

Chapter 2.

Classification of some rules of cellular automata

1 Introduction

Cellular automata are a class of mathematical systems characterized by discreteness (in space, time and state values), determinism and local interaction.

Let \mathbb{Z} and \mathbb{N} be the set of integers and the set of natural numbers. A cellular automata consists of d-dimensional lattice (\mathbb{Z}^d , $d \in \mathbb{N}$), and each site takes a state, one of the values from the set $\mathbb{Z}_k = \{0, \dots, k-1\}$, where k is a natural number. $x_i^t \in \mathbb{Z}_k$ denotes the state of a site $i \in \mathbb{Z}^d$ at time $t \in \mathbb{N}$. The state of a site i at time $t+1$ is determined from the states of its neighborhood $i-r, \dots, i+r$, at time t , i.e.

$$x_i^{t+1} = f(x_{i-r}^t, \dots, x_{i-1}^t, x_i^t, x_{i+1}^t, \dots, x_{i+r}^t),$$

where $f: (\mathbb{Z}_k)^{2r+1} \rightarrow \mathbb{Z}_k$ represents the “rule” defining the automata, (f is called *a rule function*), and parameter r determines the “range” of the rule.

Based on investigation of a large sample of cellular automata, it suggests that many (perhaps all) cellular automata fall into four basic behavior classes. In ref. [2], four classes were defined as follows.

Class 1 Evolution leads to a homogeneous state.

Class 2 Evolution leads to a set of separated simple stable or periodic structures.

Class 3 Evolution leads to a chaotic pattern.

Class 4 Evolution leads to complex localized structures, sometimes long-lived.

In this paper we discuss classification of rules of the simplest cellular automata without simulation. The simplest cellular automata are those with $r = 1$ and $k = 2$, there automata are defined on a one-dimensional spatial lattice, and

consist of binary-valued sites evolving in time according to a nearest-neighbor interaction rule. Since the domain of f is the set of 2^3 possible 3-tuples, the rule function f is completely defined by specifying the “rule table” of values $c_i \in \{0, 1\}$ with $i = 0, 1, \dots, 7$ such that

$$000 \rightarrow c_0, 001 \rightarrow c_1, \dots, 111 \rightarrow c_7,$$

where $xyz \rightarrow c_i$ indicates that $f(xyz) = c_i$. There is a total of $2^8 = 256$ distinct rules. The conventional labeling scheme [3] assigns the integer

$$R = \sum_{i=0}^7 c_i 2^i \quad (1.1)$$

to the rule defined by f . The rule number thus assumes an integer value between 0 and 255.

We redefine four classes mathematically and in Section 3 show another expression (3.1) of rules. This expression is useful to see the property of rules for classification. Since some linear rules are studied in [1], the other linear rules are investigated in Section 4 and the nonlinear rules are investigated in Section 5 using the expression of Section 3. Finally we classify some rules of the simplest cellular automata using the results of Section 4 and 5.

2 Injectivity and linearity

A rule \mathbf{R} is said to be *linear* if the function f defining the rule \mathbf{R} satisfies additivity condition, that is, for \mathbf{y} and $\mathbf{z} \in X = \{0, 1\}^{2r+1}$,

$$f(\mathbf{y}) \dot{+} f(\mathbf{z}) = f(\mathbf{y} \dot{+} \mathbf{z}),$$

where “ $\dot{+}$ ” denotes binary addition. A rule \mathbf{R} is said to be *injective in the $(i+m)$ th component* ($m \in \{-r, \dots, -1, 0, 1, \dots, r\}$) if for every tuple

$$(x_{i-r} \dots x_{i-1} x_i x_{i+1} \dots x_{i+r}) \in X,$$

the rule table for \mathbf{R} represents a one-to-one mapping between x_{i+m} and $f(x_{i-r} \dots x_{i-1} x_i x_{i+1} \dots x_{i+r})$ when the other components x_{i+j} ($j \neq m$) are fixed.

In this section, the relationship between injectivity and linearity is discussed. It is easy to check injectivity of a rule, but difficult to check linearity of a rule by definition of \mathbf{R} (1.1). The following proposition asserts that a rule is linear if the function f defining the rule \mathbf{R} is injective and satisfies a certain condition.

Notation 1. 1. For $x \in \{0, 1\}$, let $\bar{x} := x+1$.

