Chapter 1.

Introduction

This thesis consists of the following five papers.

- (1) "Classification of some rules of cellular automata" (with F. Takeo)
- (2) "The limit set of cellular automata" (with F. Takeo)
- (3) "Convergence to the limit set of linear cellular automata" (with F. Takeo)
- (4) "A note on the property of linear cellular automata"
- (5) "Convergence to the limit set of linear cellular automata, II"

A cellular automaton consists of a finite-dimensional lattice of sites, each of which takes an element of a finite set $\mathbb{Z}_M = \{0, 1, ..., M-1\}$ $(M \in \mathbb{N})$ of integers at each time step and the value of each site at any time step is determined as a function L of the values of the neighboring sites at the previous time step. Patterns of cellular automata are investigated in many ways.

From investigation of a large sample of cellular automata, S. Wolfram suggests that many (perhaps all) cellular automata fall into four basic behavior classes.

For a linear rule L with M=2, S. J. Willson considered the set of sites which takes the value 1 when time increases from 0 to 2^n in a product space $\mathbb{Z}^d \times \mathbb{Z}_+$, that is,

$$K(n,\omega) = \{(x,t) \in \mathbb{Z}^d \times \mathbb{Z}_+ \mid 0 \le t \le 2^n, \ (L^t\omega)(x) = 1\}.$$

He showed the existence of a limit set of $\{K(n,\omega)/2^n\}$ in the sense of Kuratowski limit and that limit set does not depend on an initial condition.

In case of $M = p^r$ (p is prime and $r \in \mathbb{N}$), S. Takahashi considered the set of sites which takes the positive integer with an initial configuration δ , where δ

takes 1 at the origin and 0 at other sites, when time increases from 0 to $p^n - 1$, that is,

$$K(n,\delta) = \{(x,t) \in \mathbb{Z}^d \times \mathbb{Z}_+ \mid 0 \le t \le p^n - 1, \ (L^t \delta)(x) \ne 0\}$$

and showed the existence of the limit set $\{\frac{K(n,\omega)}{p^n}\}$. Moreover Takahashi investigated the set of sites which takes the value $j(j=1,2,\ldots,p^r-1)$ when time increases from 0 to p^n-1 , that is,

$$K_b(n,\delta) = \{(x,t) \in \mathbb{Z}^d \times \mathbb{Z}_+ \mid 0 \le t \le p^n - 1, \ (L^t \delta)(x) = b \pmod{p^r} \}$$

and showed the existence of a limit set of $\{K_b(n,\delta)\}$.

Chapter 2 concerns with Wolfram's four classes. Four classes were defined as follows.

Class 1 Evolution leads to a homogeneous state.

Class 2 Evolution leads to a set of separated simple stable or periodic structures.

Class 3 Evolution leads to a chaotic pattern.

Class 4 Evolution leads to complex localized structures, sometimes long-lived.

We redefine four classes mathematically and discuss classification of rules of the simplest cellular automata without simulation.

In Chapter 3, we consider the case M = 2. Let \mathcal{P} be the set of all configurations: $\mathbb{Z}^d \to \mathbb{Z}_2$. A map $L \colon \mathcal{P} \to \mathcal{P}$ is a transition rule if (1) L(0) = 0; and (2) there exist $v_1, \ldots, v_m \in \mathbb{Z}^d$ and a map $f : (\mathbb{Z}_2)^m \to \mathbb{Z}_2$ such that

$$(La)(x) = f(a(x + v_1), \dots, a(x + v_m))$$
 for all $x \in \mathbb{Z}^d, a \in \mathcal{P}$.

We consider the space USC of all upper semi continuous functions $g: \mathbb{R}^d \times [0,1] \to \mathbb{Z}_2$ and consider an operator on USC. We define the product space $\prod E_k$ and an operator \bar{F}_L on USC corresponding to L. We investigate whether the limit set of \bar{F}_L^k as $k \to \infty$ belongs to a certain subspace E_∞ and the relation between $\lim G_L^k a$ and $\lim \bar{F}_L^k g$ [Theorem 1 in Chapter 3]. We consider a quotient

space $\tilde{E} = \prod E_k/\sim$ and the operator \tilde{F}_L on it and investigate conditions that the \tilde{F}_L -invariant set belongs to a certain subspace. In Section 3, we consider the case of linear rules. In Section 4, we consider the case that L is non-linear. We show some conditions for L and initial configurations such that there exists a \tilde{F}_L -invariant set [Theorems 5, 6 and 7 in Chapter 3].

