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**ENUMERATION OF SPATIAL 2-BOUQUET GRAPHS
UP TO FLAT VERTEX ISOTOPY**

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ABSTRACT. We enumerate spatial 2-bouquet graphs, or spatial graphs having exactly one 4-valent vertex and no other vertices, up to flat vertex isotopy. In order to do that, we give a method of constructing all such graphs from 2-string tangles, and distinguished the resulting graphs by computing their Yamada polynomials. We then prove that there exist exactly 51 flat vertex isotopy classes of the prime spatial 2-bouquet graphs with up to six crossings.

§ 1. INTRODUCTION

A spatial graph is a graph embedded in \mathbb{R}^3 , and spatial graph theory has been considered not so much a part of graph theory as an extension of knot theory. In knot theory, classifying all knots and links is a basic theme and over six billion knots and links have been tabulated. There also in spatial graph theory exist earlier studies on classification of some spatial graphs. J. Simon [6] enumerated θ -curves with up to five crossings and K_4 -graphs with up to four crossings. R. Litherland [1] provided a table of prime θ -curves with up to seven crossings without proof, and H. Moriuchi [4] gave it a proof. Moriuchi also enumerated all the prime handcuff graphs with up to seven crossings in [2]. In these studies, spatial graphs are classified up to ambient isotopy as well as in knot theory. On the other hand, spatial graphs can be also classified up to flat vertex isotopy.

We shall recall a flat vertex isotopy. A spatial graph \tilde{G} embedded in \mathbb{R}^3 is called a *flat vertex spatial graph* if for each vertex v of \tilde{G} there exists a small neighborhood B_v of v and a plane P_v in \mathbb{R}^3 such that $\tilde{G} \cap B_v \subset P_v$. Two flat vertex spatial graphs \tilde{G}_1 and \tilde{G}_2 are *flat vertex isotopic* if there exists an ambient isotopy between them leaving the image of \tilde{G}_1 to be a flat vertex spatial graph at any level of the isotopy. The ambient isotopy in this case is called a *flat vertex isotopy*. It is easy to see that there is no difference between the classification up to flat vertex isotopy and the one up to ambient isotopy for spatial trivalent graphs. (Note that θ -curves, K_4 -graphs and handcuff graphs are all trivalent.) The simplest graph which may cause the difference between these two classifications is a *2-bouquet graph*, namely a connected graph which has exactly one 4-valent vertex and no other vertices.

The aim of this paper is to enumerate all the prime flat vertex spatial 2-bouquet graphs with up to six crossings: We construct a diagram of a flat vertex spatial 2-bouquet graph from a diagram of a 2-string tangle by connecting each end of a 4-valent flat vertex to that of the 2-string tangle diagram without increasing the number of crossings (see Figure 3 in §3).

It turns out that an arbitrary flat vertex spatial 2-bouquet graph can be obtained by this construction. We call a spatial 2-bouquet graph *prime* if an arbitrary 2-sphere which intersects the graph at two points divides it into a trivial arc and something. Because this definition is similar to that of a prime 2-string tangle, we can see the following facts: A flat vertex spatial 2-bouquet graph constructed from a prime 2-string tangle is prime, and a flat vertex spatial 2-bouquet graph constructed from a non-prime 2-string tangle is not prime. Therefore it is enough to prepare all prime 2-string tangles in order to obtain all prime flat vertex spatial 2-bouquet graphs. H. Yamano [8] gave a method of constructing prime 2-string tangles from algebraic tangles (see Lemma 1 in §3), and Moriuchi made tables of algebraic tangles with seven crossings or less in [3]. From these results, we obtain all the prime 2-string tangles with up to six crossings, and construct all the prime flat vertex spatial 2-bouquet graphs with up to six crossings. We next distinguish them by using Yamada polynomial [7], which is a one-variable Laurent polynomial associated to each spatial graph diagram.

Our main result is

Main Theorem. There exist exactly 51 flat vertex isotopy classes of the prime spatial 2-bouquet graphs with up to six crossings.

It follows easily from this theorem that the difference between the classification up to flat vertex isotopy and the one up to ambient isotopy actually occurs with spatial 2-bouquet graphs.

This paper is organized as follows. In §2 we set up notation and terminology. In §3 we give a method of constructing all prime 2-bouquet graphs from prime 2-string tangles, and construct those graphs with up to six crossings. In §4 we distinguish all those graphs constructed in §3 by calculating their Yamada polynomials, thereby proving Main Theorem. We then provide the table of prime flat vertex spatial 2-bouquet graphs in §5.