2. For $\mathbf{x} = x_{i-r}x_{i-r+1} \dots x_i \dots x_{i+r-1}x_{i+r}$, let

$$\bar{\mathbf{x}}^m := x_{i-r}x_{i-r+1} \dots \bar{x}_{i+m} \dots x_{i+r-1}x_{i+r}.$$

3. For the function f defining the rule \mathbf{R} , let

$$X_0^{\mathbf{R}} := \{x \in X \mid f(\mathbf{x}) = 0\}$$

and

$$X_1^{\mathbf{R}} := \{x \in X \mid f(\mathbf{x}) = 1\},$$

where $X = \{0, 1\}^{2r+1}$.

Proposition 2.1. *If each site takes one of the values from $\mathbb{Z}_2 = \{0, 1\}$, the following (I) and (II) are equivalent.*

(I) \mathbf{R} is a linear automata rule.

(II) Either (1) or (2) holds.

(1) A rule \mathbf{R} is injective in at least one component and for any $\mathbf{x}, \mathbf{y} \in X_0^{\mathbf{R}}$, there exists $\mathbf{z} \in X_0^{\mathbf{R}}$ such that $\mathbf{x} \dot{+} \mathbf{y} = \mathbf{z}$.

(2) $\mathbf{R} = 0$.

Proof. (II) \Rightarrow (I). It is obvious that (2) implies (I). So we show that (1) implies (I).

Let \mathbf{R} is injective in the $(i+m)$ th component ($m \in \{-r, \dots, r\}$) and \mathbf{x}, \mathbf{y} belong to $X_1^{\mathbf{R}}$. Then $\bar{\mathbf{x}}^m, \bar{\mathbf{y}}^m$ belong to $X_0^{\mathbf{R}}$. So $\bar{\mathbf{x}}^m \dot{+} \bar{\mathbf{y}}^m = \mathbf{x} \dot{+} \mathbf{y}$. Therefore

$$f(\mathbf{x} \dot{+} \mathbf{y}) = 0 = f(\bar{\mathbf{x}}^m \dot{+} \bar{\mathbf{y}}^m).$$

In other cases, we can prove in a similar way.

(I) \Rightarrow (II). Suppose a rule \mathbf{R} is not injective in any component. Then it follows that all $\mathbf{y} \in X$ are mapped to 0 by the linearity of the rule \mathbf{R} . \square

3 Definition of class

In this section, based on Wolfram's four classes, we redefine four classes mathematically.

Notation 2. 1. An initial condition $\{x_i^0 \mid -\infty < i < \infty\}$ is said to be a finite initial condition $I[M_1, M_2]$ on an infinite lattice if there exist finite numbers M_1, M_2 satisfying $x_i^0 = 0$ for $i < M_1, i > M_2$ and $x_{M_1}^0 = x_{M_2}^0 = 1$.

2. Let a member of the set $\{x_i^t \mid -\infty < i < \infty\}$ [resp. $\{x_i^t \mid t = 0, 1, \dots\}$] be called a spatial sequence S^t [resp. a temporal sequence W_i].

Definition 1. Let $\{x_i^t \mid M_1 - t < i < M_2 + t, t \geq 0\}$ be generated by the rule for initial condition $I[M_1, M_2]$. Consider the following three cases (a), (b) and (c).

Case (a) There exists a time t_0 such that $\{x_i^t \mid M_1 - t < i < M_2 + t\}_{t \geq t_0}$ is homogeneous, that is,

$$x_i^t = 0 \quad (\text{for all } t \geq t_0 \text{ and } M_1 - t < i < M_2 + t)$$

or

$$x_i^t = 1 \quad (\text{for all } t \geq t_0 \text{ and } M_1 - t < i < M_2 + t).$$

Case (b) For each site i there exist a time t_1 and a natural number m such that $x_i^t = x_i^{t+m}$ for $t \geq t_1$, and not case (a).

Case (c) There exists at least one site i such that a temporal sequence W_i is aperiodic.

Then one of the above cases occur and we define four classes as follows:

Class I Case (a) holds for any initial condition $I[M_1, M_2]$.

Class II Case (b) holds for any initial condition $I[M_1, M_2]$.

Class III Case (c) holds for any initial condition $I[M_1, M_2]$.

Class IV At least two of cases (a)~(c) occur depending on the initial condition.

To show which class a rule belongs to, we use the following theorem.

