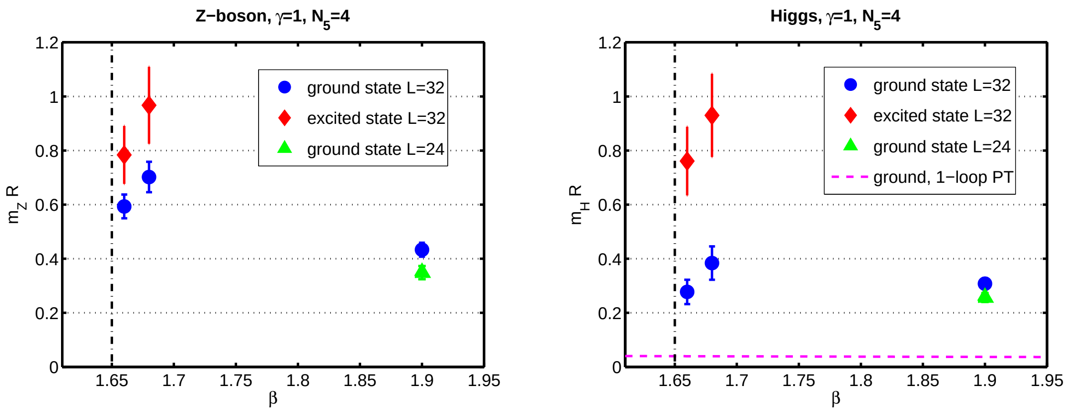



Figure6.1 Z boson mass and Higgs boson mass from Monte Carlo simulation

| $L$ | $N_5$ | $\gamma$ | $\beta$ | $m_H$ | $m_Z$ | $m^*_H$ | $m^*_Z$ |
|---|---|---|---|---|---|---|---|
| 32 | 5 | 1.0 | 1.66 | 0.217(35) | 0.466(34) | 0.598(99) | 0.616(84) |
| | | | 1.68 | 0.302(48) | 0.551(44) | 0.73(12) | 0.76(11) |
| | | | 1.9 | 0.242(16) | 0.340(20) | | |
| 24 | 5 | 1.0 | 1.9 | 0.202(13) | 0.271(14) | | |

Table6.1 The Higgs and Z boson spectrum from Monte Carlo simulation

# Chapter 7

# Conclusion

I have done the non-perturbative study of GHU using Mean-Field expansion and MC simulation. I worked on the pure $SU(2)$ gauge theory with orbifold boundary conditions and found out there is SSB even if there is no fermions. The most interesting parameter region in the Mean-Field is where the anisotropy parameter is $\gamma < 0.6$ near the critical point. In this parameter region Higgs boson can have the mass which is consistent with the Standard Model mass and we can take continuum limit along LCPs. Usually, 5-dimensional theory is non-renormalizable, so the theory depens on the cut-off, however, it is possible in the Mean-Field to take cut-off independent continuum limit in this model. Because there is 2nd order transition line in small $\gamma$ regime. Also, there is possibility to verify the model by experiments because 1st exited state of the $Z$ boson is around 1TeV in continuum limit. The spectrum computed from MC simulation confirms SSB as found in the Mean-Field.

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