# Chapter 3.

# The limit set of cellular automata

## 1 Introduction

A cellular automaton consists of d-dimensional lattice ($\mathbb{Z}^d, d \in \mathbb{N}$), and each site takes a state, one of a finite set of possible values. The value of each site evolves in discrete time steps and it is determined by the previous values of a neighborhood of sites around it.

Let $\mathcal{P}^d$ be the set of all configurations : $\mathbb{Z}^d \to \mathbb{Z}/p$. A map $L\colon \mathcal{P}^d \to \mathcal{P}^d$ is a *transition rule* if (1) $L(0) = 0$; and (2) there exist $v_1, \ldots, v_m \in \mathbb{Z}^d$ and a map $f : (\mathbb{Z}/p)^m \to \mathbb{Z}/p$ such that

$$(La)(x) = f(a(x+v_1), \ldots, a(x+v_m)) \text{ for all } x \in \mathbb{Z}^d, a \in \mathcal{P}^d. \tag{1.1}$$

To consider space-time patterns of cellular automata, we shall study the sequence $a$, $La$, $L^2a = L(La)$, $L^3a, \ldots$. If $a$ is any finite nonzero configuration, for any $k$, putting $a, La$, $L^2a$, $\ldots$, $L^ka$ on (d+1)-dimensional lattice in oder, contracting by $1/2^k$, one obtains $G_L^k a$ as a subset of $\mathbb{R}^d \times [0,1]$. S. Willson [3] studied when $L$ is linear modulo 2 and showed there exists a stable limit set of $G_L^k a$ as $k \to \infty$ and the limit set is independent of an initial configuration $a$, if $a$ is finite and nonzero .

When $L$ is non-linear, their behavior becomes complicated. Based on a large sample of cellular automata, it suggests that many cellular automata fall into four basic behavior classes and S.Wolfram [4] classified cellular automata with levels of prediction of the outcome of the evolution from particular initial states. If $L$ is non-linear, the existence of the limit set may depend on the initial configuration or there may exist no limit set for any initial configuration. We discussed the behavior of cellular automata in the case of $m = 3$ in (1.1) [1]. In [3], it was discussed when $p$ is 2 and a transition rule $L$ is linear. When $p$ is 2, the state of each site is a zero or a one and the set theory plays an important role. When $p$ is greater than 3, it is useful to consider a finite-valued function instead of the set theory and it may be helpful to use the operator theory.

In this paper, we shall investigate the structure of cellular automata by using the operator theory. We consider the space $USC$ of all upper semi continuous functions $g : \mathbb{R}^d \times [0,1] \to \mathbb{Z}/p$ and consider an operator on $USC$. In section 2, we define the product space $\prod E_k$ and the operator $\bar{F}_L$ on it corresponding to $L$. We investigate whether the limit set of $\bar{F}_L^k$ as $k \to \infty$ belongs to a certain subspace $E_\infty$ and the relation between $\lim G_L^k a$ and $\lim \bar{F}_L^k g$ [Theorem 1]. We consider a quotient space $\tilde{E} = \prod E_k / \sim$ and the operator $\tilde{F}_L$ on it and investigate conditions that the $\tilde{F}_L$-invariant set belongs to a certain subspace. In section 3, we consider the case of linear rules. In section 4, we consider the case that $L$ is non-linear. We show some conditions for $L$ and initial configurations such that there exists a $\tilde{F}_L$-invariant set when the term of non-linear is only quadratic [Theorems 5 and 6] and when the rule $L$ contains triadic non-linear terms [Theorem 7].

## 2 Operators on the space $USC$ and their limit

We shall consider cellular automata taking the value $\mathbb{Z}/2$. A *configuration* $a$ on $\mathbb{Z}^d$ is a map $a : \mathbb{Z}^d \to \mathbb{Z}/2$ and $\mathcal{P}^d$ is the set of all configurations on $\mathbb{Z}^d$. A configuration $a$ is *finite* provided $a(v) = 1$ for only finitely many $v$. We define two kinds of addition: If $a, b \in \mathcal{P}^d$ we may define $a + b \in \mathcal{P}^d$ by $(a+b)(v) = a(v) + b(v)$ mod 2 for $v \in \mathbb{Z}^d$. If $x, v \in \mathbb{Z}^d$, we may define the *translate* of $a \in \mathcal{P}^d$ by $v$ as $a \tilde{+} v$ where $(a \tilde{+} v)(x) = a(x - v)$. For $x \in \mathbb{Z}^d$, we define $\delta_x \in \mathcal{P}^d$ as

$$\delta_x(y) = \begin{cases} 1 & x = y, \\ 0 & x \neq y. \end{cases}$$

$\mathcal{P}_+^{d+1}$ is the set of all maps $w : \mathbb{Z}^d \times \mathbb{N} \to \mathbb{Z}/2$, and $\mathcal{P}_{+,k}^{d+1}$ is the set of $w \in \mathcal{P}_+^{d+1}$ such that $w(x,t) = 0$ for $t \geq k$.

$G_{L,k} : \mathcal{P}_{+,2^k}^{d+1} \to \mathcal{P}_{+,2^{k+1}}^{d+1}$ is defined by

$$G_{L,k} w(x,t) = \begin{cases} w(x,t) & 0 \leq t \leq 2^k - 1, \\ (L^{t+1-2^k} w_0)(x,t) & 2^k \leq t \leq 2^{k+1} - 1 \text{ for } w_0(x) = w(x, 2^k - 1), \\ 0 & 2^{k+1} \leq t. \end{cases}$$

Let $USC(\mathbb{R}^d \times [0,1])$ be the set of all upper semi continuous functions $g : \mathbb{R}^d \times [0,1] \to \{0,1\}$. The map $\phi_k : \mathcal{P}^{d+1}_{+,2^k} \to USC(\mathbb{R}^d \times [0,1])$ is defined by

$$\phi_k(w)(x,t) = \inf\left\{\psi(x,t) \mid \psi \in USC(\mathbb{R}^d \times [0,1]), \psi(x,t) \geq w([2^k x],[2^k t])\right\}$$

for $w \in \mathcal{P}^{n+1}_{+,2^k}$, where $[2^k x] = ([2^k x_1],[2^k x_2],\ldots,[2^k x_n])$ for $x = (x_1,x_2,\ldots,x_n)$ and $[2^k x_j]$ means the Gauss's symbol.

$G^k_L : \mathcal{P}^d \to USC(\mathbb{R}^d \times [0,1])$ is defined by

$$G^k_L a = \phi_k\left(\prod_{j=0}^{k-1} G_{L,j} a\right) \tag{2.1}$$

for $a \in \mathcal{P}^d$.

*Remark 1.* If $L$ is linear, then $G^k_L$ is also linear.

We define $f \geq g$ for $f,g \in USC(\mathbb{R}^d \times [0,1])$, if $f(x) \geq g(x)$ for all $x \in \mathbb{R}^d \times [0,1]$. Then $USC(\mathbb{R}^d \times [0,1])$ is a complete lattice [2, chap. 2]. For any $\{f_n\} \subset USC(\mathbb{R}^d \times [0,1])$, the relation $\bigwedge_{n=1}^{\infty}\bigvee_{k\geq n} f_k \geq \bigvee_{n=1}^{\infty}\bigwedge_{k\geq n} f_k$ holds. If they are equal we denote them both by $\lim_{n\to\infty} f_n$ in $USC(\mathbb{R}^d \times [0,1])$.