Notation 3. For a state $x_i^t \in \mathbb{Z}_2$, let $x_i^t \cdot x_j^t := x_i^t \times x_j^t$.

Theorem 3.1. Let $B_0 = \{0\}$, $B_1 = \{1, 2, 4\}$, $B_2 = \{3, 5, 6\}$ and $B_3 = \{7\}$, and define $\{a_i\}_{i=0}^7$ by using $\{c_i\}_{i=0}^7$ in (1.1) as follows.

(1) For $i \in B_0$, let $a_i = c_i$.

(2) For $i \in B_1$, let $a_i = c_0 \dot{+} c_i$.

(3) For $i \in B_2$,

$$a_i = \begin{cases} c_i \dot{+} c_0 & \text{if } 4a_4 + 2a_2 + a_1 = i \text{ or } 4a_4 + 2a_2 + a_1 = 7 - i, \\ c_i \dot{+} c_0 \dot{+} 1 & \text{otherwise.} \end{cases}$$

(4) For $i \in B_3$,

$$a_7 = \begin{cases} 1 \dot{+} c_7 & \text{if } \sum_{i=0}^6 a_i = 1 \pmod{2}, \\ c_7 & \text{if } \sum_{i=0}^6 a_i = 0 \pmod{2}. \end{cases}$$

Then the rule function f can also be expressed as follows:

$$\begin{aligned} f(x_{i-1}^t \cdot x_i^t \cdot x_{i+1}^t) = & a_0 \dot{+} a_1 x_{i+1}^t \dot{+} a_2 x_i^t \dot{+} a_3 x_i^t \cdot x_{i+1}^t \\ & \dot{+} a_4 x_{i-1}^t \dot{+} a_5 x_{i-1}^t \cdot x_{i+1}^t \dot{+} a_6 x_{i-1}^t \cdot x_i^t \\ & \dot{+} a_7 x_{i-1}^t \cdot x_i^t \cdot x_{i+1}^t, \end{aligned} \quad (3.1)$$

where $a_j \in \{0, 1\}$ with $j = 0, 1, \dots, 7$ and “ $\dot{+}$ ” denotes addition modulo 2.

Proof. To show that the function (3.1) represents the rule \mathbf{R} defined by (1.1), it is enough to show that $f(x_2 x_1 x_0) = c_{2^2 x_2 + 2x_1 + x_0}$ holds for any $x_2 x_1 x_0 \in X$.

For $\mathbf{x} = 000$, $f(000) = a_0 = c_0$ by (1). For $\mathbf{x} \in \{001, 010, 100\}$, let $\mathbf{x} = x_2x_1x_0$, $x_k = 1(k \in \{0, 1, 2\})$. Then $2^k \in B_1$ and by (2),

$$\begin{aligned} f(x_2x_1x_0) &= a_4x_2 \dot{+} a_2x_1 \dot{+} a_1x_0 \dot{+} a_0 \\ &= a_{2^k} \dot{+} a_0 = (c_0 \dot{+} c_{2^k}) \dot{+} a_0 \\ &= c_{2^k}. \end{aligned}$$

For $x \in \{011, 101, 110\}$ let $\mathbf{x} = x_2x_1x_0$, $x_k = x_m = 1(k \neq m, k, m \in \{0, 1, 2\})$. Then $2^k + 2^m \in B_2$. We show $f(x_2x_1x_0) = c_{2^k+2^m}$. Now

$$f(x_2x_1x_0) = a_{2^k}x_k \dot{+} a_{2^m}x_m \dot{+} a_{2^k+2^m}x_m \cdot x_k \dot{+} a_0.$$

If either $4a_4 + 2a_2 + a_1 = i$ or $4a_4 + 2a_2 + a_1 = 7 - i$, then $a_{2^k} = a_{2^m}$. By (3),

$$\begin{aligned} f(\mathbf{x}) &= a_{2^k+2^m} \dot{+} a_0 \\ &= c_{2^k+2^m} \dot{+} c_0 \dot{+} a_0 \\ &= c_{2^k+2^m}. \end{aligned}$$

Otherwise, since $a_{2^k} \neq a_{2^m}$, $a_{2^k} \dot{+} a_{2^m} = 1$. By (3),

$$\begin{aligned} f(\mathbf{x}) &= 1 \dot{+} a_{2^k+2^m} \dot{+} a_0 \\ &= 1 \dot{+} (c_{2^k+2^m} \dot{+} c_0 \dot{+} 1) \dot{+} a_0 \\ &= c_{2^k+2^m}. \end{aligned}$$

Therefore $f(x_2x_1x_0) = c_{2^k+2^m}$.