In Chapter 4, we investigate the case M=p, where p is prime. Let \mathcal{P} be the set of all configurations $a: \mathbb{Z}^d \to \mathbb{Z}_p$ with compact support. We define $\delta \in \mathcal{P}$ as

$$\delta(x) = \begin{cases} 1 & (x = 0), \\ 0 & (x \neq 0) \end{cases}$$

and a linear cellular automata rule $L \colon \mathcal{P} \to \mathcal{P}$ as follows:

$$(La)(x) = \sum_{j \in G} \alpha_j a(x + k_j) \pmod{p} \quad \text{for } a \in \mathcal{P}, \tag{1.1}$$

where G is a finite subset of \mathbb{Z} with $\sharp G \geq 2$, $k_j \in \mathbb{Z}^d$ $(j \in G)$ is a neighboring site of origin, $\alpha_k \in \mathbb{Z}_p \setminus \{0\}$.

Let

$$X_n = \{ (\frac{x}{p^n}, \frac{t}{p^n}) \in \mathbb{R}^d \times [0, 1] \mid x \in \mathbb{Z}^d, t \in \mathbb{Z}_+, 0 \le t \le p^n \}$$

for $n \in \mathbb{N}$. For $j \in \mathbb{N}$, put

$$G_j = \{ \ell \in \mathbb{Z}^d \mid (L^j \delta)(\ell) \neq 0 \}.$$

Define a map ψ_n from \mathcal{P} to the function space on $\mathbb{R}^d \times [0,1]$ for $a \in \mathcal{P}$ and $n \in \mathbb{N}$ by

$$(\psi_n(a))(\frac{x}{p^n}, \frac{t}{p^n}) = \begin{cases} (L^t a)(x) & \text{if } (\frac{x}{p^n}, \frac{t}{p^n}) \in X_n, \\ 0 & \text{if } (\frac{x}{p^n}, \frac{t}{p^n}) \in (\mathbb{R}^d \times [0, 1]) \setminus X_n. \end{cases}$$

 $\psi_n(a)$ represents the state $L^t(a)(x)$ for $0 \le t \le p^n$ and $x \in \mathbb{Z}^d$. We shall define an operator T as follows.

Define a map $S_{\ell,j} \colon \mathbb{R}^d \times [0,1] \to \mathbb{R}^d \times \left[\frac{j}{n}, \frac{j+1}{n}\right]$ by

$$S_{\ell,j}(x,t) = (\frac{x}{p}, \frac{t}{p}) + (\frac{\ell}{p}, \frac{j}{p}).$$

By using maps $S_{\ell,j}$ define an operator T on the space of functions on $\mathbb{R}^d \times [0,1]$ by

$$Tg(y,q) = \sum_{\ell \in G_j} (L^j \delta)(\ell) g(S_{\ell,j}^{-1}(y,q)) \pmod{p}$$

for $\frac{j}{p} < q \le \frac{j+1}{p}$ with $0 \le j \le p-1$ and

$$Tg(y,0) = g(py,0)$$

for a function g on $\mathbb{R}^d \times [0,1]$. Then $T(\psi_n) = \psi_{n+1}$ holds for all $n \in \mathbb{N}$.

We have the following theorem which concerns the limit function $\{\psi_n(a)\}$ in the pointwise topology.

Theorem I (Theorem 2.5 in Chapter 4). For $a \in \mathcal{P}$ with $a(0) \neq 0$, we have the following assertions:

- (1) $\psi_n(a)$ converges to a function on $\mathbb{R}^d \times [0,1]$ in the pointwise topology.
- (2) The limit function g_a of the sequence $\{\psi_n(a)\}$ in the pointwise topology is T-invariant, that is, $Tg_a = g_a$.
- (3) As for the limit functions g_{δ} and g_{a} of $\{\psi_{n}(\delta)\}$ and $\{\psi_{n}(a)\}$ respectively, we have $a(0)g_{\delta}=g_{a}$.

In order to investigate convergence of $\{\psi_n(\omega)\}$, we shall introduce two metrics d_f , D_f in the space of \mathbb{Z}_p -valued upper semi-continuous functions on a compact subset of $\mathbb{R}^d \times [0,1]$. Let USC be the space of \mathbb{Z}_p -valued upper semi-continuous functions on $\mathbb{R}^d \times [0,1]$, where \mathbb{Z}_p -valued upper semi-continuous functions mean upper semi-continuous functions embedded in \mathbb{R} -valued function spaces.