*Remark 2.* The existence of $\lim_{n\to\infty} G^n_L a$ depends on $L$ and on the initial configuration $a$.

In order to investigate the existence of the limit set, we shall consider a product space. Let $E_k = \phi_k(\mathcal{P}^{d+1}_{+,2^k})$, then $E_0 \subset E_1 \subset E_2 \subset \cdots \subset USC(\mathbb{R}^d \times [0,1])$. $F_{L,k}\colon E_k \to E_{k+1}$ is defined by

$$F_{L,k}(g) = \phi_{k+1} G_{L,k} \phi_k^{-1}(g) \qquad \text{for } g \in E_k.$$

Let $\prod E_k$ be the product space of $\{E_k\}$ and

$$E_\infty = \{\{g_k\} \in \prod E_k \mid \text{There exists } \lim_{k\to\infty} g_k \text{ in } USC(\mathbb{R}^d \times [0,1])\}.$$

The following relation holds:

$$\begin{array}{ccc} E_k & \xrightarrow{F_{L,k}} & E_{k+1} \\ {\scriptstyle\phi_k}\uparrow & & \uparrow{\scriptstyle\phi_{k+1}} \\ \mathcal{P}^{d+1}_{+,2^k} & \xrightarrow[G_{L,k}]{} & \mathcal{P}^{d+1}_{+,2^{k+1}} \end{array}$$

$\bar{F}_L\colon \prod E_k \to \prod E_k$ is defined by

$$\bar{F}_L(\bar{g}) = \{\lambda_k\}_{k=0}^{\infty} \quad \text{for } \bar{g} = \{g_k\},$$

where

$$\begin{cases} \lambda_0 & = & g_0, \\ \lambda_{k+1} & = & F_{L,k}(g_k) \quad k \geq 0. \end{cases}$$

The distance $d(\bar{g}, \bar{h})$ between $\bar{g} = \{g_k\}$ and $\bar{h} = \{h_k\} \in \prod E_k$ is defined by

$$d(\bar{g}, \bar{h}) = \sum_{k=0}^{\infty} \frac{1}{2^k} d(g_k, h_k),$$

where

$$d(g_k, h_k) = \begin{cases} 1 & g_k \neq h_k, \\ 0 & g_k = h_k. \end{cases}$$

For $\{\bar{g}^n\}_n \subset \prod E_k$ with $\bar{g}^n = \{g_k^n\}_k$, we shall define $\lim_{n\to\infty} \bar{g}^n$ in $\prod E_k$ by $\bar{h} \in \prod E_k$ if $\lim_{n\to\infty} d(\bar{g}^n, \bar{h}) = 0$. The following theorem holds.

**Theorem 1.** *The following statements hold:*

(a) *$\bar{F}_L$ is a contraction on the metric space $H_a := \{\bar{g} = \{g_k\} \in \prod E_k \mid g_0 = \phi_0(a)\}$ for any finite and nonzero $a \in \mathcal{P}^d$.*

(b) *There exists $\lim_{n\to\infty} \bar{F}_L^n \bar{g}$ in $\prod E_k$ for any $\bar{g} \in \prod E_k$.*

(c) *$\lim_{n\to\infty} \bar{F}_L^n \bar{g}$ in $\prod E_k$ is $\bar{F}_L$- invariant.*

(d) *The following* (d-1) *and* (d-2) *are equivalent for finite and nonzero $a \in \mathcal{P}^d$:*

(d-1) *There exists $\lim_{n\to\infty} G_L^n a$ in $USC(\mathbb{R}^d \times [0,1])$.*

(d-2) *$\lim_{n\to\infty} \bar{F}_L^n \bar{g}$ in $\prod E_k$ belongs to $E_\infty$ for $\bar{g} = \{\phi_0(a), 0, 0, \ldots\}$.*

*Proof.* Let $\bar{g} = \{g_k\}$, $\bar{h} = \{h_k\} \in H_a$. Then

$$\bar{F}_L \bar{g} = \{\phi_0(a), F_{L,0}\phi_0(a), F_{L,1}g_1, F_{L,2}g_2, \ldots\}$$

and

$$\bar{F}_L \bar{h} = \{\phi_0(a), F_{L,0}\phi_0(a), F_{L,1}h_1, F_{L,2}h_2, \ldots\}$$

hold. If there exists $k \geq 2$ such that $g_k = h_k$, then $F_{L,k}g_k = F_{L,k}h_k$. Thus

$$d(\bar{F}_L \bar{g}, \bar{F}_L \bar{h}) < \frac{1}{2} d(\bar{g}, \bar{h}),$$

which means (a). (b) follows from (a) and (c) is obvious.

We show (d-1) implies (d-2). For $k \leq n$,

$$\begin{aligned}
\lambda_k^n &= \Pi_{s=1}^k F_{L,s} g_0 = F_{L,k} F_{L,k-1} \cdots F_{L,1} g_0 \\
&= \phi_{k+1} G_{L,k} \phi_k^{-1} \phi_k G_{L,k-1} \phi_{k-1}^{-1} \phi_{k-1} G_{L,k-2} \phi_{k-2}^{-1} \cdots \phi_1 G_{L,1} \phi_0^{-1} g_0 \\
&= \phi_{k+1} \left( \Pi_{s=1}^k G_{L,s} \phi_0^{-1}(g_0) \right) \\
&= G_L^{k+1} \phi_0^{-1} g_0 \\
&= G_L^{k+1} a.
\end{aligned}$$

Then $\lim_{n\to\infty} \bar{F}_L^n \bar{g}$ belongs to $E_\infty$. It can be proved in the same way as above that (d-2) implies (d-1). $\square$

Since there isn't a one-to-one correspondence between the set $\{\lim_{n\to\infty} G_L^n a \mid a \in \mathcal{P}^d\}$ and the set $\{\bar{h} \in E_\infty \mid \bar{g} = \{\phi_0(a), 0, 0, \ldots\}$ such that $\bar{h} = \lim_{n\to\infty} \bar{F}_L^n \bar{g}\}$, we consider a quotient space. We define the following equivalence relation. The equivalence relation "$\sim$" for $\bar{g} = \{g_k\}, \bar{h} = \{h_k\} \in \prod E_k$ is defined by

$$\bar{g} \sim \bar{h} \iff \lim_{k\to\infty} g_k = \lim_{k\to\infty} h_k \quad \text{in} \quad USC(\mathbb{R}^d \times [0,1]). \tag{2.2}$$

Let $\tilde{E} = \prod E_k / \sim$ be a quotient space, $\pi : \prod E_k \to \tilde{E}$ be the canonical quotient map. Because $\bar{g} \sim \bar{h}$ implies $\bar{F}_L \bar{g} \sim \bar{F}_L \bar{h}$, we can define a map $\tilde{F}_L : \tilde{E} \to \tilde{E}$ by

$$\tilde{F}_L(\pi \bar{g}) = \pi(\bar{F}_L \bar{g}).$$

$$\begin{tikzcd}
\prod E_k \arrow[r, "\bar{F}_L"] \arrow[d, "\pi"'] & \prod E_k \arrow[d, "\pi"] \\
\tilde{E} \arrow[r, "\tilde{F}_L"'] & \tilde{E}
\end{tikzcd}$$

# 3 Linear rules

In this section we show that there exists $\lim_{n\to\infty} G_L^n a$ in $USC(\mathbb{R}^1 \times [0,1])$ for $L$ and the limit set is independent of an initial configuration $a$.