For $\mathbf{x} = 111$, let $\mathbf{x} = x_2x_1x_0$. Then $2^0 + 2^1 + 2^2 \in B_3$ and by (4),

$$\begin{aligned} f(111) &= a_0 \dot{+} a_1 \dot{+} \cdots \dot{+} a_7 \\ &= 1 \dot{+} (1 \dot{+} c_7) \\ &= c_7, \end{aligned}$$

where $\sum_{i=0}^6 a_i = 1 \pmod{2}$, and

$$f(111) = a_0 \dot{+} a_1 \dot{+} \cdots \dot{+} a_7 = c_7,$$

where $\sum_{i=0}^6 a_i = 0 \pmod{2}$.

Therefore $f(x_2x_1x_0) = c_{2^{x_2+2x_1+x_0}}$ holds for all $x_2x_1x_0 \in X$. \square

4 Linear rule

In this section, the properties of linear rules are investigated. A rule is linear if and only if $a_0 = a_3 = a_5 = a_6 = a_7 = 0$ in (3.1) holds, i.e. one of the following rules

$$0, 60, 90, 102, 150, 170, 204, 240.$$

Proposition 4.1. *Suppose the rules evolve on lattice with arbitrary initial condition $I[M_1, M_2]$.*

- (i) *If \mathbf{R} is a rule with $a_4 = 1$ in (3.1), then $x_{M_2+t}^t = 1$ holds for any $t \geq 1$.*
- (ii) *If \mathbf{R} is a rule with $a_1 = 1$ in (3.1), then $x_{M_1-t}^t = 1$ holds for any $t \geq 1$.*

Proof. (i) Let \mathbf{R} be a rule with $a_4 = 1$ in (3.1). By induction, we can show

$$x_{M_2+t}^t = 1 \text{ for any } t \geq 1.$$

- (ii) It is obtained in the same way as in (i).

□

Proposition 4.2. *Let \mathbf{R} be a rule with $a_1 + a_2 + a_4 = 1$ and $a_0 + a_3 + a_5 + a_6 + a_7 = 0$ in (3.1).*

- (i) *For an initial condition $I[M_1, M_2]$ and any $i \in \mathbb{Z}$, there exists $T_{(i)} \in \mathbb{N}$ such that $x_i^t = x_i^{t+1}$ holds for $t \geq T_{(i)}$.*
- (ii) *For an initial condition $I[M_1, M_2]$ and any $t \geq 0$,*

$$x_{M_2+t}^t = 1, x_{M_1-t}^t = 1 \text{ or } x_{M_1}^t = 1 \text{ holds.}$$

Proof. Let \mathbf{R} be a rule with

$$x_i^{t+1} = a_1 x_{i+1}^t + a_2 x_i^t + a_4 x_{i-1}^t,$$

where $a_1 + a_2 + a_4 = 1$.

(i) Let \mathbf{R} a rule with $a_1 = 1$. By $x_i^{t+1} = x_{i+1}^t$, $x_{M_2+i-t}^t = 0$ for any $i, t \in \mathbb{N}$. So $x_{M_2-j}^{t+j+1} = x_{M_2-j}^{t+j+2}$ and $x_{M_2+j}^t = 0$ for $j, t \in \mathbb{N}$. Therefore there exists $T_{(i)} \in \mathbb{N}$ such that $x_i^t = x_i^{t+1}$ for $t > T_{(i)}$. For a rule \mathbf{R} with $a_4 = 1$ or $a_2 = 1$, the conclusion will be obtained similarly.

(ii) Let \mathbf{R} be a rule with $a_1 = 1$ [resp. $a_4 = 1$]. By Proposition 4.1, $x_{M_2+t}^t = 1$ holds for any $t \in \mathbb{N}$ [resp. $x_{M_1-t}^t = 1$].

Let \mathbf{R} be a rule with $a_2 = 1$. By $x_i^{t+1} = x_i^t$, $x_{M_1}^t = 1$ holds for any $t \in \mathbb{N}$.

□

Remark 1. Rules 170, 204 and 240 satisfy the condition of Proposition 4.2.