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§2. A FLAT VERTEX SPATIAL 2-BOUQUET GRAPH

A graph is called a *2-bouquet graph* if it is homeomorphic to the union of two circles with a single point in common: A 2-bouquet graph is a finite graph which has exactly one 4-valent vertex, some 2-valent vertices and no other vertices. For convenience, we assume in the following that a 2-bouquet graph has no 2-valent vertices as in the leftmost picture of Figure 1.

In this section, we first define a flat vertex graph and consider a flat vertex 2-bouquet graph. We next define a flat vertex spatial graph as a flat vertex graph embedded in \mathbb{R}^3 on a certain condition. Primeness of a flat vertex spatial 2-bouquet graph is defined in the end of this section.

Let G be a graph, which is not necessarily a 2-bouquet graph, and let v be a vertex of G . We assign a cyclic order to the edges gathering at v ,

and call this order a *cyclic order at v* . We shall call a graph having a cyclic order at each vertex a *flat vertex graph*. Two flat vertex graphs G_1 and G_2 are *flat-homeomorphic* if there exists a homeomorphism $f : G_1 \rightarrow G_2$ which preserves or reverses the cyclic order at each vertex. We call such a homeomorphism a *flat-homeomorphism*, and does not discriminate among flat-homeomorphic graphs in the following.

Let us assign a cyclic order to the unique 4-valent vertex of a 2-bouquet graph, and we call the resulting graph a *flat vertex 2-bouquet graph*. By the definition of a flat-homeomorphism, we can easily see that there exist exactly two flat-homeomorphism classes of 2-bouquet graphs: a flat vertex 2-bouquet graph of *type K* (see the upper middle picture of Figure 1), and a flat vertex 2-bouquet graph of *type L* (see the lower middle picture of Figure 1).

We then define a flat vertex spatial graph and consider a flat vertex spatial 2-bouquet graph.

Definition 1. Let G be a flat vertex graph and $f : G \rightarrow \mathbb{R}^3$ an embedding. The spatial graph $\tilde{G} = f(G)$ is called a *flat vertex spatial graph* of G if f satisfies the condition: There exists a small neighborhood $B_{f(v)}$ of $f(v)$ and a plane $P_{f(v)}$ in \mathbb{R}^3 for each vertex v of G such that the set $f(G) \cap B_{f(v)}$ is included in $P_{f(v)}$ and the edges gathering at v are mapped on $P_{f(v)}$ according to the cyclic order.

We shall call $\tilde{G} = f(G)$ a flat vertex spatial 2-bouquet graph of type K (respectively, type L) if G is a flat vertex 2-bouquet graph of type K (respectively, type L). Note that a knot is obtained from a flat vertex spatial 2-bouquet graph of type K by replacing the vertex of the graph locally with a crossing, and that a link is obtained from a flat vertex spatial 2-bouquet graph of type L in the same way (see the right pictures of Figure 1).

From now on, we call a flat vertex spatial 2-bouquet graph a *2-bouquet* for short.

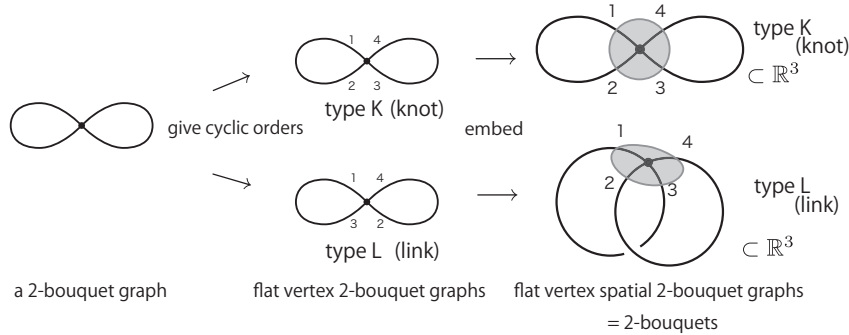


Figure 1

A flat vertex isotopy, which is also called a rigid vertex isotopy, is defined as follows.