**Theorem 2.** *Let $L$ be a linear modulo 2 and $d = 1$. Then for a finite nonzero configuration $a \in \mathcal{P}^1$ there exists a limit set $\lim_{n\to\infty} G_L^n a$ in $USC(\mathbb{R}^1 \times [0,1])$, which is independent of $a$.*

Theorem 2 follows from Lemma 1. At first, we shall state Proposition 1 in order to prove Lemma 1.

**Proposition 1 ( S. Willson [3] ).** *Let $L$ be a linear transition rule. Then the following are true:*

(a) *For any positive integer $q$, $L^q$ is linear.*

(b) *$x \in L\delta_0$ if and only if $2^q x \in L^{2^q}\delta_0$.*

(c) *If $a \in \mathcal{P}^d$, then $La = \sum_{x\in a} L(\delta_0 \tilde{+} x) \subset \cup_{x\in a} L(\delta_0 \tilde{+} x)$.*

**Lemma 1.** *Suppose a map $G_L^k : \mathcal{P}^1 \to USC(\mathbb{R}^1 \times [0,1])$ is defined by the equation (2.1) and $a \in \mathcal{P}^1$ is finite and nonzero. Then the following are true:*

(a) $\displaystyle\bigwedge_{n=1}^{\infty} \bigvee_{k\geq n} G_L^k(\delta_0) = \bigwedge_{n=1}^{\infty} \bigvee_{k\geq n} G_L^k(a)$;

(b) $\displaystyle\bigvee_{n=1}^{\infty} \bigwedge_{k\geq n} G_L^k(\delta_0) = \bigvee_{n=1}^{\infty} \bigwedge_{k\geq n} G_L^k(a)$;

(c) $\displaystyle\bigwedge_{n=1}^{\infty} \bigvee_{k\geq n} G_L^k(\delta_0) = \bigvee_{n=1}^{\infty} \bigwedge_{k\geq n} G_L^k(\delta_0)$.

*Proof.* We first show that if $\bigwedge_{n=1}^{\infty} \bigvee_{k\geq n} G_L^k(\delta_0)(x,y) = 1$, then

$$\bigwedge_{n=1}^{\infty} \bigvee_{k\geq n} G_L^k(a)(x,y) = 1.$$

Assume that for any $n \in \mathbb{N}$, $\varepsilon > 0$, there is $k_n \geq n$, $(x',y') \in \mathbb{R}^1 \times [0,1]$ such that $|x'-x| + |y'-y| < \varepsilon$ and $G_L^{k_n}(\delta_0)(x',y') = 1$. We show that there is $(x'',y'') \in \mathbb{R}^1 \times [0,1]$ such that $|x''-x| + |y''-y| < \varepsilon$ and $G_L^{k_n}(a)(x'',y'') = 1$.

Let $(x', y') \in \left[\frac{j_n}{2^{k_n}}, \frac{j_n+1}{2^{k_n}}\right] \times \left[\frac{i_n}{2^{k_n}}, \frac{i_n+1}{2^{k_n}}\right]$, then $j_n \in L^{i_n}\delta_0$. Let $|a| < 2^b$, where $|a|$ is the diameter of $a$. Since $L$ is linear, $2^b j_n \in L^{2^b i_n}\delta_0$ by Proposition 1 (b). We obtain $L^{2^b i_n}\delta_0 \subset L^{2^b i_n}a$, that is , $2^b j_n \in L^{2^b i_n}a$, since $L^{2^b i_n}a = \Sigma L^{2^b i_n}\delta_0 \tilde{+} n_j = L^{2^b i_n}\delta_0 + \sum_{j=2}^{k}(L^{2^b i_n}\delta_0 \tilde{+} n_j)$ by Proposition 1(c) and $n_j < 2^b$. Thus there is $(x'', y'') \in \left[\frac{2^b j_n}{2^b 2^{k_n}}, \frac{2^b j_n+1}{2^b 2^{k_n}}\right] \times \left[\frac{2^b i_n}{2^b 2^{k_n}}, \frac{2^b i_n+1}{2^b 2^{k_n}}\right]$ and $|x''-x| + |y''-y| < |x''-x'| + |y''-y'| + |x'-x| + |y''-y| < 2/2^{k_n} + \varepsilon$ . Let $k_n > n$ for sufficiently large $n$, $|x''-x| + |y''-y| < \varepsilon$ , then for any $\varepsilon > 0, n \in \mathbb{N}$, $G_L^{k_n+b}(a)(x'', y'') = 1$.

Conversely, assume that for $n \in \mathbb{N}$, $\varepsilon > 0$, there is $k_n \geq n$, $(x', y')$ such that $|x'-x| + |y'-y| < \varepsilon$ and $G_L^{k_n}(a)(x', y') = 1$. Let $(x', y') \in \left[\frac{j_n}{2^{k_n}}, \frac{j_n+1}{2^{k_n}}\right] \times \left[\frac{i_n}{2^{k_n}}, \frac{i_n+1}{2^{k_n}}\right]$. Since $j_n \in L^{i_n}a \subset \bigcup(L^{i_n}\delta_0 \tilde{+} n_j)$ by Proposition 1 (c), there is $n_j$ such that $j_n \in L^{i_n}\delta_0 \tilde{+} n_j$. Let $(x'', y'') \in \left[\frac{j_n - n_j}{2^{k_n}}, \frac{j_n - n_j + 1}{2^{k_n}}\right] \times \left[\frac{i_n}{2^{k_n}}, \frac{i_n+1}{2^{k_n}}\right]$. Since $|x''-x'| + |y''-y'| \leq \frac{n_j}{2^{k_n}} + \frac{1}{2^{k_n}}$, $|x''-x| + |y''-y| < \varepsilon$ . Thus, we obtain $G_L^{k_n}(\delta_0)(x'', y'') = 1$. So (a) holds.

(b) can be proved in a similar way to (a).

It is clear that if $\bigvee_{n=1}^{\infty}\bigwedge_{k\geq n} G_L^k(\delta_0)(x, y) = 1$, then

$$\bigwedge_{n=1}^{\infty}\bigvee_{k\geq n} G_L^k(\delta_0)(x, y) = 1.$$

Assume that for $n \in \mathbb{N}$, $\varepsilon > 0$, there is $k_n \geq n$, $(x', y') \in \mathbb{R}^1 \times [0, 1]$ such that $|x'-x| + |y'-y| < \varepsilon$ and $G_L^{k_n}(\delta_0)(x', y') = 1$. We show that for any $\varepsilon > 0$, there is $m \in \mathbb{N}$, $(x'', y'') \in \mathbb{R}^1 \times [0, 1]$ such that $|x''-x| + |y''-y| < \varepsilon$ and $G_L^k(\delta_0)(x'', y'') = 1$ for all $k \geq m$.