Proposition 4.3. *Let \mathbf{R} be a rule with $a_2 = 1$, $a_1 + a_4 = 1$, $a_0 + a_3 + a_5 + a_6 + a_7 = 0$ in (3.1). Then every temporal sequence W_i generated by the rule with an initial condition $I[M_1, M_2]$ on an infinite lattice is either 1 periodic or $2m$ periodic (m is a natural number).*

Proof. Let \mathbf{R} a rule with $a_1 = 1$. Then $x_i^{t+1} = x_{i-1}^t \dot{+} x_i^t$.

(1) For $i < M_1$, if $t = 1$, then $x_i^1 = x_{i-1}^0 \dot{+} x_i^0 = 0 \dot{+} 0 = 0$. We assume $x_i^t = 0$ for any $t \leq k$ with some $k \in \mathbb{N}$. For $t = k + 1$, $x_i^{k+1} = x_{i-1}^k \dot{+} x_i^k = 0 \dot{+} 0 = 0$ by assumption. Since $x_i^t = 0$ for any $t \geq 1$, any temporal sequence W_i is 1 periodic.

(2) For $i = M_1$ and $t = 1$, $x_{M_1}^1 = x_{M_1-1}^0 \dot{+} x_{M_1}^0 = 1$ by assumption. We assume that $x_i^t = 1$ for any $t \leq k$ with $k \in \mathbb{N}$. For $t = k + 1$, $x_{M_1}^{k+1} = x_{M_1-1}^k \dot{+} x_{M_1}^k = 0 \dot{+} 1 = 1$ by assumption. Since $x_{M_1}^t = 1$ for any $t \geq 1$, a temporal sequence W_{M_1} is 1 periodic.

(3) For $i > M_1$, we assume that a temporal sequence W_i is either 1 periodic or $2m$ periodic for any $i \leq i_0$ with $i_0 \geq M_1$. Let W_{i_0} be of p periodic with $p \in \{1, 2m\}$ and $k > p$. Then $x_{i_0}^{k-1} \dot{+} x_{i_0}^{k-2} \dot{+} x_{i_0}^{k-3} \dot{+} \dots \dot{+} x_{i_0}^{k-p} = a$ with $a \in \{0, 1\}$

holds for $k > p$. So

$$\begin{aligned}
x_{i_0+1}^k &= x_{i_0}^{k-1} \dot{+} x_{i_0+1}^{k-1} = x_{i_0}^{k-1} \dot{+} x_{i_0}^{k-2} \dot{+} x_{i_0+1}^{k-2} \\
&= x_{i_0}^{k-1} \dot{+} x_{i_0}^{k-2} \dot{+} x_{i_0}^{k-3} \dot{+} x_{i_0+1}^{k-3} \\
&\quad \vdots \\
&= x_{i_0}^{k-1} \dot{+} x_{i_0}^{k-2} \dot{+} \cdots \dot{+} x_{i_0}^{k-p} \dot{+} x_{i_0+1}^{k-p} \\
&= a \dot{+} x_{i_0+1}^{k-p}.
\end{aligned}$$

Therefore if $a = 1$, a temporal sequence W_{i_0+1} is $2p$ periodic, since $x_{i_0+1}^{k-p} = x_{i_0+1}^{k+p}$. If $a = 0$, a temporal sequence W_{i_0+1} is p periodic, since $x_{i_0+1}^{k-p} = x_{i_0+1}^k$.

Therefore every temporal sequence W_i is either 1 periodic or $2m$ periodic (m is a natural number).

For $a_4 = 1$, the conclusion will be obtained similarly. \square

Proposition 4.4 ([1]). *Let \mathbf{R} be an injective rule in its $(i+1)$ th component with $100 \in X_1^{\mathbf{R}}$ (or injective in its $(i-1)$ th component with $001 \in X_1^{\mathbf{R}}$). Then with arbitrary finite initial conditions, there can exist at most one periodic temporal sequence.*

Remark 2. Rules 90 and 150 satisfy the condition of Proposition 4.4.

5 Nonlinear rule

In this section, the property of nonlinear rules are investigated by using Theorem 3.1.