Definition 2. Let \tilde{G}_1 and \tilde{G}_2 be flat vertex spatial graphs of a flat vertex graph G . \tilde{G}_1 and \tilde{G}_2 are *flat vertex isotopic* if there exists a continuous map $f : \mathbb{R}^3 \times I \rightarrow \mathbb{R}^3$, called a flat vertex isotopy, satisfying the following conditions:

- (1) $f_0 = \text{id}_{\mathbb{R}^3}$.
 - (2) $f_1(\tilde{G}_1) = \tilde{G}_2$.
 - (3) $f_t : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a homeomorphism for each $t \in I$.
 - (4) $f_t(\tilde{G}_1)$ is a flat vertex spatial graph of G for each $t \in I$.
- Here $f_t : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is defined by $f_t(x) = f(x, t)$ ($x \in \mathbb{R}^3$) for each $t \in I$.

It is easy to see that there is no difference between the classification up to flat vertex isotopy and the one up to ambient isotopy for spatial trivalent graphs. Hence the difference between these two classifications may occur for the first time with spatial 4-valent graphs. It motivates us to classify 2-bouquets, the simplest spatial 4-valent graphs, up to flat vertex isotopy.

In the end of this section we give a definition of primeness of a 2-bouquet.

Definition 3. A 2-bouquet \tilde{G} in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ is called *prime* if at least either of $\tilde{G} \cap B_1$ and $\tilde{G} \cap B_2$ is a trivial arc for an arbitrary 2-sphere S in S^3 which intersects \tilde{G} at exactly two points, where B_1 and B_2 are the 3-balls in S^3 satisfying $B_1 \cap B_2 = \partial B_1 = \partial B_2 = S$.

We can construct any non-prime 2-bouquet by connecting a prime 2-bouquet with some knots (see Figure 2 for example). Therefore we consider only prime 2-bouquets.

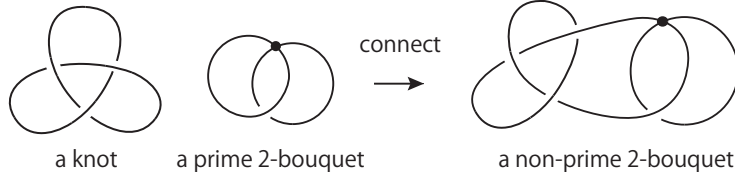


Figure 2

§ 3. CONSTRUCTION OF ALL PRIME 2-BOUQUETS

In this section, we give a method of constructing all prime 2-bouquets from prime 2-string tangles, and actually construct all those graphs with up to six crossings.

We shall recall the definition of a 2-string tangle. Let B be the unit 3-ball in \mathbb{R}^3 . A *2-string tangle* $T = (B, t)$ is a pair of disjoint arcs t in B with $t \cap \partial B = \partial t = \{(1/\sqrt{2}, 1/\sqrt{2}, 0), (1/\sqrt{2}, -1/\sqrt{2}, 0), (-1/\sqrt{2}, 1/\sqrt{2}, 0), (-1/\sqrt{2}, -1/\sqrt{2}, 0)\}$. Two 2-string tangles $T = (B, t)$ and $T' = (B, t')$ are *equivalent* if there exists an ambient isotopy $f : B \times I \rightarrow B$ from T to T' such that $f_t|_{\partial B} : \partial B \rightarrow \partial B$ is an orientation preserving congruent transformation for each $t \in I$. Here $f_t : B \rightarrow B$ is defined by $f_t(x) = f(x, t)$ ($x \in B$) for each $t \in I$. Up to this equivalence Moriuchi [3] classified algebraic tangles with seven crossings or less (see [3] for more details).

Our method of constructing 2-bouquets from 2-string tangles is as follows: Given a diagram of a 2-string tangle, we prepare one 4-valent flat vertex (see Figure 3), and connect each end of the 4-valent flat vertex to that of the 2-string tangle diagram without increasing the number of crossings. Consequently we obtain a diagram of a 2-bouquet as illustrated below. It is easy to see that the resulting 2-bouquets constructed from equivalent 2-string tangles are flat vertex isotopic.

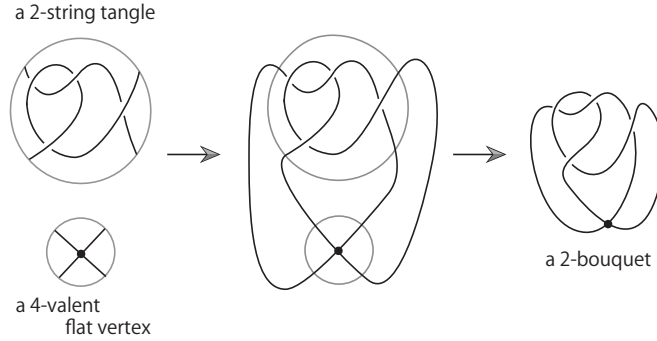


Figure 3

We then show that an arbitrary 2-bouquet can be obtained by this construction. Given a diagram D of an arbitrary 2-bouquet \tilde{G} . If necessary we prepare a solid torus containing \tilde{G} , with the hole of the solid torus is located quite near the vertex of \tilde{G} , and rotate the solid torus along the meridian through 180 degrees. Then we can obtain a diagram D' of \tilde{G} in the shape of some 2-string tangle connected with one 4-valent flat vertex (see Figure 4). Note that the number of crossings of the 2-string tangle is equal to that of D .