Let $(x', y') \in \left[\frac{j_{n_1}}{2^{k_n}}, \frac{j_n+1}{2^{k_n}}\right] \times \left[\frac{i_n}{2^{k_n}}, \frac{i_n+1}{2^{k_n}}\right]$, then $j_n \in L^{i_n}\delta_0$. Since $L$ is linear, $2^b j_n \in L^{2^b i_n}\delta_0$ for any $b \in \mathbb{Z}$. Let $(x'', y'') \in \left[\frac{2^b j_n}{2^b 2^{k_n}}, \frac{2^b j_n+1}{2^b 2^{k_n}}\right] \times \left[\frac{2^b i_n}{2^b 2^{k_n}}, \frac{2^b i_n+1}{2^b 2^{k_n}}\right]$, then

$$\begin{aligned} |x''-x| + |y''-y| &< |x''-x'| + |y''-y'| + |x'-x| + |y'-y| \\ &< \frac{2}{2^{2k_n}} + \varepsilon \, . \end{aligned}$$

Thus, for sufficiently large n, $|x''-x| + |y''-y| < 2\,\varepsilon$ . Let $k_n = m$, then $G_L^k(\delta_0)(x'', y'') = 1$ for all $k \geq m$. This implies that (c) holds. This completes the proof. □

The next theorem follows from Theorem 1(c) and Theorem 2.

**Theorem 3.** *Let $L$ be linear modulo 2 and $d = 1$ and $\bar{g} = \{g_k\} \in \prod E_k$. If $g_0 \in USC(\mathbb{R}^1 \times [0,1])$ has a compact support and nonzero, then there exists $\lim_{n\to\infty} \bar{F}_L^n \bar{g}$ in $\prod E_k$ and it belongs to $E_\infty$.*

The next theorem follows from Theorem 2 and Theorem 3.

**Theorem 4.** *Let $L$ be linear modulo 2 and $d = 1$.*

(a) *The $\tilde{F}_L$-invariant set in $\pi(E_\infty)$ consists of one element $\tilde{h}$.*

(b) *For any $\bar{g} = \{g_k\} \in \prod E_k$, there exists the limit set $\lim_{n\to\infty} \tilde{F}_L^n(\pi\bar{g})$, which is equal to $\tilde{h}$ in* (a), *if $g_0 \neq 0$ has a compact support.*

# 4 Non-linear rules

Consider a transition rule $L:\mathcal{P}^1 \to \mathcal{P}^1$ mod 2 as follows:

$$\begin{aligned} La(x) &= \sum_{k=1}^{m} \alpha_k a(x+v_k) + \sum_{i<j} \beta_{i,j} a(x+v_i)a(x+v_j) \\ &\quad + \sum_{i_1<i_2<i_3} \gamma_{i_1,i_2,i_3} a(x+v_{i_1})a(x+v_{i_2})a(x+v_{i_3}) \\ &= L_0 a(x) + L_1 a(x) + L_2 a(x), \end{aligned}$$

that is, $L_0$ is linear and $L_1$ and $L_2$ are non-linear. Let $A = \{i \mid \alpha_i \neq 0\}$, $B = \{(i,j) \mid \beta_{i,j} \neq 0\}$, $C = \{(i_1,i_2,i_3) \mid \gamma_{i_1,i_2,i_3} \neq 0\}$, then we can rewrite

$$\begin{aligned} La(x) &= \sum_{k\in A} a(x+v_k) + \sum_{(i,j)\in B} a(x+v_i)a(x+v_j) \\ &\quad + \sum_{(i_1,i_2,i_3)\in C} a(x+v_{i_1})a(x+v_{i_2})a(x+v_{i_3}). \end{aligned} \tag{4.1}$$

Let $\bar{\delta}_0 = \{\phi_0(\delta_0), 0, 0, \ldots\} \in \prod E_k$. We shall investigate conditions of $L$ and an initial configuration $g_0$ such that $\lim_{n\to\infty} \bar{F}_L^n \bar{g}$ in $\prod E_k$ belongs to $E_\infty$ for $\bar{g} = \{\phi_0(g_0), 0, 0, \ldots\}$.

We define

$$V_q = \left\{ \sum_{j=0}^{k} \delta_{n_j \cdot q} \mid k \geq 0, n_0 = 0, n_j \geq 1, (j \geq 1), n_{j+1} > n_j \right\},$$

$$m(b) = n_k \cdot q \quad \text{for } b = \sum_{j=0}^{k} \delta_{n_j \cdot q} \in V_q$$

and

$$W = \left\{ \sum_{j=1}^{k} \delta_j \mid k \geq 1 \right\}.$$

**Proposition 2.** *The following statements hold.*

(1) *If $L$ is linear, for $n \in \mathbb{N}, a \in \mathcal{P}^1$*

$$L^n a(x) = \sum_{s_1 \in A} \sum_{s_2 \in A} \cdots \sum_{s_n \in A} a(x + v_{s_1} + \cdots + v_{s_n}).$$

(2) *Suppose $C = \emptyset$ for $C$ in (4.1). If there is $q \geq 2$ such that*

(i) *$k_1, k_2 \in A$ implies $q|(v_{k_1} - v_{k_2})$,*

(ii) *$(i,j) \in B$ implies $0 < |v_i - v_j| < q$,*

*then*

$$L^n g_0(x) = L_0^n g_0(x) \text{ holds for any } x \in \mathbb{Z}, n \in \mathbb{N}, g_0 \in V_q.$$

*Proof.* (1) It is clear that $La(x) = \sum_{s_1 \in A} a(x + v_{s_1})$ for $n = 1$.

Suppose $L^n a(x) = \sum_{s_1 \in A} \sum_{s_2 \in A} \cdots \sum_{s_n \in A} a(x + v_{s_1} + \cdots + v_{s_n})$ holds. Then

$$\begin{aligned} L^{n+1} a(x) &= L(L^n a)(x) \\ &= \sum_{s_{n+1} \in A} L^n a(x + v_{s_{n+1}}) \\ &= \sum_{s_{n+1} \in A} \sum_{s_1 \in A} \sum_{s_2 \in A} \cdots \sum_{s_n \in A} a(x + v_{s_1} + \cdots + v_{s_{n+1}}). \end{aligned}$$

(2) For $n = 1$, we have

$$\begin{aligned} L_1(g_0)(x) &= \sum_{(i,j) \in B} g_0(x + v_i) g_0(x + v_j) \\ &= g_0(x + v_{i_1}) g_0(x + v_{j_1}) + g_0(x + v_{i_2}) g_0(x + v_{j_2}) + \cdots \end{aligned}$$

Suppose $L_1(g_0)(x) = 1$, then there is $(i_s, j_s) \in B$ such that $g_0(x + v_{i_s}) g_0(x + v_{j_s}) = 1$, and we have

$$g_0(x + v_{i_s}) = 1,$$

$$g_0(x + v_{j_s}) = 1,$$

so that,

$$x + v_{i_s} = kq,$$
$$x + v_{j_s} = k'q.$$

Thus $v_{i_s} - v_{j_s} = q(k-k')$, which contradicts the assumption, Therefore $L_1 g_0(x) = 0$ which implies $Lg_0(x) = L_0 g_0(x)$.