Proposition 5.1. *Consider the rule \mathbf{R} with $a_0 = 0$ and $a_1 + a_4 \leq 1$ in (3.1). Suppose there exists $M \in \mathbb{N}$ and $i_0 \in \mathbb{N}$ such that $x_{i_0}^t = 0$ holds for $t \geq M$ with an initial condition $I[M_1, M_2]$.*

- (i) *Let \mathbf{R} be a rule with $a_1 = 0$. If there exists $k \geq M$ such that $x_{i_0+1}^k = 0$, then $x_{i_0+1}^t = 0$ holds for any $t \geq k$.*
- (ii) *Let \mathbf{R} be a rule with $a_4 = 0$. If there exists $k \geq M$ such that $x_{i_0-1}^k = 0$, then $x_{i_0-1}^t = 0$ holds for any $t \geq k$.*

Proof. (i) As \mathbf{R} is a rule with $a_1 = 0$ and $x_{i_0}^t = 0$ for $t \geq M$, for any $t \geq M$

$$\begin{aligned} x_{i_0+1}^{t+1} &= f(x_{i_0}^t x_{i_0+1}^t x_{i_0+2}^t) \\ &= a_2 x_{i_0+1}^t + a_3 x_{i_0+1}^t \cdot x_{i_0+2}^t + a_4 x_{i_0}^t + a_5 x_{i_0}^t \cdot x_{i_0+2}^t \\ &\quad + a_6 x_{i_0}^t \cdot x_{i_0+1}^t + a_7 x_{i_0}^t \cdot x_{i_0+1}^t \cdot x_{i_0+2}^t \\ &= a_2 x_{i_0+1}^t + a_3 x_{i_0+1}^t \cdot x_{i_0+2}^t. \end{aligned}$$

Therefore there exists $k \geq M$ such that $x_{i_0+1}^k = 0$ and $x_{i_0+1}^t = 0$ for any $t \geq k$.

(ii) It is obtained in a similar way to (i). □

Proposition 5.2. *Let \mathbf{R} a rule with either*

$$a_4 = 1, a_0 = a_1 = a_2 = a_3 = 0 \tag{5.1}$$

or

$$a_1 = 1, a_0 = a_2 = a_4 = a_6 = 0 \tag{5.2}$$

in (3.1). Then for an initial condition $I[M_1, M_2]$ and $i \in \mathbb{Z}$ there exists $T_{(i)} \in \mathbb{N}$ such that $x_i^t = 0$ holds for $t > T_{(i)}$.

Proof. In case of (5.1), by assumption and Theorem 3.1, we have

$$x_i^{t+1} = x_{i-1}^t + a_5 x_{i-1}^t \cdot x_{i+1}^t + a_6 x_{i-1}^t \cdot x_i^t + a_7 x_{i-1}^t \cdot x_i^t \cdot x_{i+1}^t. \tag{5.3}$$

By using (5.3), we get $x_i^1 = f(x_{i-1}^0 x_i^0 x_{i+1}^0) = 0$ for any $i < M_1$ and $x_i^t = 0$ for any $t \geq 0$, $i < M_1$. As it is obvious that if $x_{i-1}^t = 0$, then $x_i^{t+1} = 0$, we have $x_{M_1-1+t}^t = 0$ for any $i \geq M_1$. By Proposition 5.1 (i), $x_i^t = 0$ for any $i \geq M_1$, $t \geq i - M_1 + 1$ and any $i \in \mathbb{Z}$, and so there exists $T_{(i)} \in \mathbb{N}$ such that $x_i^t = x_i^{t+1} = 0$ for any $t > T_{(i)}$. In case of (5.2), it is obtained in a similar way to (a). □

Proposition 5.3. *Let \mathbf{R} be a rule with either*

$$a_2 = a_3 = a_4 = 1, a_0 = a_1 = a_5 = 0 \tag{5.4}$$

or

$$a_1 = a_2 = a_6 = 1, a_0 = a_4 = a_5 = 0 \quad (5.5)$$

in (3.1). Then for an initial condition $I[M_1, M_2]$ and $i \in \mathbb{Z}$ there exists $T_{(i)} \in \mathbb{N}$ such that $x_i^t = 0$ holds for $t > T_{(i)}$.