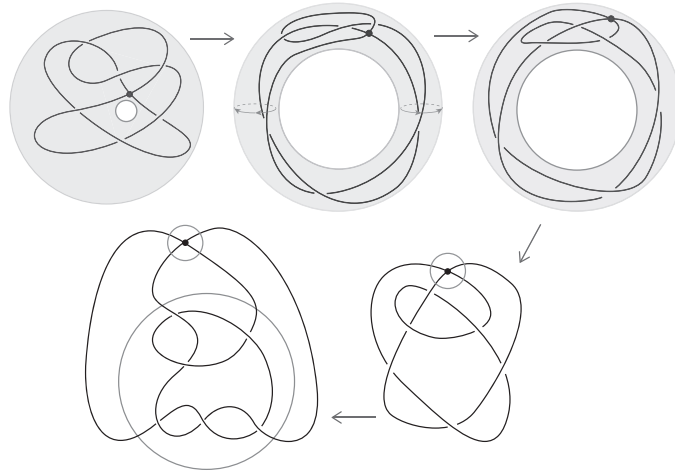


Figure 4

We shall recall a definition of primeness of a 2-string tangle (cf. [8]).

Definition 4 (H. Yamano). A 2-string tangle $T = (B, t)$ is called *prime* if $t \cap B'$ is a trivial arc for an arbitrary 3-ball B' in B whose boundary intersects t at exactly two points.

We can see the following facts: A 2-bouquet constructed from a prime 2-string tangle is prime, and a 2-bouquet constructed from a non-prime 2-string tangle is not prime. Therefore it turns out that it is enough to prepare all prime 2-string tangles in order to obtain all prime 2-bouquets.

Yamano [8] defined a *prime basic 4-regular disk graph*, which is a certain kind of planar graph inscribed inside a circle and whose vertices are all 4-valent (see Figure 5 for example). He proved the following lemma on prime 2-string tangles.

Lemma 1 (H. Yamano). For any prime 2-string tangle T with at least one crossing, there exists a prime basic 4-regular disk graph P such that a minimum crossing diagram of T is obtained by replacing a small disk neighborhood of each vertex of P with the diagram of some algebraic tangle with at least one crossing.

By using Lemma 1 and Moriuchi's table of equivalence classes of algebraic tangles in [3], we then obtain the following lemma.

Lemma 2. There exist exactly 51 equivalence classes of the prime 2-string tangles with six crossings or less, and each of those 51 equivalence classes is represented by an algebraic tangle.

Proof. It is easy to see that there exists one equivalence class of prime 2-string tangle with zero crossing. The equivalence class is represented by the algebraic tangle denoted by 0. It turns out to be enough from Lemma 1 to prepare prime basic 4-regular disk graphs with up to six vertices and algebraic tangles with up to six crossings in order to obtain all the prime 2-string tangles having at least one crossing up to six crossings.

Yamano [8] gave the list of prime basic 4-regular disk graphs with up to seven vertices, and in the list there exist exactly three graphs with up to six vertices as follows.

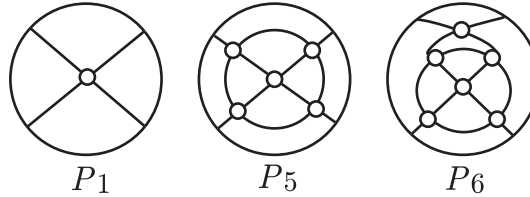


Figure 5

We now replace a disk neighborhood of each vertex of P_i ($i=1, 5$ or 6) with the diagram of an algebraic tangle as follows.

Case P_1 : We replace a disk neighborhood of the unique vertex of P_1 with the diagram of an algebraic tangle with six crossings or less.

Case P_5 -1: We replace a disk neighborhood of each of the five vertices of P_5 with an algebraic tangle diagram with one crossing.