Suppose $L^t g_0(x) = 0$. It is enough to show $L_1(L_0^t g_0)(x) = 0$, since

$$\begin{aligned} L^{t+1} g_0(x) &= L(L^t g_0)(x) \\ &= L_0(L_0^t g_0)(x) + L_1(L_0^t g_0)(x) \\ &= L_0^{t+1} g_0(x) + L_1(L_0^t g_0)(x). \end{aligned}$$

Suppose

$$L_1(L_0^t g_0)(x) = \sum_{(i,j)\in B} \{L_0^t g_0(x+v_i) L_0^t g_0(x+v_j)\} = 1,$$

then there must exist $(i', j') \in B$ such that $L_0^t g_0(x + v_{i'}) = L_0^t g_0(x + v_{j'}) = 1$. Since $L_0^t g_0(x) = \sum_{s_1 \in A} \sum_{s_2 \in A} \cdots \sum_{s_n \in A} g_0(x + v_{s_1} + \cdots + v_{s_n})$ by Proposition 2(1), there exist $\{s_i\}_{i=1}^t$, $\{s_i'\}_{i=1}^t$, for $k, k' \in \mathbb{Z}$ such that

$$\begin{aligned} x + v_{s_1} + \cdots + v_{s_t} + v_{i'} &= kq \\ x + v_{s_1'} + \cdots + v_{s_t'} + v_{j'} &= k'q. \end{aligned}$$

We have $v_{i'} - v_{j'} = q(k - k' - M)$ for some $M \in \mathbb{Z}$, which contradicts the assumption, and $L_1(L_0^t g_0)(x) = 0$ holds. $\square$

**Theorem 5.** *Suppose a transition rule $L$ is defined by (4.1) and satisfies the following properties:*

*There is $q \geq 2$ such that*

(i) *$k_1, k_2 \in A$ implies $q | (v_{k_1} - v_{k_2})$;*

(ii) *$(i, j) \in B$ implies $0 < |v_i - v_j| < q$;*

(iii) *$C = \emptyset$.*

*Then there exists* $\tilde{h} \in \pi(E_\infty)$ *such that* $\lim_{n\to\infty} \tilde{F}_L^n(\pi\bar{g}) = \tilde{h}$ *holds for any* $\bar{g} = \{\phi_0(g_0), 0, 0, \ldots\}$ *with* $g_0 \in V_q$.

*Proof.* This follows from Proposition 2 and Theorem 3. □

**Lemma 2.** *Let* $L$ *satisfy the following properties:*

(a) *There is* $q \geq 2$ *such that*

$$q|(v_{j_{l+1}} - v_{j_l}) \quad (1 \leq l \leq M-1),$$

*where* $A = \{j_1, \ldots, j_M\}$ $(j_1 < \cdots < j_M)$;

(b) $B = \{(i,j) \mid v_i = v_j - 1$ *for* $j \in \{j_2, \ldots, j_M\}\}$;

(c) $C = \emptyset$.

*If* $c \in W$, *then* $Lc = L_0\delta_1 + \sum_{t=2}^{m(c)} \delta_{-v_{j_1}+t}$ *holds.*

*Proof.* It is enough to show

$$Lc(x) = \begin{cases} 1 & \text{for } x = -v_{j_1} + y \quad \text{with } 2 \leq y \leq m(c), \\ L_0\delta_1(x) & \text{otherwise,} \end{cases}$$

since

$$\sum_{t=2}^{m(c)} \delta_{-v_{j_1}+t}(x) = \begin{cases} 1 & \text{for } x = -v_{j_1} + y \quad \text{with } 2 \leq y \leq m(c), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$L_0\delta_1(x) = 0 \text{ for } x = -v_{j_1} + y \text{ with } 1 \leq y \leq m(c).$$

(i) Suppose $x = -v_{j_1} + y$ with $2 \leq y \leq m(c)$.

Put $e =$ the number of $\{k_l \in A \mid c(x + v_{k_l}) = 1\}$. Since

$$\begin{aligned} c(x + v_{j_1}) &= c(-v_{j_1} + y + v_{j_1}) = c(y) \\ &= 1 \qquad \text{for } 2 \leq y \leq m(c), \end{aligned}$$

there is $v_{j'_1} < \cdots < v_{j'_{e-1}}$ such that $c(x + v_{j'_l}) = 1$ for $1 \leq l \leq e-1$. $c(x + v_{i'})c(x + v_{j'}) = c(x + v_{j'_l} - 1)c(x + v_{j'_l}) = 1$ for $1 \leq l \leq e-1$ by assumption $(b)$, since $x + v_{j_1} < x + v_{j'_1}$ and $c(x + v_{j_1}) = 1$. Therefore

$$e' = \text{the number of } \{\ (i,j) \in B \mid c(x + v_i)c(x + v_j) = 1\} = e - 1,$$

and $e + e' = 2e - 1$. We obtain $Lc = 1$.

(ii) Suppose $x \neq -v_{j_1} + y$ for $2 \leq y \leq m(c)$.

We can rewrite as follows:

$$
\begin{aligned}
Lc &= \sum_{k_l \in A} c(x+v_{k_l}) + \sum_{(i,j)\in B} c(x+v_i)c(x+v_j) \\
&= \sum_{k_l \in A} \delta_1(x+v_{k_l}) + \sum_{k_l \in A}\{\sum_{t=2}^{m(c)} \delta_t(x+v_{k_l})\} + \sum_{(i.j)\in B} \delta_1(x+v_i)\delta_1(x+v_j) \\
&+ \sum_{(i,j)\in B} \delta_1(x+v_i)\{\sum_{t=2}^{m(c)} \delta_t(x+v_j)\} + \sum_{(i,j)\in B} \delta_1(x+v_j)\{\sum_{t=2}^{m(c)} \delta_t(x+v_i)\} \\
&+ \sum_{(i,j)\in B} \{\sum_{t=2}^{m(c)} \delta_t(x+v_i)\}\{\sum_{t=2}^{m(c)} \delta_t(x+v_j)\}.
\end{aligned}
$$

Put $e =$ the number of $\{k_l \in A \mid \delta_t(x+v_{k_l}) = 1, \quad 2 \leq t \leq m(c)\}$. Then by assumption there is $v_{j'_1} < \cdots < v_{j'_e}$ such that

$$2 \leq x + v_{j'_1} < \cdots < x + v_{j'_e} \leq m(c) \text{ and } v_{j'_1} \neq v_{j_1}. \tag{4.2}$$

First, we calculate the term $J_1 = \sum_{(i,j)\in B} \delta_1(x+v_j)\{\sum_{t=2}^{m(c)} \delta_t(x+v_i)\}$. If $\delta_1(x+v_j) = 1$, then $x+v_i \leq 1$ since $x+v_i < x+v_j$. Thus $\sum_{t=2}^{m(c)} \delta_t(x+v_i) = 0$, that is, the term equals 0. If $\delta_1(x+v_j) = 0$, clearly the term equals 0.

Next, we calculate the term

$$J_2 = \sum_{(i,j)\in B} \delta_1(x+v_i)\{\sum_{t=2}^{m(c)} \delta_t(x+v_j)\} + \sum_{(i,j)\in B} \{\sum_{t=2}^{m(c)} \delta_t(x+v_i)\}\{\sum_{t=2}^{m(c)} \delta_t(x+v_j)\}.$$

We rewrite

$$J_2 = \sum_{(i,j)\in B} \{\sum_{t=1}^{m(c)} \delta_t(x+v_i)\}\{\sum_{t=2}^{m(c)} \delta_t(x+v_j)\}.$$

If there is $(i_0, j_0) \in B$ such that $x + v_{i_0} = 1$, then $v_{j_0} = v_{i_0} + 1$ by assumption . Thus $v_{j_0} = v_{j'_1}$ and $x + v_{j_0} = 2$.