Proof. In case of (5.4), by assumption,

$$x_i^{t+1} = f(x_{i-1}^t x_i^t x_{i+1}^t) = x_i^t \dot{+} x_i^t \cdot x_{i+1}^t \dot{+} x_{i-1}^t \dot{+} a_6 x_{i-1}^t \cdot x_i^t \dot{+} a_7 x_{i-1}^t \cdot x_i^t \cdot x_{i+1}^t.$$

By using Proposition 5.1 (i) and the relation above, we get the conclusion by induction. In case of (5.5), by assumption,

$$x_i^{t+1} = x_{i+1}^t \dot{+} x_i^t \dot{+} a_3 x_i^t \cdot x_{i+1}^t \dot{+} x_{i-1}^t \cdot x_i^t \dot{+} a_7 x_{i-1}^t \cdot x_i^t \cdot x_{i+1}^t.$$

The conclusion will be obtained in the same way as the proof of the case (5.4). \square

Proposition 5.4. Let \mathbf{R} be a rule with either

$$a_3 = a_4 = a_6 = 1, a_0 = a_1 = a_2 = 0 \quad (5.6)$$

or

$$a_1 = a_3 = a_6 = 1, a_0 = a_2 = a_4 = 0, a_5 + a_7 = 1 \quad (5.7)$$

in (3.1). Then for an initial condition $I[M_1, M_2]$ and $i \in \mathbb{Z}$ there exists $T_{(i)} \in \mathbb{N}$ such that $x_i^t = 0$ holds for $t > T_{(i)}$.

Proof. In case of (5.6),

$$x_i^{t+1} = x_{i-1}^t \dot{+} x_i^t \cdot x_{i+1}^t \dot{+} a_5 x_{i-1}^t \cdot x_{i+1}^t \dot{+} x_{i-1}^t \cdot x_i^t \dot{+} a_7 x_{i-1}^t \cdot x_i^t \cdot x_{i+1}^t,$$

where $a_5 + a_7 = 1$. By using Proposition 5.1 (i) and the relation above, we get the conclusion by induction. In case of (5.7), it is obtained in a similar way to the proof of the case (5.6). \square

Proposition 5.5. Let \mathbf{R} be a rule with $a_0 = a_1 = a_4 = 0$ in (3.1). Then for an initial condition $I[M_1, M_2]$ and $t \geq 0$, $x_i^t = 0$ holds for $i < M_1$ and $i > M_2$.

Proof. Now we have

$$x_i^{t+1} = a_2x_i^t + a_3x_i^t \cdot x_{i+1}^t + a_5x_{i-1}^t \cdot x_{i+1}^t + a_6x_{i-1}^t \cdot x_i^t + a_7x_{i-1}^t \cdot x_i^t \cdot x_{i+1}^t.$$

Let $t = 1$. Since $x_j^0 = 0$ for $j < M_1$ and $j > M_2$,

$$\begin{aligned} x_i^1 &= a_2x_{i-1}^0 \cdot x_i^0 \cdot x_{i+1}^0 + a_3x_i^0 \cdot x_{i+1}^0 + a_5x_{i-1}^0 \cdot x_{i+1}^0 + a_6x_{i-1}^0 \cdot x_i^0 + a_7x_{i-1}^0 \cdot x_i^0 \cdot x_{i+1}^0 \\ &= 0 \end{aligned}$$

for $i < M_1$ and $i > M_2$. When $t = k$, we assume that $x_j^k = 0$ for any $j < M_1$ and $j > M_2$. Then for any $i < M_1$ and $i > M_2$

$$\begin{aligned} x_i^{k+1} &= a_2x_i^k + a_3x_i^k \cdot x_{i+1}^k + a_5x_{i-1}^k \cdot x_{i+1}^k + a_6x_{i-1}^k \cdot x_i^k + a_7x_{i-1}^k \cdot x_i^k \cdot x_{i+1}^k \\ &= 0. \end{aligned}$$

Then $x_i^t = 0$ holds for $t \geq 0$, $i < M_1$ and $i > M_2$. □

Proposition 5.6. *Let \mathbf{R} be a rule with $a_0 = a_1 = a_2 = a_4 = 0$ and $a_3 + a_6 \leq 1$ in (3.1). Then for an initial condition $I[M_1, M_2]$ and $M_1 < i < M_2$ there exists $M \in \mathbb{N}$ such that $x_i^t = 0$ holds for any $t \geq M$.*

Proof. We have

$$x_i^{t+1} = a_3x_i^t \cdot x_{i+1}^t + a_5x_{i-1}^t \cdot x_{i+1}^t + a_6x_{i-1}^t \cdot x_i^t + a_7x_{i-1}^t \cdot x_i^t \cdot x_{i+1}^t,$$

where $a_3 + a_6 \leq 1$. By using Proposition 5.1 (i) and Proposition 5.5, the conclusion will be obtained by induction. □

6 Classification of some rules

In the previous sections, some propositions have been established. Using them, we classify some rules of the simplest cellular automata.