Case P_5 -2: We choose four from the five vertices of P_5 , and replace a disk neighborhood of each of the four vertices with an algebraic tangle diagram with one crossing, and replace a disk neighborhood of the other vertex with an algebraic tangle diagram with two crossings.

Case P_6 : We replace a disk neighborhood of each of the six vertices of P_6 with an algebraic tangle diagram with one crossing.

In Case P_1 , we use the table of equivalence classes of algebraic tangles. Moriuchi [3] gave a list of the equivalence classes of algebraic tangles with at least one crossing up to seven crossings, and in the list there exist exactly 68 classes up to six crossings. However we cannot use 19 classes of them to obtain 2-string tangles because of containing a loop. We then obtain exactly 49 equivalence classes of prime 2-string tangles in Case P_1 , and each of those classes is represented by an algebraic tangle.

In Case P_5 -1 and Case P_6 , we use the fact that there exist exactly two algebraic tangles with one crossing, which are denoted by 1 or $\bar{1}$. It follows that any tangle diagram obtained by replacing each vertex of P_5 -1 or P_6 with tangle 1 or $\bar{1}$ contains a loop as in Figure 6. Hence we cannot obtain any 2-string tangle diagram in either Case P_5 -1 or Case P_6 .

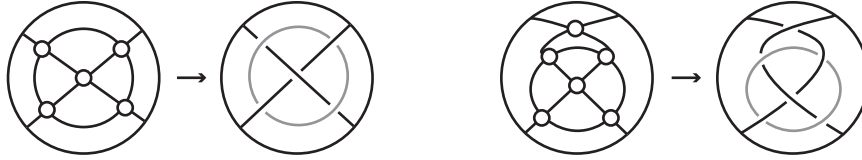


Figure 6

In Case P_5 -2, we use the fact that there exist exactly four algebraic tangles with two crossings, which are denoted by 2 , $\bar{2}$, $2\ 0$ or $\bar{2}\ 0$ as well as the fact that the algebraic tangles with one crossing are 1 and $\bar{1}$. After the replacement, we obtain a number of 2-string tangle diagrams. However all the obtained diagrams turn out to be equivalent to diagrams in Case P_1 except the diagram depicted in the middle picture of Figure 7. The diagram is not equivalent to any diagrams in Case P_1 . In fact, the pair $(6_2, 6_3)$ of numerator and denominator of this tangle is different from that of any tangle in Case P_1 , where 6_2 and 6_3 denote knots in the Rolfsen's knot table [5]. Moriuchi showed that this tangle is equivalent to the algebraic tangle $((\bar{3}, 2), 2)\ \bar{1}$ with eight crossings. We obtain another equivalence class of prime 2-string tangles in Case P_5 -2, which is represented by the algebraic tangle $((\bar{3}, 2), 2)\ \bar{1}$.

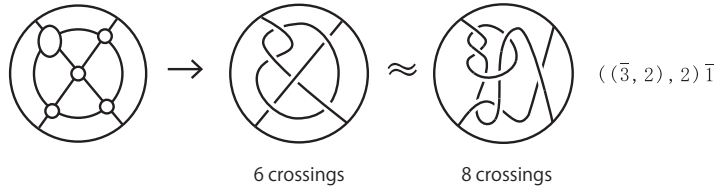


Figure 7

Therefore, there exist exactly 51 equivalence classes of prime 2-string tangles with up to six crossings.

□

We provide the table of all the prime 2-string tangles with six crossings and their resulting 2-bouquets in Table 1. All the equivalence classes of prime 2-string tangles in the first column are denoted by the notation of algebraic tangles (see [3]), and the notation of the 2-bouquets in the second column corresponds to the tables in §5. We shall prove in the next section that if two 2-string tangles in Table 1 are not equivalent, the constructed 2-bouquets are not flat vertex isotopic.

Table 1: The 2-string tangles with up to six crossings.