For $(i'_l, j'_l) \in B$ with $l = 1, \cdots, e$, $\sum_{t=1}^{m(c)} \delta_t(x+v_{i'_l}) \sum_{t=2}^{m(c)} \delta_t(x+v_{j'_l}) = 1$ by (4.2). Thus

$$\text{the number of } \{\ (i,j) \in B \mid \sum_{t=1}^{m(c)} \delta_t(x+v_i) \sum_{t=2}^{m(c)} \delta_t(x+v_j) = 1\} = e.$$

If $x+v_i \neq 1$ for any $(i,j) \in B$, then $x+v_{j_1'} > 2$ and $x+v_{i_1'} \geq 2$. Thus

$$\text{the number of } \{\ (i,j) \in B \mid \sum_{t=2}^{m(c)} \delta_t(x+v_i) \sum_{t=2}^{m(c)} \delta_t(x+v_j) = 1\} = e$$

by (4.2). Finally,

$$\begin{aligned} Lc &= \sum_{k_l \in A} \delta_1(x+v_{k_l}) + \sum_{(i.j) \in B} \delta_1(x+v_i)\delta_1(x+v_j) + \sum_{k_l \in A}\{\sum_{t=2}^{m(c)} \delta_t(x+v_{k_l})\} \\ &\quad + J_1 + J_2 \\ &= \sum_{k_l \in A} \delta_1(x+v_{k_l}) + \sum_{(i.j) \in B} \delta_1(x+v_i)\delta_1(x+v_j)c \\ &= L_0\delta_1 \end{aligned}$$

by $\sum_{(i.j)\in B} \delta_1(x+v_i)\delta_1(x+v_j) = 0$.

This completes the proof. □

**Proposition 3.** *Let* $L$ *satisfy the same conditions as in Lemma 2. If* $g_0 = b + (c\tilde{+}m(b))$ *for* $c \in W$, $b \in V_q$, *then*

$$L^n g_0 = L_0^n b + (c\tilde{+}(m(b) - nv_{j_1})). \tag{4.3}$$

*Proof.* We shall show (4.3) by induction. When $n=1$, we have

$$\begin{aligned} Lg_0 &= Lb + (c\tilde{+}m(b)) \\ &= \sum_{k \in A} \left\{b + (c\tilde{+}m(b))\right\}(x+v_k) \\ &\quad + \sum_{(i,j)\in B} \left\{b + (c\tilde{+}m(b))\right\}(x+v_i)\left\{b + (c\tilde{+}m(b))\right\}(x+v_j) \\ &= \sum_{k\in A} b(x+v_k) + \sum_{k \in A}(c\tilde{+}m(b))(x+v_k) + \sum_{(i,j)\in B} b(x+v_i)b(x+v_j) \\ &\quad + \sum_{(i,j)\in B} (c\tilde{+}m(b))(x+v_i)(c\tilde{+}m(b))(x+v_j) \\ &\quad + \sum_{(i,j)\in B} b(x+v_i)(c\tilde{+}m(b))(x+v_j) + \sum_{(i,j)\in B} (c\tilde{+}m(b))(x+v_i)b(x+v_j) \end{aligned}$$

$$
\begin{aligned}
&= L_0 b(x) + L((c\tilde{+}m(b))(x) + \sum_{(i,j)\in B} b(x+v_i)(c\tilde{+}m(b))(x+v_j) \\
&\quad + \sum_{(i,j)\in B} (c\tilde{+}m(b))(x+v_i)b(x+v_j) \qquad \text{(by } \sum_{(i,j)\in B} b(x+v_i)b(x+v_j)=0) \\
&= L_0 b(x) + \sum_{t'=2}^{m(c)} (\delta_{-v_{j_1}+t'}\tilde{+}m(b))(x) + L(\delta_1\tilde{+}m(b))(x) \\
&\quad + \sum_{(i,j)\in B} b(x+v_i)(c\tilde{+}m(b))(x+v_j) + \sum_{(i,j)\in B} (c\tilde{+}m(b))(x+v_i)b(x+v_j)
\end{aligned}
$$

by Lemma 2.

First, we show that $\sum_{(i,j)\in B}(c\tilde{+}m(b))(x+v_i)b(x+v_j) = 0$. Suppose that there is $(i,j) \in B$ such that $b(x+v_j) = (c\tilde{+}m(b))(x+v_i) = 1$, then $0 \le x+v_j \le m(b), 1+m(b) \le x+v_i \le m(b)+m(c)$. By assumption, there is some $k_l \in A$ such that

$$
\begin{aligned}
v_i &= v_{k_l} - 1, \\
v_j &= v_{k_l},
\end{aligned}
$$

therefore, we must have

$$
0 \le x + v_{k_l} \le m(b) \quad \text{and} \quad 2 + m(b) \le x + v_{k_l} \le m(b) + m(c) + 1,
$$

which is a contradiction. Thus $\sum_{(i,j)\in B}(c\tilde{+}m(b))(x+v_i)b(x+v_j) = 0$.

Let $J_1(x) = L(\delta_1\tilde{+}m(b))(x) + \sum_{(i,j)\in B} b(x+v_i)(c\tilde{+}m(b))(x+v_j)$, then

$$
\begin{aligned}
Lg_0 &= L_0 b(x) + \sum_{t'=2}^{m(c)} (\delta_{-v_{j_1}+t'}\tilde{+}m(b))(x) + L(\delta_1\tilde{+}m(b))(x) \\
&\quad + \sum_{(i,j)\in B} b(x+v_i)(c\tilde{+}m(b))(x+v_j) \\
&= L_0 b(x) + \sum_{t'=2}^{m(c)} (\delta_{-v_{j_1}+t'}\tilde{+}m(b))(x) + J_1(x).
\end{aligned}
$$

Furthermore, we have

$$
\sum_{(i,j)\in B} b(x+v_i)(c\tilde{+}m(b))(x+v_j) = \begin{cases} 1 & \text{for } x = m(b)+1-v_{k_l},\ k_l \in A\backslash\{v_{j_1}\}, \\ 0 & \text{otherwise} \end{cases} \tag{4.4}
$$

by the assumption $(b)$, since

$$
b(x+v_i) = \begin{cases} 1 & x+v_i = m(b), \\ 0 & x+v_i \ge m(b)+1 \end{cases}
$$

and

$$(c\tilde{+}m(b))(x+v_j) = \begin{cases} 1 & \text{for } m(b)+1 \le x+v_j \le m(b)+1+m(c), \\ 0 & \text{for } x+v_j \le m(b). \end{cases}$$

Since $c\tilde{+}m(b) = \sum_{k=1}^{m(c)}(\delta_{-v_{j_1}+k}\tilde{+}m(b))$, it is enough to show

$$J_1(x) = \begin{cases} 1 & x = m(b)+1-v_{j_1}, \\ 0 & x \neq m(b)+1-v_{j_1}. \end{cases} \tag{4.5}$$

Suppose $x \neq m(b)+1-v_{k_l}$ for any $k_l \in A\backslash\{v_{j_1}\}$, then

$$\begin{aligned} &L(\delta_1\tilde{+}m(b))(x) \\ &= \sum_{k_l\in A}\delta_1(x-m(b)+v_{k_l}) + \sum_{(i,j)\in B}\delta_1(x-m(b)+v_j)\delta_1(x-m(b)+v_j) \\ &= \sum_{k_l\in A}\delta_1(x-m(b)+v_{k_l}) \qquad \text{for } (i,j)\in B \end{aligned}$$

by $v_i \neq v_j$. Suppose $x-m(b)+v_{k_l}=1$ for some $k_l \in A$, then $x = m(b)+1-v_{k_l}$. This contradicts the assumption. Thus $J_1(x)=0$ by (4.4).