Let K be a compact subset of $\mathbb{R}^d \times [0,1]$ and (y_0,q_0) be a point of $(\mathbb{R}^d \times [0,1]) \setminus K$. Let

$$USC|_K = \{g \in USC \mid \text{ support of } g \subset K\}.$$

By using the Hausdorff distance D(A, B) of non-empty compact sets A and B in $\mathbb{R}^d \times [0, 1]$, we shall define the pseudodistance $D_0(A, B)$ of A and B in

 $\mathbb{R}^d \times [0,1]$ by

$$D_0(A, B) = D(A \cup \{(y_0, q_0)\}, B \cup \{(y_0, q_0)\})$$

and metrics d_f , D_f in $USC|_K$ as follows:

$$d_f(g_1, g_2) = \max_{1 \le j \le p-1} D_0(\overline{g_1^{-1}(j)}, \overline{g_2^{-1}(j)}),$$

$$D_f(g_1, g_2) = \max_{1 \le s \le p-1} D_0(g_1^{-1}[s+], g_2^{-1}[s+]),$$

for $g_1, g_2 \in USC|_K$, where $g^{-1}[s+] = \{(x,t) \mid g(x,t) \geq s\}$ and $g_1^{-1}(j)$ is the closure of the set $g_1^{-1}(j) = \{(x,t) \mid g(x,t) = j\}$. Then the following theorem holds.

Theorem II (Theorem 3.5 in Chapter 4). For $\{f_n\} \subset USC|_K$, suppose $d_f(f_n, f_m) \to 0$ as $n, m \to \infty$. Let $g = \bigwedge_{k=1}^{\infty} \bigvee_{n \ge k} f_n$. Then we have

$$D_f(f_n,g) \to 0 \ as \ n \to \infty.$$

Using the above theorem, we have

Theorem III (Theorem 6.4 in Chapter 4). For a nonzero $a \in \mathcal{P}$, the following assertions hold:

- (1) $d_f(\psi_n(a), \psi_m(a)) \to 0 \text{ as } n, m \to \infty.$
- (2) Put $f_a = \bigwedge_{k>1} \bigvee_{n>k} \psi_n(a) \in USC$. Then we have

$$D_f(\psi_n(a), f_a) \to 0$$
 as $n \to \infty$.

and

Theorem IV (Theorem 6.5 in Chapter 4). For $a \in \mathcal{P}$ with $a(0) \neq 0$, let $Y_a = \bigcap_{k=1}^{\infty} \overline{\bigcup_{n \geq k} \frac{K(n,a)}{p^n}}$ and g_a be defined by $g_a(y,q) = \lim_{n \to \infty} (\psi_n(a))(y,q)$ in the pointwise topology. Then the following assertions hold:

(1) The characteristic function 1_{Y_a} of the set Y_a satisfies

$$\hat{g}_a = (p-1)1_{Y_a}$$

and

$$\hat{g}_a = \bigwedge_{k>1} \bigvee_{n>k} \psi_n(a),$$

where $\hat{g}(x,t) = \inf\{\phi(x,t)|\phi \in USC, \phi(x,t) \geq g(x,t)\}.$

(2) Though g_a is not necessarily the same as g_{δ} for any $a \in \mathcal{P}$ as shown in Theorem I, the upper envelope \hat{g}_a of g_a is the same, that is,

$$\hat{q}_a = \hat{q}_\delta = f_a = f_\delta$$

where
$$f_a = \bigwedge_{k>1} \bigvee_{n\geq k} \psi_n(a)$$
.

Therefore in case of M=p, the upper envelope of a limit function, \hat{g}_a , takes either 0 or p-1 and does not depend on an initial configuration.

In Chapter 5, we investigate the property of linear cellular automata in order to extend the results in Chapter 4 to the case $M = p^r$ with $r \in \mathbb{N}$. We show the following theorem for a linear cellular automata rule L satisfying some condition, which we call it the condition (A) and it will be described in Chapter 5.

Theorem V (Theorem 2.7 in Chapter 5). For a prime number p and $r \in \mathbb{N}$, let L be defined as the equation (1.1) and the summation \sum is taken as the summation with mod p^r and satisfy the condition (A). Put $t(r,j) = j(p^r - p^{r-1})$ and $i(r,j) = -(t(r,j) - p^{r-1})r_1 - p^{r-1}r_2$, where $r_1, r_2 \in G$ satisfy (I),(III),(III) and (IV) of the condition (A).

Then the set $\{L^{t(r,j)}\delta_0(i(r,j)) | 1 \le j \le p^r\}$ is a p^r -set, where the set $\{a_n | n = 1, \ldots, k\}$ is a k-set if the set has one-to-one, onto correspondence with the set $\{0, 1, \ldots, k-1\}$.

In Chapter 6, we extend the result above to mod p^r , where p is prime and $r \in \mathbb{N}$, that is, L is defined as the equation (1.1) and the summation \sum is taken as the summation with mod p^r . We have a similar result to the case of M = p.