2-string tangles	2-bouquets	2-string tangles	2-bouquets
0	0_1^k	3 1 2	6_6^k
1	1_1^l	3 1 1 1	6_3^l
2	2_1^k	2 4	6_3^k
3	3_1^l	2 3 1	6_6^l
2 1	3_1^k	2 2 2	6_{10}^k
4	4_1^k	2 2 1 1	6_{13}^k
3 1	4_2^k	2 1 3	6_2^l
2 2	4_3^k	2 1 2 1	6_{11}^k
2 1 1	4_1^l	2 1 1 2	6_5^l
5	5_1^l	2 1 1 1 1	6_{15}^k
4 1	5_1^k	3, 2+	6_9^k
3 2	5_2^l	2 1, 2+	6_{12}^k
3 1 1	5_3^k	(3, 2) 1	6_7^l
2 3	5_2^k	(3, 2) $\bar{1}$	6_{10}^l
2 2 1	5_3^l	(3, $\bar{2}$) $\bar{1}$	6_{11}^l
2 1 2	5_5^k	(2 1, 2) 1	6_8^l
2 1 1 1	5_6^k	2 2, 2	6_{14}^k
3, 2	5_4^k	2 2, $\bar{2}$	6_{18}^k
3, $\bar{2}$	5_8^k	2 1 1, 2	6_{17}^k
2 1, 2	5_7^k	3, 3	6_7^k
6	6_1^k	3, $\bar{3}$	6_{19}^k
5 1	6_2^k	3, 2 1	6_4^l
4 2	6_4^k	3, $\bar{2}$ $\bar{1}$	6_{12}^l
4 1 1	6_1^l	2 1, 2 1	6_{16}^k
3 3	6_5^k	$((\bar{3}, 2), 2) \bar{1}$	6_9^l
3 2 1	6_8^k		

§ 4. CLASSIFICATION OF ALL CONSTRUCTED 2-BOUQUETS

S. Yamada [7] introduced a 1-variable Laurent polynomial associated to each spatial graph diagram.

Definition 5. Let g be a diagram of a spatial graph. Then $R(g)(A) \in \mathbb{Z}[A, A^{-1}]$ is defined recursively as follows.

- (1) $R(\bigcirc) = A + 1 + A^{-1}$.
- (2) $R(\bigtimes) = AR(\bigcup) + A^{-1}R(\bigcap) + R(\bigtimes)$.
- (3) $R(\bigtriangleright \! \! \! \leftarrow \! \! \! \bigtriangleleft) = R(\bigtriangleright) \bigtriangleleft + R(\bigtriangleleft) \bigtriangleright + R(\bigtimes)$, where e is a non-loop edge.
- (4) $R(g_1 \sqcup g_2) = R(g_1)R(g_2)$, where $g_1 \sqcup g_2$ denotes the disjoint union of spatial graph diagrams g_1 and g_2 .
- (5) $R(g_1 \vee g_2) = -R(g_1)R(g_2)$, where $g_1 \vee g_2$ denotes a wedge at a vertex of spatial graph diagrams g_1 and g_2 .

The *Yamada polynomial* $R(g)$ is a flat vertex isotopy invariant for a spatial graph up to multiplying $(-A)^n$ for some integer n (see [7] for more details).

We compute the Yamada polynomial of each 2-bouquet diagram constructed in §3. The results are listed in Table 2. The first number, which appears in the curly brackets, is the minimum degree of the polynomial. The next sequence of numbers gives the coefficients of the polynomial, beginning with the coefficient of the minimum degree term. For example, $\{-2\}(-1, -2, -3, -2, -1)$ denotes the polynomial $-A^{-2} - 2A^{-1} - 3 - 2A - A^2$.

Table 2: The Yamada polynomials of prime 2-bouquets with up to six crossings.

2-bouquets	Yamada polynomials
0_1^k	$\{-2\}(-1, -2, -3, -2, -1)$
2_1^k	$\{0\}(-1, -1, -2, -2, -2, -1)$
3_1^k	$\{-4\}(1, 1, 0, 0, -1, -1, -2, -2, -2, -2, -1)$
4_1^k	$\{2\}(-1, -1, -2, -1, -1, -1, -1, -1)$
4_2^k	$\{-6\}(1, 0, -1, 0, 0, 1, 0, 0, -1, -2, -2, -2, -2, -1)$
4_3^k	$\{-8\}(-1, -1, 0, 0, 0, 0, -1, -1, -2, -1, -1, -1)$
5_1^k	$\{-8\}(1, 0, 0, 1, 0, 0, -1, 0, 0, 0, 0, -1, -2, -2, -2, -2, -1)$
5_2^k	$\{-10\}(-1, -1, -1, -2, -2, -2, -2, -1, -1, 1, 1, 1)$
5_3^k	$\{-6\}(1, 1, 0, 0, 0, -1, -1, 0, -1, 0, -1, -1, -2, -2, 0, -1, -1)$
5_4^k	$\{-8\}(-1, -2, -2, -2, -3, -2, -1, -1, 0, 1, 0, 1, 1, 2, 1, 0, 1, -1, -1)$
5_5^k	$\{-4\}(-1, -1, -1, -2, -2, -1, -2, -1, -1, 0, 1, 1, 2, 0, -1)$
5_6^k	$\{-7\}(1, 0, -2, 0, 0, -1, 0, -1, -1, -2, -1, 0, -1, 0, 1, -1, -1)$
5_7^k	$\{-9\}(-1, -1, 1, 1, 0, 2, 1, 0, 1, 0, 0, -2, -1, -1, -3, -2, -1, -2, -1)$
5_8^k	$\{-6\}(1, 0, 0, 0, -1, 0, -1, -1, -2, -2, -1, -1, -1)$