Suppose $x = m(b)+1-v_{k_l}$ for $k_l \in A\backslash\{v_{j_1}\}$, then

$$\begin{aligned} &L(\delta_1\tilde{+}m(b))(x) \\ &= \sum_{k_l\in A}\delta_1(x+v_k-m(b)) + \sum_{(i,j)\in B}\delta_1(x-m(b)+v_i)\delta_1(x-m(b)+v_j) \\ &= \sum_{k_l\in A}\delta_1(x+v_k-m(b)) + \sum_{(i,j)\in B}\delta_1(v_{k_l}+v_j)\delta_1(1-v_{k_l}+v_j) \\ &= 1, \end{aligned}$$

thus $J_1(x)=0$ by (4.4).

Suppose $x = m(b)+1-v_{j_1}$, then we have

$$\begin{aligned} L(\delta_1\tilde{+}m(b))(x) &= \sum_{k_l\in A}\delta_1(x+v_k-m(b)) \\ &\quad + \sum_{(i,j)\in B}\delta_1(x-m(b)+v_i)\delta_1(x-m(b)+v_j) \\ &= \sum_{k_l\in A}\delta_1(x+v_k-m(b)) \\ &\quad + \sum_{(i,j)\in B}\delta_1(v_{j_1}+v_j)\delta_1(1-v_{j_1}+v_j) \\ &= 1, \end{aligned}$$

thus $J_1(x) = 1$. Finally we obtain (4.5) and $Lg_0 = L_0 b + \sum_{k=1}^{m(c)}(\delta_{-v_{j_1}+1}\tilde{+}m(b))$.

Assume $L^t g_0 = L_0^t b + (c\tilde{+}(m(b) - tv_{j_1}))$ for $t \leq n$. Then

$$
\begin{aligned}
&L^{n+1}g_0(x) = L(L^n g_0(x)) \\
&= \sum_{k\in A} L^n g_0(x+v_k) + \sum_{(i,j)\in B} L^n g_0(x+v_i)L^n g_0(x+v_j) \\
&= L_0^{n+1}b(x) + Lc(x - m(b) + nv_{j_1}) + \sum_{(i,j)\in B} c(x+v_i - m(b) + nv_{j_1})L_0^n b(x+v_j) \\
&\quad + \sum_{(i,j)\in B} L_0^n b(x+v_i)c(x+v_j - m(b) + nv_{j_1}) + \sum_{(i,j)\in B} L_0^n b(x+v_i)L_0 b^n(x+v_j) \\
&= L_0^{n+1}b(x) + L_0\delta_1(x - m(b) + nv_{j_1}) + \sum_{h=2}^{m(c)} \delta_{-v_{j_1}+h}(x - m(b) + nv_{j_1}) \\
&\quad + \sum_{(i,j)\in B} c(x+v_i - m(b) + nv_{j_1})L_0^n b(x+v_j) + \sum_{(i,j)\in B} L_0^n b(x+v_i)L_0^n b(x+v_j) \\
&\quad + \sum_{(i,j)\in B} L_0^n b(x+v_i)c(x+v_j - m(b) + nv_{j_1}) \quad \text{(by Lemma 2)} \\
&= L_0^{n+1}b(x) + L_0\delta_1(x - m(b) + nv_{j_1}) + \sum_{h=2}^{m(c)} \delta_h(x - m(b) + (n+1)v_{j_1}) \\
&\quad + \sum_{(i,j)\in B} c(x+v_i - m(b) + nv_{j_1})L_0^n b(x+v_j) + \sum_{(i,j)\in B} L_0^n b(x+v_i)L_0^n b(x+v_j) \\
&\quad + \sum_{(i,j)\in B} L_0^n b(x+v_i)c(x+v_j - m(b) + nv_{j_1}).
\end{aligned}
$$

Since $L_0$ is linear,

$$
L_0^n g_0(x) = \sum_{s_1\in A}\sum_{s_2\in A}\cdots\sum_{s_n\in A} g_0(x + v_{s_1} + \cdots + v_{s_n})
$$

by Proposition 3. Suppose $\sum_{(i,j)\in B} L_0^n b(x+v_i)L_0^n b(x+v_j) = 1$ holds, then $\{s_i\}_{i=1}^n$, $\{s_i'\}_{i=1}^n$, for $k, k' \in \mathbb{Z}$ such that

$$
\begin{aligned}
x + v_{s_1} + \cdots + v_{s_n} + v_{i'} &= kq \\
x + v_{s_1'} + \cdots + v_{s_n'} + v_{j'} &= k'q.
\end{aligned}
$$

We get $q(k - k' - M) = v_{i'} - v_{j'} = -1$ for $M \in \mathbb{Z}$ by assumption (b), but this contradicts $q \geq 2$ and $k - k' - M \in \mathbb{Z}$.

Thus it is enough to show

$$\begin{aligned}
J_2(x) &= L_0\delta_1(x - m(b) + nv_{j_1}) + \sum_{(i,j)\in B} c(x + v_i - m(b) + nv_{j_1})L_0^n b(x + v_j) \\
&+ \sum_{(i,j)\in B} L_0^n b(x + v_i)c(x + v_j - m(b) + nv_{j_1}) + \sum_{(i,j)\in B} L_0^n b(x + v_i)L_0^n(x + v_j) \\
&= \begin{cases} 1 & \text{for } x = m(b) + 1 - (n+1)v_{j_1}, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

which will be proved in a similar way to the case $n = 1$.

So (4.3) holds for any $n \in \mathbb{N}$. This completes the proof. □

We define the set $J$ by

$$J = \{g_0 \mid g_0 = b + (c \tilde{+} m(b)) \quad \text{for} \quad c \in W,\ b \in V_q\}.$$

**Theorem 6.** *Suppose a transition rule $L$ is defined by (4.1) and satisfies the following properties:*

(a) *There is $q \geq 2$ such that*

$$q | (v_{j_{l+1}} - v_{j_l}) \quad \textit{for } 1 \leq l \leq M - 1,$$

*where $A = \{j_1, \ldots, j_M\}$ $(j_1 < \cdots < j_M)$;*

(b) $B = \{(i,j) \mid v_i = v_j - 1$ *for* $j \in \{j_2, \ldots, j_M\}\}$ ;

(c) $C = \emptyset$.

*Then there exists $\tilde{h} \in \pi(E_\infty)$ such that $\lim_{n\to\infty} \tilde{F}_L^n(\pi\bar{g}) = \tilde{h}$ holds for any $\bar{g} = \{\phi_0(g_0), 0, 0, \ldots\}$ with $g_0 \in J$.*

*Proof.* This follows from Proposition 3 and Theorem 3. □

We investigate the most simplest non-linear rule which contains the triadic term. We consider the conditions for $\bar{g}$ and $L$ such that $\lim_{n\to\infty} \tilde{F}_L^n(\pi\bar{g})$ exists and it belongs to $\tilde{h} \in \pi(E_\infty)$, when the rule satisfies $v_1 = -r$, $v_2 = -r + 1$, $\cdots$, $v_{2r+1} = r$.

**Lemma 3.** *Let $a \in \mathcal{P}^1$ be finite and nonzero. Suppose a transition rule $L$ is defined by (4.1) and satisfies the following properties:*

(a) $A = \{1, 2r+1\}$;

(b) $B = \{(1, r+1)\}$ *or* $B = \{(r+1, 2r+1)\}$;

(c) $C = \{(1, r+1, 2r+1)\}$.