Theorem VI (Theorem 2.3 in Chapter 6). For $a \in \mathcal{P}$ with $a(0) \neq 0$, we have the following assertions:

- (1) The sequence $\{\psi_n(a)\}$ converges to a function on $\mathbb{R}^d \times [0,1]$ in the pointwise topology.
- (2) The limit function g_a of the sequence $\{\psi_n(a)\}$ in the pointwise topology is T-invariant, that is, $Tg_a = g_a$, where

$$Tg(y,q) = \sum_{\ell \in G_{jp^{r-1}}} (L^{jp^{r-1}} \delta)(\ell) g(S_{\ell,j}^{-1}(y,q))$$

for $\frac{j}{p} < q \le \frac{j+1}{p}$ with $0 \le j \le p-1$ and

$$Tg(y,0) = g(py,0)$$

and

$$S_{\ell,j}(x,t) = (\frac{x}{p}, \frac{t}{p}) + (\frac{\ell}{p^r}, \frac{j}{p}).$$

- (3) As for the limit functions g_{δ} and g_{a} of $\{\psi_{n}(\delta)\}$ and $\{\psi_{n}(a)\}$ respectively, we have $a(0)g_{\delta} = g_{a}$.
- S. Takahashi investigated the set of "b-state"

$$K_b(n,\delta) = \{(x,t) \in \mathbb{Z}^d \times \mathbb{Z}_+ \mid 0 \le t \le p^n - 1, \ (L^t \delta)(x) = b \pmod{p^r} \}$$

for $b \in \{1, \dots, p^r - 1\}$ and the set

$$K^f(n,\delta) = \{(x,t) \in \mathbb{Z}^d \times \mathbb{Z}_+ \mid 0 \le t \le p^n - 1, \ (L^t \delta)(x) \ne 0 \pmod{p^f}\}$$

for $f \in \{1, 2, ..., r\}$ and for $b \in \mathbb{Z}_{p^r}$ satisfying $b/p^{f-1} \in \mathbb{N}$ and $b/p^f \notin \mathbb{N}$, he showed a limit set of $\{K^f(n, \delta)/p^n\}$ is equal to a limit set of $\{K_b(n, \delta)/p^n\}$ in the sense of Kuratowski limit.

We show the relationship between the limit function and the limit set of $\{\frac{K^f(n,\omega)}{n^n}\}$. We define that an element $j \in G$ is prime if $\alpha_j/p \notin \mathbb{N}$.

Theorem VII (Theorem 5.2 in Chapter 6). Suppose that the set G in (1.1) with mod p^r has at least two prime elements. Let the function g_{δ} be defined by $g_{\delta}(y,q) = \lim_{n\to\infty} (\psi_n(\delta))(y,q)$ and Y_f be the limit set of $\{K^f(n,\delta)/p^n\}$ in the sense of Kuratowski limit.

Then

(1) the relation between \hat{g}_{δ} and $\{Y_f\}$ is as follows:

$$\hat{g}_{\delta} = \sum_{1 \le f \le r} (p^{r+1-f} - 1) p^{f-1} 1_{Y_f \setminus \bigcup_{i=1}^{f-1} Y_i},$$

where $\hat{g}(x,t) = \inf\{\phi(x,t)|\phi\in USC, \phi(x,t)\geq g(x,t)\}.$

(2) The relation between \hat{g}_{δ} and $\{\psi_n(\delta)\}$ is as follows:

$$\hat{g}_{\delta} = \bigwedge_{k=1}^{\infty} \bigvee_{n>k} \psi_{n+r-1}(\delta).$$

Theorem VIII (Theorem 5.3 in Chapter 6). Suppose that the set G in (1.1) with mod p^r has at least two prime elements. Put $g_a(y,q) = \lim_{n\to\infty} (\psi_n(a))(y,q)$ for $a \in \mathcal{P}$ with $a(0) = kp^j$ $(k/p \notin \mathbb{N} \ and \ j \in \{0,1,\ldots,r-1\})$. Then we have the following assertions.

(1) the relation between \hat{g}_a and $\{Y_f\}$ is as follows:

$$\hat{g}_a = \sum_{1 \le f \le r-j} (p^{r+1-f-j} - 1) p^{f-1+j} 1_{Y_f \setminus \bigcup_{i=1}^{f-1} Y_i}.$$

(2) The relation between \hat{g}_a and $\{\psi_n(a)\}$ is as follows:

$$\bigwedge_{k=1}^{\infty} \bigvee_{n\geq k} \psi_{n+r-1}(a) = \bigvee_{x\in G_a} \hat{g}_{\tau_x(a)}.$$

Though Takahashi showed the existence of the limit set of each value of \mathbb{Z}_{p^r} separately, we can consider the convergence to the limit sets of some values simultaneously by using the functions $\{\psi_n\}$.