Table 2: The Yamada polynomials of prime 2-bouquets with up to six crossings (continued).

2-bouquets	Yamada polynomials
6_1^k	$\{4\}(-1, -1, -2, -1, -1, 0, 0, -1, -1, -1)$
6_2^k	$\{-10\}(1, 0, 0, 0, -1, 0, 0, 1, 0, 0, 0, 0, -1, -2, -2, -2, -2, -1)$
6_3^k	$\{-12\}(1, 1, 1, -1, 0, 0, 0, 0, -1, 0, -1, -1, -1, -1)$
6_4^k	$\{-7\}(-1, 0, -1, -2, 0, -1, -1, -2, -1, 0, 0, 1, 1, 0, 0, 0, -1, -1)$
6_5^k	$\{-11\}(-1, -1, -1, -2, -2, -1, -1, 0, 1, 0, 0, -2, 0, 0, 0, 1)$
6_6^k	$\{-5\}(-1, -1, -1, -2, -2, -1, -1, -1, 1, 0, 1, 0, 0, 0, -1, 1, 0, -1)$
6_7^k	$\{-9\}(-1, -2, -2, -2, -3, -2, 0, 0, 1, 2, 1, 0, 1, -1, -1, -2, 0, 1, 0, 2, 0, -1)$
6_8^k	$\{-8\}(1, 0, -1, 1, 1, -3, 0, -2, -2, 0, 2, 3, -1, -1, 0, -3, -2, 0, -1, -1)$
6_9^k	$\{-10\}(-1, -1, 0, -1, -2, 0, -1, -4, 0, -2, -1, 0, 2, 3, -1, 0, 1, -2, 0, 1)$
6_{10}^k	$\{-6\}(-1, 0, 0, -4, 0, -1, -1, 0, 1, 1, -2, -2, 0, -1, 0, 2, 0, -1)$
6_{11}^k	$\{-10\}(-1, -1, 1, 0, -2, 0, -1, -3, -1, 0, -1, 0, 0, 1, -1, 0)$
6_{12}^k	$\{-8\}(1, 1, -1, 0, 0, -3, -2, -1, -3, -1, -1, 0, -1, -1, 6, 0, 0, 2, 0, -1)$
6_{13}^k	$\{-8\}(1, 0, -2, 0, 1, -1, 1, 2, 0, 1, 0, 0, -2, -2, 0, -3, -3, 0, -1, -1)$
6_{14}^k	$\{-10\}(-1, -1, 1, 1, -1, 1, 1, -2, 0, 0, -2, -1, -1, 0, -2, -1, 2, -1, -1, 1, -1, -1)$
6_{15}^k	$\{-8\}(1, 0, -2, 1, 1, -2, 0, 0, -3, -2, -2, -1, -2, -1, 2, -1, 0, 3, 0, -1)$
6_{16}^k	$\{-10\}(-1, -2, -1, -2, -4, -1, -1, -3, 1, 3, 1, 2, 1, 1, -2, -2, 1, -2, 0, 3, 0, -1)$
6_{17}^k	$\{-10\}(-1, -1, 1, 0, -2, 1, 0, -3, 0, 0, -2, -1, -1, 0, -2, 0, 3, -1, 0, 2, -1, -1)$
6_{18}^k	$\{-7\}(1, 0, -1, 0, -1, -1, 0, -1, 0, -1, -1, -1, -2, 0, 0, -1)$
6_{19}^k	$\{-8\}(1, 0, 1, -1, 0, 2, -3, -1, -6, -2, -2, 0, 1, 0, 0, 1)$
1_1^l	$\{-1\}(1, 1, 1)$
3_1^l	$\{1\}(1, 1, 2, 1, 0, -1, -1)$
4_1^l	$\{-5\}(1, 1, 0, 0, 0, -1, 0, 0, 1, 1, 1, 1, -1, -1)$
5_1^l	$\{3\}(1, 1, 2, 1, 1, 0, -1, -1, -1)$
5_2^l	$\{-5\}(-1, 0, 0, -1, 1, 1, 2, 1, 1, 1, 0, 0, 0, -1, -1)$
5_3^l	$\{-7\}(1, 0, -1, 1, 1, 0, 1, 0, 0, -1, 0, 1, 0, 1, 1, -1, -1)$
6_1^l	$\{-7\}(1, 1, 0, 0, 0, -1, -1, 0, 0, 0, 1, 0, 1, 1, 2, 1, -1, 0, -1, -1)$
6_2^l	$\{-3\}(1, 1, 1, 1, 0, -1, 0, -1, 0, 0, 0, 1, 0, 1, 0, -1)$
6_3^l	$\{-9\}(1, 0, -1, 2, 0, -2, 0, -1, 0, 0, 2, 2, 0, 1, 1, -1, 0, 1, -1, -1)$
6_4^l	$\{-11\}(-1, -1, 1, 0, 0, 3, 1, 0, 2, 0, -1, -3, -1, 0, -1, 2, 2, 0, 1, 1, -1, -1)$
6_5^l	$\{-11\}(-1, 0, 2, 0, 0, 1, -1, -1, 0, 0, 1, 0, 2, 1, -1, 1, 0, -1)$
6_6^l	$\{-9\}(1, 0, 0, 2, 0, -1, 1, 0, 0, -1, 0, 0, -1, 1, 1, 0, 1, 1, -1, -1)$
6_7^l	$\{-12\}(-1, 0, 1, -1, 1, 1, 0, 0, -1, 0, -1, 0, 1, 0, 0, 1, 0, 0, 1, 1)$
6_8^l	$\{-11\}(-1, 0, 2, -1, -1, 2, 0, 0, 2, 1, 1, -1, 1, 0, -2, 1, 0, -2, 0, 1)$
6_9^l	$\{-10\}(-1, -1, 2, 1, -2, 2, 2, -2, 1, 2, -1, 0, 0, 1, -2, -1, 3, -2, -1, 3, 0, -1)$
6_{10}^l	$\{-7\}(-1, 0, 0, 0, 1, 0, 0, -1, 0, 1, 1, 1, 1)$
6_{11}^l	$\{-7\}(1, 1, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, -1)$
6_{12}^l	$\{-7\}(-1, -1, 0, -1, 0, 3, 3, -6, 2, 0, 0, -2, -1, -1, -3, 1, 1)$