*Then*

(i) *If $a(x)a(x+r+2rl) = 0$ for any $l \in \mathbb{N} \cup \{0\}$ and any $x \in \mathbb{Z}$, then*

$$L^n a(x) L^n a(x+r+2rl) = 0 \qquad (n \in \mathbb{N},\ l \in \mathbb{N} \cup \{0\}).$$

(ii) *If there is $M \in \mathbb{N}$ such that*

$$a(x)a(x+r+2rl) = 0 \qquad (x \in \mathbb{Z},\ 0 \leq l \leq M),$$

*then*

$$L^k a(x) L^k a(x+r+2rl) = 0 \quad (k \leq M,\ 0 \leq l \leq M-k).$$

*Proof.* Suppose $B = \{1, r+1\}$.

(i) Since $a(x)a(x+r+2rl) = 0$ for $n = 1$,

$$\sum_{(i,j)\in B} a(x+v_i)a(x+v_j) + \sum_{(i_1,i_2,i_3)\in C} a(x+v_{i_1})a(x+v_{i_2})a(x+v_{i_3}) = 0.$$

Therefore

$$\begin{aligned}
&La(x)La(x+r+2rl)\\
&\quad = \{a(x-r)+a(x+r)\}\{a(x+2rl)+a(x+r+2rl+r)\}\\
&\quad = a(x-r)a(x+2rl) + a(x-r)a(x+r+2rl+r)\\
&\qquad + a(x+r)a(x+2rl) + a(x+r)a(x+r+2rl+r)\\
&\quad = 0.
\end{aligned}$$

Proceeding inductively, we have $L^n a(x) L^n a(x+r+2rl) = 0$.

(ii) Let $k = 1$, then

$$
\begin{aligned}
&La(x)La(x+r+2rl) \\
&= \{a(x-r)+a(x+r)\}\{a(x+2rl)+a(x+r+2rl+r)\} \\
&= a(x-r)a(x+2rl)+a(x-r)a(x+r+2rl+r) \\
&\qquad + a(x+r)a(x+2rl)+a(x+r)a(x+r+2rl+r) \\
&= a(x-r)a(x-r+r+2r(l+1)).
\end{aligned}
$$

We have $a(x-r)a(x+r+2rl+r) = 0$ for $0 \le l \le M-1$ by assumption. Then

$$La(x)La(x+r+2rl) = 0 \text{ for } 0 \le l \le M-1.$$

Using an induction argument, we have $L^k a(x)L^k a(x+r+2rl) = 0$ for $k \le M$, $0 \le l \le M-k$.

For $B = \{r+1, 2r+1\}$, we can prove (i) and (ii) in the same way as above. $\square$

**Proposition 4.** *Suppose a transition rule $L$ is defined by (4.1) and satisfies the same conditions as in Lemma 3. Let $a \in \mathcal{P}^1$ be finite and nonzero. The following are equivalent:*

(i) *$a(x)a(x+r+2rl) = 0$ holds for any $l \in \mathbb{N} \cup \{0\}$, any $x \in \mathbb{Z}$.*

(ii) *$L^n a = L_0^n a$ holds for any $n \in \mathbb{N}$.*

*Proof.* Suppose $B = \{(1, r+1)\}$.

First we show (i) implies (ii). $L^n a(x)L^n a(x+r+2rl) = 0$ holds for $n \in \mathbb{N}$, $l \in \mathbb{N} \cup \{0\}$ by Lemma 3(i). Thus $La = L_0 a$ for $n = 1$ by $a(x)a(x+r+2rl) = 0$.
Assume $L^{n-1}a = L_0^{n-1}$.

$$
\begin{aligned}
L^n a(x) &= \textstyle\sum_{v_k \in A} L^{n-1}a(x+v_k) + L^{n-1}a(x+v_1)L^{n-1}a(x+v_{r+1}) \\
&\quad + L^{n-1}a(x+v_1)L^{n-1}a(x+v_{r+1})L^{n-1}a(x+v_{2r+1}) \\
&= \textstyle\sum_{v_k \in A} L^{n-1}a(x+v_k) \\
&= L_0^n a.
\end{aligned}
$$

Thus $L^n a = L_0^n a$ for any $n \in \mathbb{N}$.

Conversely, suppose $L^n a = L_0^n a$ hold for any $n \in \mathbb{N}$, then $a(x-r)a(x)\{1 + a(x+r)\} = 0$ by $La = L_0 a$, that is, either (I) $a(x-r)a(x) = 1$ and $a(x+r) = 1$ or (II) $a(x-r)a(x) = 0$ holds for $x \in \mathbb{Z}$. Let (I) holds for some $x \in \mathbb{Z}$. Set

$x' = \max\{x \mid x \text{ satisfies (I)}\}$, then $a(x'+r-r)a(x'+r) = 1$, but $a(x'+r+r) = 0$. Thus $x'+r$ doesn't satisfy (I) and (II). This contradicts the assumption. Therefore if $La = L_0 a$, then $a(x)a(x+r) = 0$ for $x \in \mathbb{Z}$.

If $L^{n-1}a = L_0^{n-1}a$, then $a(x)a(x+r+2r(n-1)) = 0$ for $x \in \mathbb{Z}$. $L_0^{n-1}a(x)L_0^{n-1}a(x+r) = 0$ holds by $L^n a = L_0^n a$. Therefore

$$
\begin{aligned}
& L_0^{n-1}a(x) + L_0^{n-1}a(x+r) \\
&= \{L_0^{n-2}a(x-r) + L_0^{n-2}a(x+r)\}\{L_0^{n-2}a(x) + L_0^{n-2}a(x+2r)\} \\
&= L_0^{n-2}a(x-r)L_0^{n-2}a(x+2r) \\
&= \{L_0^{n-3}a(x-2r) + L_0^{n-3}a(x)\}\{L_0^{n-3}a(x+r) + L_0^{n-3}a(x+3r)\} \\
&= L_0^{n-3}a(x-2r)L_0^{n-3}a(x+3r) \\
&\quad \vdots \\
&= L_0^{n-k}a(x-(k-1)r)L_0^{n-k}a(x+kr) \\
&\quad \vdots \\
&= a(x-(n-1)r)a(x+nr) \\
&= 0
\end{aligned}
$$

by Lemma 3 (ii).

For $B = \{(r+1, 2r+1)\}$, we may show the equivalence of (i) and (ii) in the same way as above. □

**Theorem 7.** *Suppose a transition rule $L$ is defined by (4.1) and satisfies the following properties:*

(a) $A = \{1, 2r+1\}$;

(b) $B = \{(1, r+1)\}$ *or* $B = \{(r+1, 2r+1)\}$;

(c) $C = \{(1, r+1, 2r+1)\}$.

*Let $g_0 \in \mathcal{P}^1$ be finite and nonzero. If $g_0(x)g_0(x+r+2rl) = 0$ for any $l \in \mathbb{N} \cup \{0\}$ and any $x \in \mathbb{Z}$, then there exists $\tilde{h} \in \pi(E_\infty)$ such that $\lim_{n\to\infty} \tilde{F}_L^n(\pi\bar{g}) = \tilde{h}$.*

*Proof.* This follows from Proposition 4 and Theorem 3. □

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