Theorem 6.1. *Some rules of the simplest cellular automata are classified as follows.*

<i>class</i>	<i>rule</i>
<i>class I</i>	0, 8, 32, 40, 64, 96, 128, 136, 160, 168, 192, 224
<i>class II</i>	2, 10, 16, 24, 34, 38, 42, 46, 48, 52, 56, 60, 66, 80, 98, 102, 112, 116, 130, 138, 144, 162, 166, 170, 174, 176, 180, 204, 208, 240
<i>class III</i>	18, 30, 86, 90, 150, 154, 210

Proof. Let

$$A_1 = \{2, 10, 16, 34, 42, 48, 80, 112, 130, 138, 144, 162, 176, 208\},$$

$$A_2 = \{38, 46, 52, 116, 166, 174, 180, 244\},$$

$$A_3 = \{24, 56, 66, 98\},$$

$$A_4 = \{8, 32, 40, 64, 96, 128, 136, 160, 168, 192, 224\},$$

$$A_5 = \{0, 170, 204, 240\},$$

$$A_6 = \{30, 86, 90, 150, 154, 210\},$$

$$A_7 = \{18\},$$

$$A_8 = \{60, 102\}.$$

- (i) For a rule $\mathbf{R} \in A_1$, there exists $T_{(i)} \in \mathbb{N}$ such that $x_i^t = 0$ for any $t > T_{(i)}$ and any $i \in \mathbb{Z}$ with an initial condition $I[M_1, M_2]$ by Proposition 5.2. In addition, either $x_{M_1-t}^t = 1$ for any $t \geq 1$ or $x_{M_2+t}^t = 1$ for any $t \geq 1$ by Proposition 4.1. Since every temporal sequence W_i generated by the rule \mathbf{R} does not satisfy (a) Definition 1 but satisfy (b), it belongs to Class II.
- (ii) For a rule $\mathbf{R} \in A_2$, by Proposition 4.1 and Proposition 5.3, every temporal sequence W_i generated by the rule \mathbf{R} does not satisfy (a) of Definition 1 but satisfy (b). Therefore it belongs to Class II.
- (iii) For a rule $\mathbf{R} \in A_3$, by Proposition 4.1 and Proposition 5.4, every temporal sequence W_i generated by the rule \mathbf{R} does not satisfy (a) of Definition 1 but satisfy (b). Therefore it belongs to Class II.
- (iv) For a rule $\mathbf{R} \in A_4$, there exists $M \in \mathbb{N}$ such that $x_i^t = 0$ for any $t > M$ and $M_1 - t < i < M_2 + t$ for an initial condition $I[M_1, M_2]$ by Proposition 5.5 and Proposition 5.6. Since the rule \mathbf{R} satisfies (a) of Definition 1, it belongs to Class I.

- (v) For a rule $\mathbf{R} \in A_5$ with $\mathbf{R} \neq 0$, either $x_{M_1-i}^t = 0$ or $x_{M_2+i}^t = 0$ holds for any $t, i \in \mathbb{N}$. So the rule \mathbf{R} belongs to Class II by Proposition 4.2. Rule 0 belongs to Class I obviously.
- (vi) For a rule $\mathbf{R} \in A_6$, almost all temporal sequence W_i generated by the rule \mathbf{R} is aperiodic for an initial condition $I[M_1, M_2]$ by Proposition 4.4. Since the rule \mathbf{R} satisfies (c) of Definition 1, it belongs to Class III.
- (vii) For a rule $\mathbf{R} \in A_7$, we get the conclusion by the following proposition.

Proposition 6.2 ([1]). *For arbitrary finite initial condition of even length on an infinite lattice, every temporal sequence generated by rule 18 is aperiodic. For arbitrary finite initial condition of odd length on an infinite lattice, every temporal sequence -with the exception of the trivial case-is aperiodic. The trivial case is the center temporal sequence of all 0's generated by rule 18 from a finite spatial sequence that is spatially symmetric, with all 0-blocks of odd length.*

- (viii) For a rule $\mathbf{R} \in A_8$, the conclusion will be obtained by Proposition 4.3.

□

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