We see that the polynomials listed above are mutually distinct up to multiplying $(-A)^n$ for some integer n . Therefore all the 2-bouquets we constructed in §3 are not flat vertex isotopic.

We finally investigate whether each of the obtained 2-bouquets is of type K or of type L , to obtain the following theorem.

Main Theorem. There exist exactly 33 flat vertex isotopy classes of the prime 2-bouquets of type K and exactly 18 classes of type L with up to six crossings.

§ 5. TABLE

We provide the table of prime 2-bouquets of type K and that of type L as follows. The 2-bouquets are listed in order of the crossing numbers.

The prime 2-bouquet graphs of type K with up to six crossings


 0_1^k

 2_1^k

 3_1^k

 4_1^k

 4_2^k

 4_3^k

 5_1^k

 5_2^k

 5_3^k

 5_4^k

 5_5^k

 5_6^k

 5_7^k

 5_8^k

 6_1^k

 6_2^k

The prime 2-bouquet graphs of type K with up to six crossings (continued)



The prime 2-bouquet graphs of type L with up to six crossings


 1_1^l

 3_1^l

 4_1^l

 5_1^l

 5_2^l

 5_3^l

 6_1^l

 6_2^l

 6_3^l

 6_4^l

 6_5^l

 6_6^l

 6_7^l

 6_8^l

 6_9^l

 6_{10}^l

 6_{11}^l

 6_{12}^l

Remark Yamano gave a list of the prime basic 4-regular disk graphs with seven vertices or less in [8], and Moriuchi gave a list of the algebraic tangles with seven crossings or less up to equivalence in [3]. Therefore we can obtain all the prime 2-string tangles with seven crossings in the same way, and also construct all the prime 2-bouquets with seven crossings. However we provide the tables with up to only six crossings in this paper because the situation becomes much more complicated. In fact, there exists three more prime basic 4-regular disk graphs with seven vertices and 132 algebraic tangles with seven crossings.

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